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Spherical fuzzy power partitioned Maclaurin Symmetric Mean Operators and their application in Multiple Attribute Group Decision Making

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Spherical fuzzy sets (SFSs) provide more free space for decision makers (DMs) to express preference information from four aspects: approval, objection, abstention and refusal. The partitioned Maclaurin symmetric mean (PMSM) operator is an effective information fusion tool, which can fully capture the interrelationships among any multiple attributes in the same block whereas attributes in different block are unrelated. Therefore, in this paper, we first extend PMSM operator to spherical fuzzy environment and develop spherical fuzzy PMSM (SFPMSM) operator as well as spherical fuzzy weighted PMSM (SFWPMSM) operator. Meanwhile, we discuss some properties and special cases of these two operators. To diminish the impact of extreme evaluation values on decision-making results, then we integrate power average (PA) operator and PMSM operator to further develop spherical fuzzy power PMSM (SFPPMSM) operator and spherical fuzzy weighted power PMSM (SFWPPMSM) operator and also investigate their desirable properties. Subsequently, a new multiple attribute group decision making (MAGDM) method is established based on SFWPPMSM operator under spherical fuzzy environment. Finally, two numerical examples are used to illustrate the proposed method, and comparative analysis with the existing methods to further testy the validity and superiority of the proposed method.

Key words: spherical fuzzy sets, partitioned Maclaurin symmetric mean operator, power average operator, multiple attribute group decision making

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1. Introduction

MAGDM is an important part of decision theory, which is to evaluate the alternative through a group of DMs on the premise of considering multiple attributes to get the optimal alternative. However, in the face of increasingly complex decision-making environment and uncertainty of evaluation information, how to effectively obtain attribute information is an important challenge for MAGDM. Thus Zadeh [1] proposed fuzzy sets (FSs) to reflect the fuzziness of things by defining membership degree (MD) function. Subsequently, Atanassov [2] proposed intuitionistic FSs (IFSs) on the basis of FSs. IFSs could describe things more accurately by adding non-membership degree (N-MD), which have been broadly used in many ways [3–10]. Nonetheless, IFSs also had some problems. For example, they required that the sum of MD and N-MD did not exceed 1, which limited the expression of DMs. To overcome above problem, Yager [11] proposed Pythagorean FSs (PyFSs) which satisfied that the sum of squares of MD and N-MD was no more than 1 and provided a wider range of MD and N-MD. However, abstinence degree (AD) depended on MD and N-MD in IFSs and PyFSs, which leaded to many unsatisfactory results. To do so, Cuong [12] presented picture fuzzy sets (PFSs) which utilized three indices (MD $M(s)$, N-MD $N(s)$, AD $I(s)$) with the condition $0 \leqslant M(s) + N(s) + I(s) \leqslant 1$. Obviously, PFSs were more reasonable than IFSs and PyFSs for dealing with ambiguous information. So far, the concept of PFSs has been widely used in many multiple attribute decision-making (MADM) problems [13–19].

Whereas in many cases, DMs tend to encounter situation which is invalid to use PFSs for example $M(s) + N(s) + I(s) > 1$. In such situation, a new type of FSs called SFSs were proposed by Mahmood, Ullah, Khan and Jan [20]. SFSs could not only express the attitude of DMs towards things from four aspects (yes, no, abstain, refusal), but also satisfy $0 \leqslant M^2(s) + N^2(s) + I^2(s) \leqslant 1$, which provided DMs with expansive space of information expression. Therefore, SFSs were better at capturing the fuzziness of things. Since SFSs were put forward, they have attracted the attention of many researchers. Ali [21] proposed a new score function based on CRITIC-MARCOS approach under SFSs. Dogan [22] designed spherical fuzzy AHP and sensitivity analysis for process mining technology selection. Fernandez-Martinez and Sanchez-Lozano [23] utilized SFSs to evaluate near-earth asteroid deflection techniques. Peng and Li [24] extended combined compromise solution approach in SFSs for IIoT industry evaluation. Zhang, Wei and Chen [25] designed spherical fuzzy CPT-MABAC method for green supplier selection. Seyfi-Shishavan, Gundogdu, Donyatalab, Farrokhizadeh and Kahraman [26] developed bi-objective linear assignment approach on basis of SFSs on insurance options selection. Zhang, Wei and Wei [27] presented spherical fuzzy TOPSIS approach based on cumulative prospect theory (CPT) for solving residential location issue. Wei, Wang, Lu, Wu and Wei [28] proposed similarity measures of SFSs by cosine function for MAGDM issues. Oztaysi, Onar, Gundogdu and Kahraman [29] used spherical fuzzy AHP-VIKOR approach for choosing ad position. Zhang, Wei and Chen [30] designed GRA method based on CPT in spherical fuzzy environment for emergency supplies supplier selection. Aydogdu and Gul [31] designed a novel entropy proposition of SFSs to MADM.

In decision system, aggregation operators (AOs) occupy a vital position in information fusion. Recently, numerous AOs have been developed by researchers to aggregate spherical fuzzy information. For example, Ashraf, Abdullah, Mahmood, Ghani and Mahmood [32] proposed spherical weighted averaging aggregation (SFNWAA) operator and spherical weighted geometric aggregation (SFNWGA) operator. Gundogdu and Kahraman [33] developed spherical weighted arithmetic mean (SWAM) and spherical weighted geometric mean (SWGM) operators. Donyatalab, Farokhizadeh, Garmroodi and Shishavan [34] introduced Harmonic mean AOs in spherical fuzzy environment. Ashraf, Abdullah and Mahmood [35] presented spherical fuzzy Dombi AOs for MADGD problems. Sindhu, Rashid and Kashif [36] established Hamy mean AOs of SFSs. Zhang, Wei and Chen [37] proposed spherical fuzzy Dombi power Heronian mean AOs for MAGDM issues. Ashraf, Abdullah and Aslam [38] put forward symmetric sum AOs for spherical fuzzy information, Farrokhizadeh, Seyfi Shishavan, Donyatalab, Kutlu Gündoğdu and Kahraman [39] designed spherical fuzzy Bonferroni mean (SFBM) AOs to MADM. However, most aforementioned operators are based on algebraic product and algebraic sum. In addition, these operators fail to model this situation where attributes are divided into some partitions and multiple attributes exist the interrelationships in each partition. As an extension of Maclaurin symmetric mean (MSM) [40] operator, PMSM [41] operator can not only reflect the relationship between attributes, but also capture the interrelationships among any multiple attributes in each category. The advantages of PMSM operator in information fusion have been elaborated in many literatures [42–45]. In view of these, in order to fuse spherical fuzzy information more effectively, we extend PMSM operator to spherical fuzzy environment and present SFPMSM as well as SFWPMSM operators. Meanwhile, we investigate some properties and special cases of these two operators. To minish the influence of extreme values on results in the evaluation process, then we integrate PA operator and PMSM operator to further develop SFPPMSM operator as well as SFWPPMSM operator and discuss some desirable properties of the developed operators. Whereafter, we establish a novel MAGDM approach by SFWPPMSM operator for settling spherical fuzzy uncertain problems. Eventually, we testify the availability and superiority of the established approach with existing approaches.

The motivation of this paper: SFSs can provide DMs with more free space to express preference information. At present, most spherical fuzzy AOs can only capture the relationship between two attribute values. As an effective information aggregation tool, PMSM operator can not only reflect the interrelationships between multiple attribute values but also capture the interrelationships among any multiple attribute values in each category. Thus, in order to better carry out information fusion, we extend PMSM operator to SFSs and present SFPMSM operator and SFWPMSM operator. Meanwhile, we investigate some properties and special cases of SFPMSM and SFWPMSM operators. In addition, considering the impact of extreme data on results in the evaluation process, we embed PA into PMSM operator to further develop SFPPMSM operator and SFWPPMSM operator. Then we design a new MAGDM method based on SFWPPMSM operator for settling spherical fuzzy uncertain problems. Finally, the availability and superiority for the designed approach are certified via comparative analysis with existing methods.

