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# New stability tests for fractional positive descriptor linear systems

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The asymptotic stability of fractional positive descriptor continuous-time and discrete-time linear systems is considered. New sufficient conditions for stability of fractional positive descriptor linear systems are established. The efficiency of the new stability conditions are demonstrated on numerical examples of fractional continuous-time and discrete-time linear systems.

**Key words:** descriptor, fractional, positive, continuous-time, discrete-time, system, stability

## 1. Introduction

In positive systems state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1, 2, 5, 9]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of fractional calculus are given in the monographs [11–14]. The fractional positive linear systems have been addressed in [6, 7, 9, 10]. Descriptor positive systems have been analyzed in [3, 8, 10] and descriptor fractional linear systems in [6, 7, 10, 18, 19]. The two-dimensional continuous CFD pseudo-fractional systems described by the Roesser model have been considered in [15]. The stability of fractional linear discrete-time systems has been investigated in [10, 16, 17]. Robust stabilization of discrete-time positive switched systems with uncertainties and average dwell time switching has been

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analyzed in [20]. The asymptotic stability of positive descriptor continuous-time and discrete-time linear systems has been investigated in [4].

In this paper the asymptotic stability of fractional positive descriptor continuous-time and discrete-time linear systems will be addressed.

The paper is organized as follows. Some basic properties of fractional linear systems and elementary operations in matrices are presented in section 2. In section 3 the new stability tests for positive descriptor linear continuous-time systems are presented. The corresponding stability tests for positive descriptor discrete-time linear systems are given in section 4. Concluding remarks are given in section 5.

The following notation will be used:  $\mathfrak{R}$  – the set of real numbers,  $\mathfrak{R}^{n \times m}$  – the set of  $n \times m$  real matrices,  $\mathfrak{R}_+^{n \times m}$  – the set of  $n \times m$  real matrices with nonnegative entries and  $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$ ,  $I_n$  – the  $n \times n$  identity matrix.

## 2. Positive descriptor continuous-time linear systems

Consider the fractional descriptor continuous-time linear system

$$E \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t), \quad 0 < \alpha < 1, \quad (1)$$

where  $x(t) \in \mathfrak{R}^n$  the state vector,  $E, A \in \mathfrak{R}^{n \times n}$

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau, \quad \dot{x}(\tau) = \frac{dx(\tau)}{d\tau} \quad (2)$$

is the Caputo fractional derivative and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0 \quad (3)$$

is the gamma function [9, 13].

**Lemma 1** *If  $x(t) \in \mathfrak{R}^n$  is continuous-time vector function of  $t$  and  $0 < \alpha \leq 1$  then*

$$\int_0^\infty \left[ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{dx(\tau)}{d\tau} (t-\tau)^{-\alpha} d\tau \right] dt = x(\infty) - x(0). \quad (4)$$

Proof is given in [11].

**Definition 1** [9, 13] *The fractional descriptor system (1) is called (internally) positive if  $x(t) \in \mathfrak{X}_+^n$ ,  $t \geq 0$  for any initial conditions  $x(0) \in \mathfrak{X}_+^n$ .*

A real matrix  $A = [a_{ij}] \in \mathfrak{X}^{n \times n}$  is called Metzler matrix if its off-diagonal entries are nonnegative, i.e.  $a_{ij} \geq 0$  for  $i \neq j$ . The set of  $n \times n$  Metzler matrices will be denoted by  $M_n$ .

The system (1) satisfies the conditions  $\det E = 0$  and

$$\det[Es - A] \neq 0. \quad (5)$$

The following elementary operations on real matrices will be used [5]:

1. Multiplication of any  $i$ -th row (column) by the number  $a$ . This operation will be denoted by  $L[i \times a]$  for row operation and by  $R[i \times a]$  for column operation.
2. Addition to any  $i$ -th row (column) of the  $j$ -th row (column) multiplied by any number  $b$ . This operation will be denoted by  $L[i + j \times b]$  for row operation and by  $R[i + j \times b]$  for column operation.
3. The interchange of rows  $i$  and  $j$  will be denoted by  $L[i, j]$  and for columns by  $R[i, j]$ .

The elementary operations do not change the rank of the matrices [5].