The main contributions of this article are: (1) to extend PMSM operator to SFSs and develop SFPMSM operator and SFWPMSM operator. Meanwhile, some properties and special cases of SFPMSM and SFWPMSM operators are researched; (2) to integrate PA operator and PMSM operator and further develop SFPPMSM operator as well as SFWPPMSM operator. Meanwhile, some desirable properties of these two operators are discussed; (3) to establish a new MAGDM approach based on SFWPPMSM operator to settle uncertain problems; (4) to utilize numerical examples to illustrate the established approach and use comparative analysis by existing approaches to demonstrate the feasibility and superiority for the established approach.

To do this, the remainders of this article are constructed as following: Part two briefly reviews some basic knowledge of SFSs and definitions of PMSM and PA operators. Part three develops SFPMSM and SFWPMSM operators and investigates some properties and special cases of these two operators. Part four integrates PA and PMSM operator to develop SFPPMSM and SFWPPMSM operators. Part five establishes a novel spherical fuzzy MAGDM method by SFWPPMSM operator. Part six provides numerical examples to certify the established method and gives a comparative analysis to illustrate the superiority of the established method. At last, we briefly summarize this article.

2. Preliminaries

2.1. SFSs

Definition 1 [20]. *The SFS $\vec{\vec{\Theta}}$ of the universe of discourse $\vec{\vec{F}}$ is defined as:*

$$\vec{\vec{\Theta}} = \left\{ \left\langle \vec{\vec{f}},\ \left(\vec{\vec{\mu}}_{\vec{\vec{\Theta}}}(\vec{\vec{f}}),\ \vec{\vec{\pi}}_{\vec{\vec{\Theta}}}\left(\vec{\vec{f}}\right),\ \vec{\vec{\upsilon}}_{\vec{\vec{\Theta}}}\left(\vec{\vec{f}}\right) \right) \mid \vec{\vec{f}} \in \vec{\vec{F}} \right\rangle \right\}, \tag{1}$$

where $\vec{\mu}_{\vec{\bar{\Theta}}}: \vec{\bar{F}} \to [0,1]$, $\vec{\pi}_{\vec{\bar{\Theta}}}: \vec{\bar{F}} \to [0,1]$, $\vec{\upsilon}_{\vec{\bar{\Theta}}}: \vec{\bar{F}} \to [0,1]$ *are MD of* $\vec{\bar{f}}$ *to* $\vec{\bar{\Theta}}$*, AD of* $\vec{\bar{f}}$ *to* $\vec{\bar{\Theta}}$ *and N-MD of* $\vec{\bar{f}}$ *to* $\vec{\bar{\Theta}}$ *respectively. Also* $\vec{\mu}_{\vec{\bar{\Theta}}}$, $\vec{\pi}_{\vec{\bar{\Theta}}}$, $\vec{\upsilon}_{\vec{\bar{\Theta}}}$ *meet the conditions:* $\forall \vec{\bar{f}} \in \vec{\bar{F}}$, $0 \leqslant \vec{\mu}^2_{\vec{\bar{\Theta}}}(\vec{\bar{f}}) + \vec{\pi}^2_{\vec{\bar{\Theta}}}(\vec{\bar{f}}) + \vec{\upsilon}^2_{\vec{\bar{\Theta}}}(\vec{\bar{f}}) \leqslant 1$. *In addition,* $\vec{\tau}_{\vec{\bar{\Theta}}}(\vec{\bar{f}}) = \sqrt{1 - \vec{\mu}^2_{\vec{\bar{\Theta}}}(\vec{\bar{f}}) - \vec{\pi}^2_{\vec{\bar{\Theta}}}(\vec{\bar{f}}) - \vec{\upsilon}^2_{\vec{\bar{\Theta}}}(\vec{\bar{f}})}$ *denotes the refusal degree.*

The triple component $\vec{\bar{\Theta}} = \left(\vec{\mu}_{\vec{\bar{\Theta}}}(\vec{\bar{f}}), \vec{\pi}_{\vec{\bar{\Theta}}}(\vec{\bar{f}}), \vec{\upsilon}_{\vec{\bar{\Theta}}}(\vec{\bar{f}})\right)$ *is said to spherical fuzzy number (SFN), which is denoted by* $\vec{\bar{\Theta}} = (\vec{\mu}, \vec{\pi}, \vec{\upsilon})$*, satisfying* $\vec{\mu}, \vec{\pi}, \vec{\upsilon} \in [0,1]$ *and* $0 \leqslant \vec{\mu}^2 + \vec{\pi}^2 + \vec{\upsilon}^2 \leqslant 1$.

Definition 2 [46]. *Suppose there are three SFNs,* $\vec{\bar{\Theta}} = (\vec{\mu}, \vec{\pi}, \vec{\upsilon})$, $\vec{\bar{\Theta}}_1 = (\vec{\mu}_1, \vec{\pi}_1, \vec{\upsilon}_1)$ *and* $\vec{\bar{\Theta}}_2 = (\vec{\mu}_2, \vec{\pi}_2, \vec{\upsilon}_2)$ *respectively, then:*

$$i. \quad \left(\vec{\bar{\Theta}}\right)^c = \left(\vec{\upsilon}, \vec{\pi}, \vec{\mu}\right), \tag{2}$$

$$ii. \quad \vec{\bar{\Theta}}_1 \oplus \vec{\bar{\Theta}}_2 = \left(\left(\vec{\mu}_1^2 + \vec{\mu}_2^2 - \vec{\mu}_1^2\vec{\mu}_2^2\right)^{1/2}, \vec{\pi}_1\vec{\pi}_2, \vec{\upsilon}_1\vec{\upsilon}_2\right), \tag{3}$$

$$iii. \quad \vec{\bar{\Theta}}_1 \otimes \vec{\bar{\Theta}}_2 = \left(\vec{\mu}_1\vec{\mu}_2, \left(\vec{\pi}_1^2 + \vec{\pi}_2^2 - \vec{\pi}_1^2\vec{\pi}_2^2\right)^{1/2}, \left(\vec{\upsilon}_1^2 + \vec{\upsilon}_2^2 - \vec{\upsilon}_1^2\vec{\upsilon}_2^2\right)^{1/2}\right), \tag{4}$$

$$iv. \quad \delta \cdot \vec{\bar{\Theta}} = \left(\left(1 - \left(1 - \vec{\mu}^2\right)^{\delta}\right)^{1/2}, \vec{\pi}^{\delta}, \vec{\upsilon}^{\delta}\right), \quad \delta > 0, \tag{5}$$

$$v. \quad \vec{\bar{\Theta}}^{\delta} = \left(\vec{\mu}^{\delta}, \left(1 - \left(1 - \vec{\pi}^2\right)^{\delta}\right)^{1/2}, \left(1 - \left(1 - \vec{\upsilon}^2\right)^{\delta}\right)^{1/2}\right), \quad \delta > 0. \tag{6}$$

Obviously, the above operations have the following rules:

(1) $\vec{\bar{\Theta}}_1 \oplus \vec{\bar{\Theta}}_2 = \vec{\bar{\Theta}}_2 \oplus \vec{\bar{\Theta}}_1$;

(2) $\vec{\bar{\Theta}}_1 \otimes \vec{\bar{\Theta}}_2 = \vec{\bar{\Theta}}_2 \otimes \vec{\bar{\Theta}}_1$;

(3) $\delta(\vec{\bar{\Theta}}_1 \oplus \vec{\bar{\Theta}}_2) = \delta\vec{\bar{\Theta}}_1 \oplus \delta\vec{\bar{\Theta}}_2, \quad \delta > 0$;

(4) $(\delta_1\vec{\bar{\Theta}} \oplus \delta_2\vec{\bar{\Theta}}) = (\delta_1 + \delta_2)\vec{\bar{\Theta}}, \quad \delta_1, \delta_2 > 0$;

(5) $\vec{\bar{\Theta}}^{\delta_1} \otimes \vec{\bar{\Theta}}^{\delta_2} = \vec{\bar{\Theta}}^{\delta_1 + \delta_2}, \quad \delta_1, \delta_2 > 0$;

(6) $\vec{\bar{\Theta}}_1^{\delta} \otimes \vec{\bar{\Theta}}_2^{\delta} = (\vec{\bar{\Theta}}_1 \otimes \vec{\bar{\Theta}}_2)^{\delta}, \quad \delta > 0$.

Definition 3 [32]. *Let $\vec{\vec{\Theta}}_1 = (\vec{\vec{\mu}}_1, \vec{\vec{\pi}}_1, \vec{\vec{\upsilon}}_1)$, $\vec{\vec{\Theta}}_2 = (\vec{\vec{\mu}}_2, \vec{\vec{\pi}}_2, \vec{\vec{\upsilon}}_2)$ are two SFNs, the score function $Sf(\vec{\vec{\Theta}}_1)$ and accuracy function $Af(\vec{\vec{\Theta}}_1)$ are given as:*

$$Sf(\vec{\vec{\Theta}}_1) = \frac{\left(2 + \vec{\vec{\mu}}_1 - \vec{\vec{\pi}}_1 - \vec{\vec{\upsilon}}_1\right)}{3}, \qquad Sf(\vec{\vec{\Theta}}_1) \in [0, 1], \tag{7}$$

$$Af(\vec{\vec{\Theta}}_1) = \vec{\vec{\mu}}_1 - \vec{\vec{\upsilon}}_1, \qquad Af(\vec{\vec{\Theta}}_1) \in [0, 1]. \tag{8}$$

And, they have the following comparison rules:

1. *If $Sf(\vec{\vec{\Theta}}_1) < Sf(\vec{\vec{\Theta}}_2)$, then $\vec{\vec{\Theta}}_1 < \vec{\vec{\Theta}}_2$;*
2. *If $Sf(\vec{\vec{\Theta}}_1) = Sf(\vec{\vec{\Theta}}_2)$ and $Af(\vec{\vec{\Theta}}_1) < Af(\vec{\vec{\Theta}}_2)$, then $\vec{\vec{\Theta}}_1 < \vec{\vec{\Theta}}_2$;*
3. *If $Sf(\vec{\vec{\Theta}}_1) = Sf(\vec{\vec{\Theta}}_2)$ and $Af(\vec{\vec{\Theta}}_1) = Af(\vec{\vec{\Theta}}_2)$, then $\vec{\vec{\Theta}}_1 = \vec{\vec{\Theta}}_2$.*

Definition 4 [46]. *Let $\vec{\vec{\Theta}}_1 = (\vec{\vec{\mu}}_1, \vec{\vec{\pi}}_1, \vec{\vec{\upsilon}}_1)$ and $\vec{\vec{\Theta}}_2 = (\vec{\vec{\mu}}_2, \vec{\vec{\pi}}_2, \vec{\vec{\upsilon}}_2)$ are two SFNs respectively, the normalized Hamming distance between $\vec{\vec{\Theta}}_1$ and $\vec{\vec{\Theta}}_2$ is given as:*

$$Dis = (\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2) = \frac{1}{3}\left(\left|\vec{\vec{\mu}}_1^2 - \vec{\vec{\mu}}_2^2\right| + \left|\vec{\vec{\pi}}_1^2 - \vec{\vec{\pi}}_2^2\right| + \left|\vec{\vec{\upsilon}}_1^2 - \vec{\vec{\upsilon}}_2^2\right|\right) \tag{9}$$

2.2. PMSM operators

Definition 5 [41]. *Let $\vec{\vec{A}} = \{\vec{\vec{\gamma}}_1, \vec{\vec{\gamma}}_2, \cdots, \vec{\vec{\gamma}}_q\}$ be a set of nonnegative real numbers, which are divided into e different partitions $\vec{\vec{Y}}_1, \vec{\vec{Y}}_2, \ldots, \vec{\vec{Y}}_e$ with $\vec{\vec{Y}}_\eta \cap \vec{\vec{Y}}_x = \emptyset$ and $\bigcup_{b=1}^{e} \vec{\vec{Y}}_b = \vec{\vec{A}}$, then PMSM operator is defined as following:*

$$PMSM^{(g_1,g_2,\ldots,g_e)}\left(\vec{\vec{\gamma}}_1, \vec{\vec{\gamma}}_2, \cdots, \vec{\vec{\gamma}}_q\right) = \frac{1}{e}\left(\sum_{b=1}^{e}\left(\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}\left(\sum_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}}\prod_{x=1}^{g_b}\vec{\vec{\gamma}}_{\eta_x}\right)\right)^{\frac{1}{g_b}}\right), \tag{10}$$

where $|\vec{\vec{Y}}_b|$ is the cardinality of $\vec{\vec{Y}}_b (b = 1, 2, \cdots, e)$ and $\sum_{b=1}^{e} |\vec{\vec{Y}}_b| = q$, g_b is the parameter in the partition $\vec{\vec{Y}}_b$ and $g_b = 1, 2, \cdots, |\vec{\vec{Y}}_b|$. $(\eta_1, \eta_2, \ldots, \eta_{g_b})$ traverses *all the g_b-tuple combination of* $(1, 2, \ldots, |\vec{\bar{Y}}_b|)$*, and* $C^{g_b}_{|\vec{\bar{Y}}_b|}$ *represents the binomial coefficient satisfying* $C^{g_b}_{|\vec{\bar{Y}}_b|} = \dfrac{|\vec{\bar{Y}}_b|!}{g_b!(|\vec{\bar{Y}}_b| - g_b)!}$.

In addition, the PMSM operator has the following properties:

(1) $PMSM^{(g_1,g_2,\cdots,g_e)}(0, 0, \cdots, 0) = 0$, $PMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\gamma}}, \vec{\bar{\gamma}}, \cdots, \vec{\bar{\gamma}}\right) = \vec{\bar{\gamma}}$.

(2) $PMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\gamma}}_1, \vec{\bar{\gamma}}_2, \cdots, \vec{\bar{\gamma}}_q\right) \leqslant PMSM^{(g_1,g_2,\cdots,g_e)}\vec{\bar{\gamma}}'_1, \vec{\bar{\gamma}}'_2, \cdots, \vec{\bar{\gamma}}'_q$, if $\vec{\bar{\gamma}}_\eta \leqslant \vec{\bar{\gamma}}'_\eta$ for all η.

(3) $\min_\eta\{\vec{\bar{\gamma}}_\eta\} \leqslant PMSM^{(g_1,g_2,\cdots g_e)}\left(\vec{\bar{\gamma}}_1, \vec{\bar{\gamma}}_2, \cdots, \vec{\bar{\gamma}}_q\right) \leqslant \max_\eta\{\vec{\bar{\gamma}}_\eta\}$.

2.3. PA operator

In 2001, PA operator was developed by Yager [47]. As a nonlinear weighted average aggregation operator, which integrates information by considering the support degree between input values and effectively reduces the impact of extreme values on assessment results.