### 3. Stability of fractional positive descriptor continuous-time linear systems

Consider the fractional positive system (1). Performing elementary row operations on the array

$$E, A \quad (6)$$

or equivalently on (1) we obtain

$$\begin{bmatrix} E_1 & A_1 \\ 0 & A_2 \end{bmatrix} \quad (7)$$

and

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{d^\alpha}{dt^\alpha} x_1(t) \\ \frac{d^\alpha}{dt^\alpha} x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (8)$$

where

$$\begin{aligned} x_1(t) \in \mathfrak{X}^{n_1}, \quad x_2(t) \in \mathfrak{X}^{n_2}, \quad A_1 = [A_{11} \ A_{12}], \quad A_2 = [A_{21} \ A_{22}], \\ A_{11} \in \mathfrak{X}^{n_1 \times n_1}, \quad A_{12} \in \mathfrak{X}^{n_1 \times n_2}, \\ A_{21} \in \mathfrak{X}^{n_2 \times n_1}, \quad A_{22} \in \mathfrak{X}^{n_2 \times n_2}, \quad n_1 + n_2 = n. \end{aligned} \quad (9)$$

**Lemma 2** *The system (1) is (internally) positive if the matrix  $E_1 \in \mathfrak{R}_+^{n_1 \times n}$  contains  $n_1$  monomial columns and*

$$\begin{aligned} A_{11} &\in M_{n_1}, & A_{12} &\in \mathfrak{R}_+^{n_1 \times n_2}, \\ A_{21} &\in \mathfrak{R}_+^{n_2 \times n_1}, & A_{22} &\in M_{n_2}, \quad n_1 + n_2 = n. \end{aligned} \quad (10)$$

**Proof.** If the matrix  $E_1$  contains  $n_1$  monomial columns then by elementary column operations the remaining its columns may be eliminated and we obtain  $E_1 = [E_{11} \ 0]$ , where  $E_{11} \in M_{n_1}$ . In this case from (8) we obtain  $\frac{d^\alpha}{dt^\alpha} x_1(t) = E_{11}^{-1} A_{11} x_1(t) + E_{11}^{-1} A_{12} x_2(t)$ , where  $E_{11}^{-1} A_{11} \in M_{n_1}$ ,  $E_{11}^{-1} A_{12} \in \mathfrak{R}_+^{n_1 \times n_2}$ . This completes the proof.  $\square$

**Theorem 1** *The positive descriptor system (1) is asymptotically stable if and only if there exists strictly positive vectors  $\lambda_1 \in \mathfrak{R}_+^{n_1}$ ,  $\lambda_2 \in \mathfrak{R}_+^{n_2}$  such that*

$$A_{11}\lambda_1 + A_{12}\lambda_2 < 0 \quad \text{and} \quad A_{21}\lambda_1 + A_{22}\lambda_2 = 0. \quad (11)$$

**Proof.** Integrating (8) we obtain

$$E_1 \int_0^\infty \begin{bmatrix} \frac{d^\alpha}{dt^\alpha} x_1(t) \\ \frac{d^\alpha}{dt^\alpha} x_2(t) \end{bmatrix} dt = A_{11} \int_0^\infty x_1(t) dt + A_{12} \int_0^\infty x_2(t) dt, \quad (12)$$

$$0 = A_{21} \int_0^\infty x_1(t) dt + A_{22} \int_0^\infty x_2(t) dt. \quad (13)$$

Taking into account that

$$E_1 \int_0^\infty \begin{bmatrix} \frac{d^\alpha}{dt^\alpha} x_1(t) \\ \frac{d^\alpha}{dt^\alpha} x_2(t) \end{bmatrix} dt = E_1 \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} - E_1 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad (14)$$

and for asymptotically stable system  $x_1(\infty) = 0$ ,  $x_2(\infty) = 0$  from (12) and (13) we obtain

$$-E_1 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = [A_{11} \ A_{12}] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} < 0 \quad (15)$$

and

$$[A_{21} \ A_{22}] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0, \quad (16)$$

where  $\lambda_1 = \int_0^\infty x_1(t) dt$ ,  $\lambda_2 = \int_0^\infty x_2(t) dt$  which is equivalent (11).