Definition 6 *[47]. Let $\vec{\bar{\gamma}}_\eta$ $(\eta = 1, 2, \ldots, q)$ be a set of nonnegative real numbers, PA operator is defined as following:*

$$PA(\vec{\bar{\gamma}}_1, \vec{\bar{\gamma}}_2, \cdots, \vec{\bar{\gamma}}_q) = \frac{\sum_{\eta=1}^{q}(1 + T(\vec{\bar{\gamma}}_\eta))\vec{\bar{\gamma}}_\eta}{\sum_{\eta=1}^{q}(1 + T(\vec{\bar{\gamma}}_\eta))}, \tag{11}$$

where

$$T(\vec{\bar{\gamma}}_\eta) = \sum_{\varepsilon=1,\varepsilon\neq\eta}^{q} Sup\left(\vec{\bar{\gamma}}_\eta, \vec{\bar{\gamma}}_\varepsilon\right) \tag{12}$$

and $Sup(\vec{\bar{\gamma}}_\eta, \vec{\bar{\gamma}}_\varepsilon)$ is the support for $\vec{\bar{\gamma}}_\eta$ from $\vec{\bar{\gamma}}_\varepsilon$, satisfying the following conditions:

(1) $Sup(\vec{\bar{\gamma}}_\eta, \vec{\bar{\gamma}}_\varepsilon) \in [0, 1]$;

(2) $Sup(\vec{\bar{\gamma}}_\eta, \vec{\bar{\gamma}}_\varepsilon) = Sup(\vec{\bar{\gamma}}_\varepsilon, \vec{\bar{\gamma}}_\eta)$;

(3) $Sup(\vec{\bar{\gamma}}_\eta, \vec{\bar{\gamma}}_\varepsilon) \geqslant Sup(\vec{\bar{\gamma}}_i, \vec{\bar{\gamma}}_j)$, if $\left|\vec{\bar{\gamma}}_\eta - \vec{\bar{\gamma}}_\varepsilon\right| \leqslant \left|\vec{\bar{\gamma}}_i - \vec{\bar{\gamma}}_j\right|$.

3. The SFPMSM AOs

In this part, we shall extend PMSM operator to SFSs and develop some SFPMSM AOs based on operation laws of SFNs.

3.1. The SFPMSM operator

Definition 7 *Let* $\vec{\tilde{N}} = \{\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\}$ *be a set of SFNs, which are divided into e different partitions* $\vec{\tilde{Y}}_1, \vec{\tilde{Y}}_2, \cdots, \vec{\tilde{Y}}_e$ *with* $\vec{\tilde{Y}}_\eta \cap \vec{\tilde{Y}}_x = \emptyset$ *and* $\bigcup_{b=1}^{e} \vec{\tilde{Y}}_b = \vec{\tilde{N}}$*, then SFPMSM operator is defined as following:*

$$SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) = \frac{1}{e}\left(\bigoplus_{b=1}^{e}\left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \bigotimes_{x=1}^{g_b} \vec{\tilde{\Theta}}_{\eta_x}\right)\right)^{\frac{1}{g_b}}\right), \tag{13}$$

where $|\vec{\tilde{Y}}_b|$ *is the cardinality of* $\vec{\tilde{Y}}_b$ $(b = 1, 2, \ldots, e)$ *and* $\sum_{b=1}^{e} |\vec{\tilde{Y}}_b| = q$*,* g_b *is the parameter in the partition* $\vec{\tilde{Y}}_b$ *and* $g_b = 1, 2, \cdots, |\vec{\tilde{Y}}_b|$*.* $(\eta_1, \eta_2, \ldots, \eta_{g_b})$ *traverses all the* g_b*-tuple combination of* $(1, 2, \ldots, |\vec{\tilde{Y}}_b|)$*, and* $C_{|\vec{\tilde{Y}}_b|}^{g_b}$ *represents the binomial coefficient satisfying* $C_{|\vec{\tilde{Y}}_b|}^{g_b} = \dfrac{|\vec{\tilde{Y}}_b|!}{g_b!(|\vec{\tilde{Y}}_b| - g_b)!}$.

Theorem 1 *Let* $\vec{\tilde{\Theta}}_\eta = \left(\vec{\tilde{\mu}}_\eta, \vec{\tilde{\pi}}_\eta, \vec{\tilde{\upsilon}}_\eta\right)$ $(\eta = 1, 2, \ldots, q)$ *be a set of SFNs with parameter vector* $(g_1, g_2, \ldots, g_e)$*, then the aggregated value by SFPMSM operator is still a SFN, and*

$$SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right)$$

$$=\left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C^{g_b}_{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{g_b}}\right)\right)^{1/e}},\right.$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\pi}}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{1/e},$$

$$\left.\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\prod_{x=1}^{g}\left(1-\vec{\tilde{\upsilon}}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{1/e}\right). \tag{14}$$

Proof. According to operational rules of SFNs in Definition 2, we have

$$\bigotimes_{x=1}^{g_b}\vec{\tilde{\Theta}}_{\eta_x}=\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x},\sqrt{1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\pi}}^2_{\eta_x}\right)},\sqrt{1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\upsilon}}^2_{\eta_x}\right)}\right)$$

and

$$\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\bigotimes_{x=1}^{g_b}\vec{\tilde{\Theta}}_{\eta_x}=\left(\sqrt{1-\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)},\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\pi}}^2_{\eta_x}\right)},\right.$$

$$\left.\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\upsilon}}^2_{\eta_x}\right)}\right).$$

Thereafter, we can get

$$\left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\bigoplus_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\bigotimes_{x=1}^{g_b}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)^{\frac{1}{g_b}}
=\left(\left(\sqrt{1-\left(1-\left(1-\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}}\right)^{\frac{1}{g_b}},\right.$$

$$\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\pi}}_{\eta_x}^2\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}},$$

$$\left.\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\upsilon}}_{\eta_x}^2\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)$$

and

$$\bigoplus_{b=1}^{e}\left(\left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\bigoplus_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\bigotimes_{x=1}^{g_b}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)^{\frac{1}{g_b}}\right)
=\left(\sqrt{1-\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)}^{\frac{1}{g_b}},\right.$$

$$\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\pi}}_{\eta_x}^2\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}},$$

$$\left.\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\upsilon}}_{\eta_x}^2\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right).$$

Therefore,

$$SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right)$$

$$= \frac{1}{e}\left(\bigoplus_{b=1}^{e}\left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\bigotimes_{x=1}^{g_b}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)^{\frac{1}{g_b}}\right)$$

$$=\left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}},\right.$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\tilde{\pi}}_{\eta_x}^2\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left.\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\prod_{x=1}^{g}\left(1-\vec{\tilde{\upsilon}}_{\eta_x}^2\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right).$$

Moreover, SFPMSM operator have the following properties:

Theorem 2 (Idempotence) *Let* $\vec{\tilde{\Theta}}_\eta = (\vec{\tilde{\mu}}_\eta, \vec{\tilde{\pi}}_\eta, \vec{\tilde{\upsilon}}_\eta)$ *(*$\eta = 1,2,\ldots,q$*) be a set of SFNs with parameter vector* $(g_1,g_2,\ldots,g_e)$*, if* $\vec{\tilde{\Theta}}_\eta = \vec{\tilde{\Theta}} = (\vec{\tilde{\mu}}, \vec{\tilde{\pi}}, \vec{\tilde{\upsilon}})$ *for all* η*, then*

$$SFPMSM^{(g_1,g_2,\cdots g_e)}(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q) = \vec{\tilde{\Theta}}. \quad (15)$$

Proof. Since $\vec{\tilde{\Theta}}_q = \vec{\tilde{\Theta}} = (\vec{\mu}, \vec{\pi}, \vec{\upsilon})$, then

$$SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right)$$

$$= \left(\sqrt{1 - \left(\prod_{b=1}^{e}\left(1 - \left(1 - \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1 - \left(\prod_{x=1}^{g_b}\vec{\mu}\right)^2\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}\right)\right)}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1 - \left(1 - \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1 - \prod_{x=1}^{g_b}\left(1 - \vec{\pi}^2\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left.\left(\prod_{b=1}^{e}\sqrt{1 - \left(1 - \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1 - \prod_{x=1}^{g}\left(1 - \vec{\upsilon}^2\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right)$$

$$= \left(\sqrt{1 - \left(\prod_{b=1}^{e}\left(1 - \left(1 - \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1 - \vec{\mu}^{2g_b}\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}\right)\right)}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1 - \left(1 - \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1 - \left(1 - \vec{\pi}^2\right)^{g_b}\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left.\left(\prod_{b=1}^{e}\sqrt{1 - \left(1 - \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1 - \left(1 - \vec{\upsilon}^2\right)^{g_b}\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right)$$

$$
= \left(\sqrt{1 - \left(\prod_{b=1}^{e} \left(1 - \left(1 - \left(\left(1 - \vec{\mu}^{2g_b} \right)^{C^{g_b}_{|\vec{Y}_b|}} \right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}} \right)^{\frac{1}{g_b}} \right) \right)^{\frac{1}{e}}} , \right.
$$

$$
\left(\prod_{b=1}^{e} \sqrt{1 - \left(1 - \left(\left(1 - \left(1 - \vec{\pi}^2 \right)^{g_b} \right)^{C^{g_b}_{|\vec{Y}_b|}} \right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}} \right)^{\frac{1}{g_b}}} \right)^{\frac{1}{e}} ,
$$

$$
\left. \left(\prod_{b=1}^{e} \sqrt{1 - \left(1 - \left(\left(1 - \left(1 - \vec{\upsilon}^2 \right)^{g_b} \right)^{C^{g_b}_{|\vec{Y}_b|}} \right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}} \right)^{\frac{1}{g_b}}} \right)^{\frac{1}{e}} \right)
$$

$$
= \left(\sqrt{1 - \left(\prod_{b=1}^{e} \left(1 - \vec{\mu}^2 \right) \right)^{\frac{1}{e}}} , \left(\prod_{b=1}^{e} \sqrt{1 - \left(1 - \vec{\pi}^2 \right)} \right)^{\frac{1}{e}} , \left(\prod_{b=1}^{e} \sqrt{1 - \left(1 - \vec{\upsilon}^2 \right)} \right)^{\frac{1}{e}} \right)
$$

$$
= \left(\vec{\mu}, \vec{\pi}, \vec{\upsilon} \right) = \vec{\Theta} .
$$

And that completes the proof. □

Theorem 3 (Monotonicity) *Let* $\vec{\Theta}_\eta = \left(\vec{\mu}_\eta, \vec{\pi}_\eta, \vec{\upsilon}_\eta \right)$ *and* $\hat{\Theta}_\eta = \left(\hat{\mu}_\eta, \hat{\pi}_\eta, \hat{\upsilon}_\eta \right)$ *($\eta = 1, 2, \ldots, q$) be two sets of SFNs with same partitions and parameter vector $(g_1, g_2, \ldots, g_e)$, if $\vec{\mu}_\eta \geqslant \hat{\mu}_\eta, \vec{\pi}_\eta \leqslant \hat{\pi}_\eta, \vec{\upsilon}_\eta \leqslant \hat{\upsilon}_\eta$ for all η, then,*

$$
SFPMSM^{(g_1,g_2,\cdots,g_e)} \left(\vec{\Theta}_1, \vec{\Theta}_2, \cdots, \vec{\Theta}_q \right) \geqslant SFPMSM^{(g_1,g_2,\cdots,g_e)} \left(\hat{\Theta}_1, \hat{\Theta}_2, \cdots, \hat{\Theta}_q \right) . \tag{16}
$$

Proof. Let

$$
SFPMSM^{(g_1,g_2,\cdots,g_e)} \left(\vec{\Theta}_1, \vec{\Theta}_2, \cdots, \vec{\Theta}_q \right) = \vec{\Theta} = \left(\vec{\mu}, \vec{\pi}, \vec{\upsilon} \right)
$$

and

$$
SFPMSM^{(g_1,g_2,\cdots,g_e)} \left(\hat{\Theta}_1, \hat{\Theta}_2, \cdots, \hat{\Theta}_q \right) = \hat{\Theta} = \left(\hat{\mu}, \hat{\pi}, \hat{\upsilon} \right) .
$$

Since $g_b \geqslant 1$, $\vec{\vec{\mu}}_\eta \geqslant \hat{\hat{\mu}}_\eta \geqslant 0$, $\hat{\hat{\pi}}_\eta \geqslant \vec{\vec{\pi}}_\eta \geqslant 0$, $\hat{\hat{\upsilon}}_\eta \geqslant \vec{\vec{\upsilon}}_\eta \geqslant 0$, then we have $\vec{\vec{\mu}}_{\eta_x} \geqslant \hat{\hat{\mu}}_{\eta_x} \geqslant 0$, $\hat{\hat{\pi}}_{\eta_x} \geqslant \vec{\vec{\pi}}_{\eta_x} \geqslant 0$, $\hat{\hat{\upsilon}}_{\eta_x} \geqslant \vec{\vec{\upsilon}}_{\eta_x} \geqslant 0$. Therefore, we can obtain

$$
\begin{aligned}
&1-\left(\prod_{x=1}^{g_b}\vec{\vec{\mu}}_{\eta_x}\right)^2 \leqslant 1-\left(\prod_{x=1}^{g_b}\hat{\hat{\mu}}_{\eta_x}\right)^2 \\
&\Rightarrow \prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\vec{\mu}}_{\eta_x}\right)^2\right) \leqslant \prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}}\left(1-\left(\prod_{x=1}^{g_b}\hat{\hat{\mu}}_{\eta_x}\right)^2\right) \\
&\Rightarrow 1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}}\left(1-\left(\prod_{x=1}^{g_b}\vec{\vec{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}} \geqslant 1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}}\left(1-\left(\prod_{x=1}^{g_b}\hat{\hat{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}} \\
&\Rightarrow \prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\vec{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right) \\
&\quad\leqslant \prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\hat{\hat{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right) \\
&\Rightarrow \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\vec{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}} \\
&\quad\geqslant \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\hat{\hat{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}} \\
&\Rightarrow \vec{\vec{\mu}} \geqslant \hat{\hat{\mu}}\,.
\end{aligned}
$$

Furthermore,

$$\hat{\pi}_{\eta_x} \geqslant \vec{\pi}_{\eta_x} \Rightarrow 1-\hat{\pi}^2_{\eta_x} \leqslant 1-\vec{\pi}^2_{\eta_x}$$

$$\Rightarrow \left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{Y}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\hat{\pi}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}} \geqslant \left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{Y}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\pi}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}$$

$$\Rightarrow \left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\hat{\pi}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}$$

$$\leqslant \left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\pi}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}$$

$$\Rightarrow \left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\hat{\pi}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}$$

$$\geqslant \left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\hat{\pi}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}$$

$$\Rightarrow \hat{\pi} \geqslant \vec{\pi}\,.$$

Similarly, we have $\hat{\upsilon} \geqslant \vec{\upsilon}$.

Based on Definition 3, we have

$$Sf(\vec{\Theta}) - Sf(\hat{\Theta}) = \frac{2+\vec{\mu}-\vec{\pi}-\vec{\upsilon}}{3} - \frac{2+\hat{\mu}-\hat{\pi}-\hat{\upsilon}}{3} = \left(\vec{\mu}-\hat{\mu}\right) + \left(\hat{\pi}-\vec{\pi}\right) + \left(\hat{\upsilon}-\vec{\upsilon}\right) \geqslant 0,$$

so $\vec{\Theta} \geqslant \hat{\Theta}$ that is

$$SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\Theta}_1,\vec{\Theta}_2,\cdots,\vec{\Theta}_q\right) \geqslant SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\hat{\Theta}_1,\hat{\Theta}_2,\cdots,\hat{\Theta}_q\right).$$

Theorem 4 (Boundness) *Let* $\vec{\tilde{\Theta}}_\eta = (\vec{\tilde{\mu}}_\eta, \vec{\tilde{\pi}}_\eta, \vec{\tilde{\upsilon}}_\eta)$ *(*$\eta = 1, 2, \ldots, q$*) be a set of SFNs with parameter vector* $(g_1, g_2, \ldots, g_e)$*, if* $\vec{\tilde{\Theta}}^{+} = \left(\max_\eta \vec{\tilde{\mu}}_\eta, \min_\eta \vec{\tilde{\pi}}_\eta, \min_\eta \vec{\tilde{\upsilon}}_\eta\right)$ *and* $\vec{\tilde{\Theta}}^{-} = \left(\min_\eta \vec{\tilde{\mu}}_\eta, \max_\eta \vec{\tilde{\pi}}_\eta, \max_\eta \vec{\tilde{\upsilon}}_\eta\right)$*, then*

$$\vec{\tilde{\Theta}}^{-} \leqslant SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) \leqslant \vec{\tilde{\Theta}}^{+}. \tag{17}$$