If  $A_{22} \in M_{n_2}$  is asymptotically stable then  $-A_{22}^{-1} \in \mathfrak{X}_+^{n_2 \times n_2}$  and from (16) we obtain

$$\lambda_2 = -A_{22}^{-1}A_{21}\lambda_1 \in \mathfrak{X}_+^{n_2}. \quad (17)$$

Substituting (17) into  $A_{11}\lambda_1 + A_{12}\lambda_2 = 0$  we obtain

$$\hat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21} \in M_{n_1} \quad (18)$$

which is asymptotically stable. This completes the proof.  $\square$

In particular case when the system is standard and  $\det E \neq 0$  then from (1) we have  $\frac{d^\alpha}{dt^\alpha}x(t) = \bar{A}x$ ,  $\bar{A} = E^{-1}A$  and the relations (11) take the well-known form [9]

$$\bar{A}\lambda < 0, \quad (19)$$

where  $\lambda \in \mathfrak{X}_+^n$  is strictly positive vector.

**Example 1** Consider the autonomous descriptor system (1) with the matrices

$$E = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 3 \\ 0 & -1 & 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 2 & -8 & 2 & 1 \\ -3 & -9 & 5 & 0 \\ -3 & 8 & -3 & 2 \end{bmatrix}. \quad (20)$$

The system satisfies the assumption (5) since

$$\det[Es - A] = \begin{vmatrix} 2s+4 & -1 & s-1 & 2s \\ -2 & s+8 & -2 & s-1 \\ 2s+3 & s+9 & s-5 & 3s \\ 3 & -s-8 & 3 & -s-2 \end{vmatrix} = 22s^2 + 175s + 37. \quad (21)$$

Performing on the array

$$E, A = \left[ \begin{array}{cccc|cccc} 2 & 0 & 1 & 2 & -4 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -8 & 2 & 1 \\ 2 & 1 & 1 & 3 & -3 & -9 & 5 & 0 \\ 0 & -1 & 0 & -1 & -3 & 8 & -3 & 2 \end{array} \right], \quad (22)$$

the following elementary operations  $L[4+2]$ ,  $L[3+1 \times (-1)]$ ,  $L[3+2 \times (-1)]$  and we obtain

$$\begin{array}{c} E_1 \\ 0 \end{array} \begin{array}{c} A_1 \\ A_2 \end{array} = \left[ \begin{array}{cccc|cccc} 2 & 0 & \vdots & 1 & 2 & -4 & 1 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 0 & 1 & 2 & -8 & \vdots & 2 & 1 \\ \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & \vdots & 0 & 0 & -1 & -2 & \vdots & 2 & -1 \\ 0 & 0 & \vdots & 0 & 0 & -1 & 0 & \vdots & -1 & 3 \end{array} \right]. \quad (23)$$

From (17) and (23) for  $\lambda_1 = [1 \ 1]^T$  we obtain

$$\lambda_2 = -A_{22}^{-1}A_{21}\lambda_1 = -\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \quad (24)$$

In this case we have

$$A_{11}\lambda_1 + A_{12}\lambda_2 = \begin{bmatrix} -4 & 1 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad (25)$$

and

$$\begin{aligned} \hat{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{bmatrix} -4 & 1 \\ 2 & -8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -2 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -3.2 & 2.2 \\ 4.2 & -5.2 \end{bmatrix}. \end{aligned} \quad (26)$$

The descriptor system with (20) satisfies the condition (11) for  $\lambda_1 = [1 \ 1]^T$ ,  $\lambda_2 = [2 \ 1]^T$  and by Theorem 1 it is asymptotically stable.

#### 4. Stability of fractional positive descriptor discrete-time linear systems

Consider the autonomous descriptor discrete-time linear system

$$E\Delta^\alpha x_{i+1} = Ax_i, \quad (27)$$

where  $x_i \in \mathfrak{R}^n$  is the state vector and  $E \in \mathfrak{R}^{n \times n}$ ,  $A \in \mathfrak{R}^{n \times n}$ .

It is assumed that the system (27) satisfies the conditions  $\det E = 0$  and

$$\det[Ez - A] \neq 0. \quad (28)$$

Using

$$\Delta^\alpha x_{i+1} = \sum_{k=0}^i (-1)^k \binom{\alpha}{k} x_{i-k}, \quad i = 0, 1, \dots, \quad (29)$$

where

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for } k = 1, 2, \dots \end{cases} \quad (30)$$

from (27) we obtain

$$Ex_{i+1} = A_\alpha x_i - \sum_{k=2}^{i+1} Ec_k x_{i-k+1}, \quad (31)$$

where

$$A_\alpha = A + E\alpha, \quad c_k = (-1)^{k+1} \binom{\alpha}{k} > 0, \quad k = 1, 2, \dots \quad (32)$$