Proof. By Theorem 2 and Theorem 4, we have

$$\begin{aligned} & SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) \\ & \qquad \geqslant \vec{\tilde{\Theta}}^{-} = SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}^{-}, \vec{\tilde{\Theta}}^{-}, \cdots, \vec{\tilde{\Theta}}^{-}\right) \end{aligned}$$

and

$$\begin{aligned} & SFPMSM^{(g_1,g_2,\cdots,g_e)}\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q \\ & \qquad \leqslant \vec{\tilde{\Theta}}^{+} = SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}^{+}, \vec{\tilde{\Theta}}^{+}, \cdots, \vec{\tilde{\Theta}}^{+}\right). \end{aligned}$$

Therefore, Theorem 4 is proved. □

Next, we further study the monotonicity of SFPMSM operator in regard to parameter g_b, for which we first introduce the following lemma:

Lemma 1 *[48]. Let* $\vec{\tilde{\gamma}}_\eta > 0$*,* $\vec{\tilde{\chi}}_\eta > 0$ *(*$\eta = 1, 2, \ldots, q$*) and* $\sum_{\eta=1}^{q} \vec{\tilde{\chi}}_\eta = 1$*, then*

$$\prod_{\eta=1}^{q} \left(\vec{\tilde{\gamma}}_\eta\right)^{\vec{\tilde{\chi}}_\eta} \leqslant \sum_{\eta=1}^{q} \vec{\tilde{\chi}}_\eta \vec{\tilde{\gamma}}_\eta \tag{18}$$

with equality if and only if $\vec{\tilde{\gamma}}_1 = \vec{\tilde{\gamma}}_2 = \cdots \vec{\tilde{\gamma}}_q$*.*

Theorem 5 *Let* $\vec{\tilde{\Theta}}_\eta = (\vec{\tilde{\mu}}_\eta, \vec{\tilde{\pi}}_\eta, \vec{\tilde{\upsilon}}_\eta)$ *(*$\eta = 1, 2, \ldots, q$*) be a set of SFNs with parameter vector* $(g_1, g_2, \ldots, g_b)$*, then SFPMSM operator is monotonically decreasing in regard to the parameter* g_b *(*$g_b = 1, 2, \ldots, |\vec{\tilde{Y}}_b|$*).*

Proof. By Theorem 1, we have

$$SFPMSM^{(g_1,g_2,\cdots g_e)}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right)$$

$$=\left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\mu}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}\right)\right)}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\pi}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\prod_{x=1}^{g}\left(1-\vec{\upsilon}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\Bigg).$$

Therefore, let

$$l(g_b)=\left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\mu}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}\right)\right)}\right)^{\frac{1}{e}},$$

$$m(g_b)=\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\vec{\pi}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

and

$$n(g_b)=\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{Y}_b|}\left(1-\prod_{x=1}^{g}\left(1-\vec{\upsilon}^2_{\eta_x}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}.$$

First, we prove $l(g_b)$ about the parameter g_b is monotonically decreasing. According to Lemma 1, we can get

$$\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}$$

$$\leqslant \sum_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\frac{1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2}{C_{|\vec{\tilde{Y}}_b|}^{g_b}} = 1-\frac{\sum_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}$$

$$\Rightarrow \left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}$$

$$\geqslant \left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\sum_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{g_b}}$$

$$\Rightarrow \left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}$$

$$\leqslant \left(\prod_{b=1}^{e}\left(1-\left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\sum_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}$$

$$\Rightarrow \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\left(\prod_{x=1}^{g_b}\vec{\tilde{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}}$$

$$\geqslant \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(\frac{1}{C_{|\vec{\ddot{Y}}_b|}^{g_b}}\left(\sum_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\ddot{Y}}_b|}\left(\prod_{x=1}^{g_b}\vec{\ddot{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}}$$

$$\Leftrightarrow l(g_b) \geqslant \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(\frac{1}{C_{|\vec{\ddot{Y}}_b|}^{g_b}}\left(\sum_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\ddot{Y}}_b|}\left(\prod_{x=1}^{g_b}\vec{\ddot{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}} \tag{19}$$

Then we take the following proof by the contradiction method. We assume that $l(g_b)$ is increasing with respect to g_b, then it follows that

$$l(|\vec{\ddot{Y}}_b|) \geqslant l(|\vec{\ddot{Y}}_b|-1) \geqslant \cdots \geqslant l(1).$$

By (19), we have

$$l(1) \geqslant \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(\frac{1}{C_{|\vec{\ddot{Y}}_b|}^{1}}\left(\sum_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\ddot{Y}}_b|}\left(\prod_{x=1}^{1}\vec{\ddot{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{e}}}$$

$$= \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\frac{\sum_{\eta=1}^{|\vec{\ddot{Y}}_b|}\vec{\ddot{\mu}}_{\eta}^2}{|\vec{\ddot{Y}}_b|}\right)\right)^{\frac{1}{e}}}.$$

Thereafter, let $g_b = |\vec{\ddot{Y}}_b|$, then

$$l(|\vec{\ddot{Y}}_b|) = \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\ddot{Y}}_b|}\left(1-\left(\prod_{x=1}^{|\vec{\ddot{Y}}_b|}\vec{\ddot{\mu}}_{\eta_x}\right)^2\right)\right)^{\frac{1}{C_{|\vec{\ddot{Y}}_b|}^{|\vec{\ddot{Y}}_b|}}}\right)^{\frac{1}{|\vec{\ddot{Y}}_b|}}\right)\right)^{\frac{1}{e}}}$$

$$= \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(\prod_{\eta=1}^{|\vec{\ddot{Y}}_b|}\vec{\ddot{\mu}}_{\eta}^2\right)^{\frac{1}{|\vec{\ddot{Y}}_b|}}\right)\right)^{\frac{1}{e}}}.$$

Based on the assumption, we have

$$l(|\vec{\vec{Y}}_b|) = \sqrt[\frac{1}{e}]{1-\left(\prod_{b=1}^{e}\left(1-\left(\prod_{\eta=1}^{|\vec{\vec{Y}}_b|}\vec{\vec{\mu}}_\eta^2\right)^{\frac{1}{|\vec{\vec{Y}}_b|}}\right)\right)} \geqslant l(1) \geqslant \sqrt[\frac{1}{e}]{1-\left(\prod_{b=1}^{e}\left(1-\frac{\sum_{\eta=1}^{|\vec{\vec{Y}}_b|}\vec{\vec{\mu}}_\eta^2}{|\vec{\vec{Y}}_b|}\right)\right)}$$

$$\Rightarrow \left(\prod_{\eta=1}^{|\vec{\vec{Y}}_b|}\vec{\vec{\mu}}_\eta^2\right)^{\frac{1}{|\vec{\vec{Y}}_b|}} \geqslant \frac{\sum_{\eta=1}^{|\vec{\vec{Y}}_b|}\vec{\vec{\mu}}_\eta^2}{|\vec{\vec{Y}}_b|}.$$

However, according to Lemma 1, we have

$$\left(\prod_{\eta=1}^{|\vec{\vec{Y}}_b|}\vec{\vec{\mu}}_\eta^2\right)^{\frac{1}{|\vec{\vec{Y}}_b|}} \leqslant \frac{\sum_{\eta=1}^{|\vec{\vec{Y}}_b|}\vec{\vec{\mu}}_\eta^2}{|\vec{\vec{Y}}_b|}.$$

Obviously, it is not true that $l(g_b)$ increases monotonically with g_b increasing, so $l(g_b)$ is monotonically decreasing about g_b.

Similarly, we can also show that the functions $m(g_b)$ and $n(g_b)$ are monotonically increasing about g_b.

According to Definition 3, we have

$$Sf(g_b) = \frac{2 + l(g_b) - m(g_b) - n(g_b)}{3}.$$

Thereafter, for any $g_b = (1, 2, \ldots, |\vec{\vec{Y}}_b|)$, we can get

$$\begin{aligned} Sf(g_b+1) - Sf(g_b) &= \frac{2 + l(g_b+1) - m(g_b+1) - n(g_b+1)}{3} \\ &\quad - \frac{2 + l(g_b) - m(g_b) - n(g_b)}{3} \\ &= (l(g_b+1) - l(g_b)) + (m(g_b) - m(g_b+1)) \\ &\quad + (n(g_b) - n(g_b+1)) < 0 \end{aligned}$$

that is, $Sf(g_b+1) < Sf(g_b)$ for all g_b. For which completes the proof. □

Theorem 6 *Let* $\vec{\vec{\Theta}}_\eta = (\vec{\mu}_\eta, \vec{\pi}_\eta, \vec{\upsilon}_\eta)$ $(\eta = 1, 2, \ldots, q)$ *be a set of SFNs with parameter vector* $(g_1, g_2, \cdots, g_e)$ *and* $g_b = 1, 2, \ldots, |\vec{\vec{Y}}_b|$, *then*

$$\max\left\{SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2, \cdots, \vec{\vec{\Theta}}_q\right)\right\}$$
$$= SFPMSM^{(1,1,\cdots,1)}\left(\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2, \cdots, \vec{\vec{\Theta}}_q\right)$$
$$= \left(\sqrt{1 - \left(\prod_{b=1}^{e}\left(\left(\prod_{\eta=1}^{|\vec{\vec{Y}}_b|}\left(1 - \vec{\mu}_\eta^2\right)\right)^{\frac{1}{|\vec{\vec{Y}}_b|}}\right)\right)^{\frac{1}{e}}},\right.$$
$$\left.\left(\prod_{b=1}^{e}\sqrt{\left(\prod_{\eta=1}^{|\vec{\vec{Y}}_b|}\vec{\pi}_\eta^2\right)^{\frac{1}{|\vec{\vec{Y}}_b|}}}\right)^{\frac{1}{e}}, \left(\prod_{b=1}^{e}\sqrt{\left(\prod_{\eta=1}^{|\vec{\vec{Y}}_b|}\vec{\upsilon}_\eta^2\right)^{\frac{1}{\vec{\vec{Y}}_b|}}}\right)^{\frac{1}{e}}\right)$$
$$\min\left\{SFPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2, \cdots, \vec{\vec{\Theta}}_q\right)\right\}$$
$$= SFPMSM^{(|\vec{\vec{Y}}_1|,|\vec{\vec{Y}}_2|,\cdots,|\vec{\vec{Y}}_e|)}\left(\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2, \cdots, \vec{\vec{\Theta}}_q\right).$$

Next, we study some special circumstances for SFPMSM operator.

(1) When all inputs belong to the same partition and the types of relationship among inputs are same, in other words, $e = 1$, $|\vec{\vec{Y}}_1| = q$ and $g_1 = g = 1, 2, \ldots, q$, then SFPMSM operator becomes spherical fuzzy MSM(SFMSM)operator as follows:

$$SFPMSM^{g_1}\left(\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2, \cdots, \vec{\vec{\Theta}}_q\right) = \left(\frac{1}{C_{|\vec{\vec{Y}}_1|}^{g_1}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_1}\leqslant|\vec{\vec{Y}}_1|}}\bigotimes_{x=1}^{g_1}\vec{\vec{\Theta}}_{\eta_x}\right)\right)^{\frac{1}{g_1}}$$
$$= \left(\frac{1}{C_q^g}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_g\leqslant n}}\bigotimes_{x=1}^{g}\vec{\vec{\Theta}}_{\eta_x}\right)\right)^{\frac{1}{g}} = SFMSM^g\left(\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2, \cdots, \vec{\vec{\Theta}}_q\right)$$

(2) When $e = 1$ and $g = 1$, SFPMSM operator becomes the spherical fuzzy arithmetic averaging operator as follows:

$$SFPMSM^{1}\left(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q\right) = \left(\frac{\left(\bigoplus_{1\leqslant\eta_1\leqslant|\vec{\bar{Y}}_1|} \bigotimes_{x=1}^{1} \vec{\bar{\Theta}}_{\eta_x}\right)}{C^{1}_{|\vec{\bar{Y}}_1|}}\right)^{\frac{1}{1}}$$

$$= \frac{1}{|\vec{\bar{Y}}_1|} \bigoplus_{\eta_1=1}^{|\vec{\bar{Y}}_1|} \vec{\bar{\Theta}}_{\eta_x} = \frac{1}{q} \bigoplus_{\eta=1}^{q} \vec{\bar{\Theta}}_{\eta}.$$

(3) When $e = 1$ and $g = 2$, SFPMSM operator becomes the special SFBM operator as follows:

$$SFPMSM^{2}\left(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q\right) = \left(\frac{\left(\bigoplus_{1\leqslant\eta_1<\eta_2\leqslant|\vec{\bar{Y}}_1|} \bigotimes_{x=1}^{2} \vec{\bar{\Theta}}_{\eta_x}\right)}{C^{2}_{|\vec{\bar{Y}}_1|}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{q(q-1)} \bigoplus_{\eta,x=1,\eta\neq x}^{q} \left(\vec{\bar{\Theta}}_{\eta} \otimes \vec{\bar{\Theta}}_{x}\right)\right)^{\frac{1}{2}}$$

$$= SFBM^{1,1}(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2 \cdots, \vec{\bar{\Theta}}_q)$$

(4) When $e = 1$ and $g = q$, SFPMSM operator becomes the spherical fuzzy geometric averaging operator as follows:

$$SFPMSM^{q}(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q) = \left(\frac{\left(\bigoplus_{1\leqslant\eta_1<\cdots<\eta_{g_q}\leqslant|\vec{\bar{Y}}_1|} \bigotimes_{x=1}^{q} \vec{\bar{\Theta}}_{\eta_x}\right)}{C^{q}_{|\vec{\bar{Y}}_1|}}\right)^{\frac{1}{q}} = \left(\bigotimes_{\eta=1}^{q} \vec{\bar{\Theta}}_{\eta}\right)^{\frac{1}{q}}.$$

3.2. The SFWPMSM operator

Definition 8 *Let* $\vec{\tilde{\Theta}}_\eta = (\vec{\tilde{\mu}}_\eta, \vec{\tilde{\pi}}_\eta, \vec{\tilde{\upsilon}}_\eta)$ $(\eta = 1, 2, \ldots, q)$ *be a set of SFNs, which are divided into e different partitions* $\vec{\tilde{Y}}_1, \vec{\tilde{Y}}_2, \cdots, \vec{\tilde{Y}}_e$ *with* $\vec{\tilde{Y}}_\eta \cap \vec{\tilde{Y}}_x = \emptyset$ *and* $\bigcup_{b=1}^{e} \vec{\tilde{Y}}_b = \vec{\tilde{N}}$. *The* $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_q)^T$ *is the weight vector of* $\vec{\tilde{\Theta}}_\eta$ $(\eta = 1, 2, \ldots, q)$, *with* $\vartheta_\eta \geqslant 0$ *and* $\sum_{\eta=1}^{q} \vartheta_\eta = 1$, *then SFWPMSM operator is given as follows:*

$$SFPWMSM^{(g_1, g_2, \cdots g_e)}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) = \frac{1}{e}\left(\bigoplus_{b=1}^{e}\left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\bigoplus_{\substack{1 \leqslant \eta_1 < \cdots \\ < \eta_{g_b} \leqslant |\vec{\tilde{Y}}_b|}} \bigotimes_{x=1}^{g_b}\left(\vartheta_{\eta_x} \vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g_b}}\right), \tag{20}$$

where $|\vec{\tilde{Y}}_b|$ *is the cardinality of* $\vec{\tilde{Y}}_b$ $(b = 1, 2, \cdots, e)$ *and* $\sum_{b=1}^{e} |\vec{\tilde{Y}}_b| = q$, g_b *is the parameter in the partition* $\vec{\tilde{Y}}_b$ *and* $g_b = 1, 2, \cdots, |\vec{\tilde{Y}}_b|$. $(\eta_1, \eta_2, \ldots, \eta_{g_b})$ *traverses all the* g_b*-tuple combination of* $(1, 2, \ldots, |\vec{\tilde{Y}}_b|)$, *and* $C_{|\vec{\tilde{Y}}_b|}^{g_b}$ *represents the binomial coefficient satisfying*

$$C_{|\vec{\tilde{Y}}_b|}^{g_b} = \frac{|\vec{\tilde{Y}}_b|!}{g_b!(|\vec{\tilde{Y}}_b| - g_b)!}.$$