Performing elementary row operations on the array

$$E, A \quad (33)$$

or equivalently on the system (27) we obtain

$$\begin{array}{cc} E_1 & A_1 \\ 0 & A_2 \end{array} \quad (34)$$

and

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} + \begin{bmatrix} E_1 \\ 0 \end{bmatrix} x_{oc}(i), \quad (35)$$

where

$$\begin{aligned} x_{oc}(i) &= \sum_{k=2}^{i+1} c_k x_{i-k+1}, \quad x_{1,i} \in \mathfrak{R}^{n_1}, \quad x_{2,i} \in \mathfrak{R}^{n_2}, \\ A_1 &= [A_{11} \ A_{12}], \quad A_2 = [A_{21} \ A_{22}], \\ A_{11} &\in \mathfrak{R}^{n_1 \times n_1}, \quad A_{12} \in \mathfrak{R}^{n_1 \times n_2}, \quad A_{21} \in \mathfrak{R}^{n_2 \times n_1}, \quad A_{22} \in \mathfrak{R}^{n_2 \times n_2}, \\ E_1 &= [E_{11} \ E_{12}], \quad E_{11} \in \mathfrak{R}_+^{n_1 \times n_1}, \quad E_{12} \in \mathfrak{R}_+^{n_1 \times n_2}. \end{aligned} \quad (36)$$

The discrete-time system (27) is positive if

$$A_{11} - E_{11} \in \mathfrak{R}_+^{n_1 \times n_1}, \quad A_{12} - E_{12} \in \mathfrak{R}_+^{n_1 \times n_2}, \quad A_{21} \in \mathfrak{R}_+^{n_2 \times n_1} \quad (37)$$

and

$$A_{22} \in M_{n_2} \quad (38)$$

is asymptotically stable.

Note that  $-A_{22} \in \mathfrak{R}_+^{n_2 \times n_2}$  if and only if  $A_{22}$  is asymptotically stable Metzler matrix [8].

**Theorem 2** *The fractional positive descriptor system (27) is asymptotically stable if there exists strictly positive vectors  $\lambda_1 \in \mathfrak{R}_+^{n_1}$ ,  $\lambda_2 \in \mathfrak{R}_+^{n_2}$ , such that*

$$(A_{11} - E_{11})\lambda_1 + (A_{12} - E_{12})\lambda_2 < 0 \quad \text{and} \quad A_{21}\lambda_1 + A_{22}\lambda_2 = 0. \quad (39)$$

**Proof.** Taking into account (35) and (36) we obtain

$$E_{11} \sum_{i=0}^{\infty} x_{1,i+1} + E_{12} \sum_{i=0}^{\infty} x_{2,i+1} = A_{11} \sum_{i=0}^{\infty} x_{1,i} + A_{12} \sum_{i=0}^{\infty} x_{2,i} + E_1 \sum_{i=0}^{\infty} x_{oc}(i), \quad (40)$$

$$0 = A_{21} \sum_{i=0}^{\infty} x_{1,i} + A_{22} \sum_{i=0}^{\infty} x_{2,i}. \quad (41)$$

Note that

$$E_{11} \sum_{i=0}^{\infty} x_{1,i+1} = E_{11} \left( \sum_{i=0}^{\infty} x_{1,i} - x_{1,0} \right), \quad E_{12} \sum_{i=0}^{\infty} x_{2,i+1} = E_{12} \left( \sum_{i=0}^{\infty} x_{2,i} - x_{2,0} \right) \quad (42)$$

and using (37) and (38) and taking into account that  $x_{oc}(i) > 0, i > 0$  we obtain

$$-E_{11}x_{1,0} - E_{12}x_{2,0} = [A_{11} \quad -E_{11}] \lambda_1 + [A_{12} - E_{12}] \lambda_2 < 0, \quad (43)$$

$$A_{21}\lambda_1 + A_{22}\lambda_2 = 0, \quad (44)$$

where

$$\lambda_1 = \sum_{i=0}^{\infty} x_{1,i}, \quad \lambda_2 = \sum_{i=0}^{\infty} x_{2,i}. \quad (45)$$

This completes the proof.  $\square$

**Remark 1** The condition (43) is not satisfied if at least one row of the matrix  $A_{11} + A_{12} - E_{11} - E_{12}$  is not negative.

**Remark 2** Assuming  $\lambda_1 \in \mathfrak{X}_+^{n_1}$  is strictly positive vector we may find the desired vector  $\lambda_2 \in \mathfrak{X}_+^{n_2}$  from the equation

$$\lambda_2 = -A_{22}^{-1}A_{21}\lambda_1. \quad (46)$$

**Example 2** Consider the positive descriptor system (27) with the matrices

$$E = \begin{bmatrix} 0.7 & 0.2 & \vdots & 0.5 & 0.1 \\ 0.2 & 0.6 & \vdots & 0.2 & 0.7 \\ \dots & \dots & \vdots & \dots & \dots \\ 0.9 & 0.8 & \vdots & 0.7 & 0.8 \\ -0.2 & -0.6 & \vdots & -0.2 & -0.7 \end{bmatrix}, \quad A = \begin{bmatrix} 0.2 & 0.3 & \vdots & 0.6 & 0.2 \\ 0.1 & 0.4 & \vdots & 0.3 & 0.4 \\ \dots & \dots & \vdots & \dots & \dots \\ 0.5 & 1 & \vdots & 0.2 & 0.8 \\ 0 & -0.1 & \vdots & 0 & -1.2 \end{bmatrix}. \quad (47)$$



The system satisfied the assumption (28) since

$$\det[Et - A] = 0.3893z^2 - 0.3738z + 0.0702. \quad (48)$$

Performing on the array

$$E, A = \left[ \begin{array}{cccc|cccc} 0.7 & 0.2 & \vdots & 0.5 & 0.1 & 0.2 & 0.3 & \vdots & 0.6 & 0.2 \\ 0.2 & 0.6 & \vdots & 0.2 & 0.7 & 0.1 & 0.4 & \vdots & 0.3 & 0.4 \\ \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \vdots & \dots & \dots \\ 0.9 & 0.8 & \vdots & 0.7 & 0.8 & 0.5 & 1 & \vdots & 0.2 & 0.8 \\ -0.2 & -0.6 & \vdots & -0.2 & -0.7 & 0 & -0.1 & \vdots & 0 & -1.2 \end{array} \right], \quad (49)$$

the following elementary operations  $L[3+1 \times (-1)]$ ,  $L[3+2 \times (-1)]$ ,  $L[4+2 \times (-1)]$  we obtain

$$\left[ \begin{array}{cc|cc} E_{11} & E_{12} & A_{11} & A_{12} \\ 0 & 0 & A_{21} & A_{22} \end{array} \right] = \left[ \begin{array}{cccc|cccc} 0.7 & 0.2 & \vdots & 0.5 & 0.1 & 0.2 & 0.3 & \vdots & 0.6 & 0.2 \\ 0.2 & 0.6 & \vdots & 0.2 & 0.7 & 0.1 & 0.4 & \vdots & 0.3 & 0.4 \\ \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & \vdots & 0 & 0 & 0.2 & 0.3 & \vdots & -0.7 & 0.2 \\ 0 & 0 & \vdots & 0 & 0 & 0.1 & 0.3 & \vdots & 0.3 & -0.8 \end{array} \right]. \quad (50)$$

In this case from (46) for  $\lambda_1 = [1 \ 1]^T$  we have

$$\lambda_2 = -A_{22}^{-1}A_{21}\lambda_1 = - \begin{bmatrix} -0.7 & 0.2 \\ 0.3 & -0.8 \end{bmatrix}^{-1} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.96 \\ 0.86 \end{bmatrix} \quad (51)$$

and

$$\begin{aligned} (A_{11} - E_{11})\lambda_1 + (A_{12} - E_{12})\lambda_2 &= \left( \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \end{bmatrix} - \begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 0.6 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &+ \left( \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.4 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.7 \end{bmatrix} \right) \begin{bmatrix} 0.96 \\ 0.86 \end{bmatrix} = \begin{bmatrix} -0.218 \\ -0.462 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \quad (52)$$

Therefore, the positive descriptor system (27) with (47) is asymptotically stable.

## 5. Concluding remarks

The asymptotic stability of fractional positive descriptor continuous-time and discrete-time linear systems has been investigated. New sufficient conditions for stability of the fractional positive descriptor continuous-time linear

systems have been given in section 3 and for discrete-time systems in section 4. The efficiency of the new stability conditions are demonstrated on numerical examples of fractional positive continuous-time and discrete-time linear systems. The considerations can be extended to positive descriptor different fractional orders continuous-time and discrete-time systems.

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