Theorem 7 *Let* $\vec{\tilde{\Theta}}_\eta = (\vec{\tilde{\mu}}_\eta, \vec{\tilde{\pi}}_\eta, \vec{\tilde{\upsilon}}_\eta)$ $(\eta = 1, 2, \ldots, q)$ *be a set of SFNs with parameter vector* $(g_1, g_2, \ldots, g_e)$, *and* $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_q)^T$ *is the weight vector of* $\vec{\tilde{\Theta}}_\eta$ $(\eta = 1, 2, \ldots, q)$, *with* $\vartheta_\eta \geqslant 0$, $\sum_{\eta=1}^{q} \vartheta_\eta = 1$, *then the aggregated value by SFWPMSM operator is still a SFN, and*

$$
\begin{aligned}
&SFWPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\Theta}}_1,\vec{\bar{\Theta}}_2,\cdots,\vec{\bar{\Theta}}_q\right)\\
&=\left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\bar{\mu}}_{\eta_x}^2\right)^{\vartheta_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}},\right.\\
&\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\pi}}_{\eta_x}\right)^{2\vartheta_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},\\
&\left.\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\upsilon}}_{\eta_x}\right)^{2\vartheta_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right).
\end{aligned}
\tag{21}
$$

The proof of Theorem 7 is similar to Theorem 1. Furthermore, it is easy to obtain that SFWPMSM operator satisfies idempotence, monotonicity and boundness.

Theorem 8 *Let $\vec{\bar{\Theta}}_\eta = (\vec{\bar{\mu}}_\eta, \vec{\bar{\pi}}_\eta, \vec{\bar{\upsilon}}_\eta)$ $(\eta = 1, 2, \ldots, q)$ be a set of SFNs with parameter vector $(g_1, g_2, \ldots, g_e)$ $(g_b = 1, 2, \ldots, |\vec{\bar{Y}}_b|)$, then SFWPMSM operator is monotonically decreasing in regard to the parameter g_b.*

Theorem 9 *Let $\vec{\bar{\Theta}}_\eta = (\vec{\bar{\mu}}_\eta, \vec{\bar{\pi}}_\eta, \vec{\bar{\upsilon}}_\eta)$ $(\eta = 1, 2, \ldots, q)$ be a set of SFNs with parameter vector $(g_1, g_2, \ldots, g_e)$ $(g_b = 1, 2, \cdots, |\vec{\bar{Y}}_b|)$, and $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_q)^T$ is the weight vector of $\vec{\bar{\Theta}}_\eta$ $(\eta = 1, 2, \ldots, q)$, with $\vartheta_\eta \geqslant 0$, $\sum_{\eta=1}^{q} \vartheta_\eta = 1$. Then*

$$
\begin{aligned}
\min\left\{SFWPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\Theta}}_1,\vec{\bar{\Theta}}_2,\cdots,\vec{\bar{\Theta}}_q\right)\right\}\\
= SFWPMSM^{(|\vec{\bar{Y}}_1|,|\vec{\bar{Y}}_2|,\cdots,|\vec{\bar{Y}}_e|)}\left(\vec{\bar{\Theta}}_1,\vec{\bar{\Theta}}_2,\cdots,\vec{\bar{\Theta}}_q\right).
\end{aligned}
$$

$$\max\left\{SFWPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right)\right\}$$
$$= SFWPMSM^{(1,1,\cdots,1)}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right)$$
$$=\left(\sqrt{1-\left(\prod_{b=1}^{e}\left(\prod_{\eta=1}^{|\vec{\tilde{Y}}_b|}\left(1-\vec{\tilde{\mu}}_{\eta_x}^{2}\right)^{\vartheta_{\eta_x}}\right)^{\frac{1}{|\vec{\tilde{Y}}_b|}}\right)^{\frac{1}{e}}},\left(\prod_{b=1}^{e}\sqrt{\left(\prod_{\eta=1}^{|\vec{\tilde{Y}}_b|}\left(\vec{\tilde{\pi}}_{\eta_x}\right)^{2\vartheta_{\eta_x}}\right)^{\frac{1}{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{e}},\right.$$
$$\left.\left(\prod_{b=1}^{e}\sqrt{\left(\prod_{\eta=1}^{|\vec{\tilde{Y}}_b|}\left(\vec{\tilde{\upsilon}}_{\eta_x}\right)^{2\vartheta_{\eta_x}}\right)^{\frac{1}{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{e}}\right).$$

Next, we study some special circumstances about SFWPMSM operator.

(1) When $e = 1$, $|\vec{\tilde{Y}}_1| = q$ and $g_1 = g = 1, 2, \cdots, q$, then SFWPMSM operator becomes spherical fuzzy weighted MSM (SFWMSM) operator as follows:

$$SFWPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right)$$
$$=\left(\frac{1}{C_{|\vec{\tilde{Y}}_1|}^{g_1}}\left(\bigoplus_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_1|}\bigotimes_{x=1}^{g_1}\left(\vartheta_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g_1}}$$
$$=\left(\frac{1}{C_q^{g}}\left(\bigoplus_{1\leqslant\eta_1<\cdots<\eta_g\leqslant q}\bigotimes_{x=1}^{g}\left(\vartheta_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g}} = SFWMSM^{g}(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q).$$

(2) When $e = 1$, $g = 1$, SFWPMSM operator becomes the spherical fuzzy weighted average operator as follows:

$$SFWPMSM^{1}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right)$$
$$=\left(\frac{1}{C_{|\vec{\tilde{Y}}_1|}^{1}}\left(\bigoplus_{1\leqslant\eta_1\leqslant|\vec{\tilde{Y}}_1|}\bigotimes_{x=1}^{1}\left(\vartheta_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)=\frac{1}{q}\bigoplus_{\eta=1}^{q}\left(\vartheta_\eta\vec{\tilde{\Theta}}_\eta\right).$$

(3) When $e = 1$ and $g = 2$, SFWPMSM operator becomes the special spherical fuzzy weighted BM (SFWBM) operator as follows:

$$SFWPMSM^2\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) = \left(\frac{1}{C^2_{|\vec{\tilde{Y}}_1|}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\eta_2\\ \leqslant|\vec{\tilde{Y}}_1|}} \bigotimes_{x=1}^{2}\left(\vartheta_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{q(q-1)} \bigoplus_{\substack{\eta=x=1\\ \eta\neq x}}^{q}\left(\vartheta_\eta\vec{\tilde{\Theta}}_\eta \otimes \vartheta_x\vec{\tilde{\Theta}}_x\right)\right)^{\frac{1}{2}} = SFWBM^{1,1}(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q)/$$

(4) When $e = 1$ and $g = q$, SFWPMSM operator becomes the spherical fuzzy weighted geometric operator as follows:

$$SFWPMSM^q\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) = \left(\frac{1}{C^q_{|\vec{\tilde{Y}}_b|}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\ <\eta_{g_q}\leqslant|\vec{\tilde{Y}}_b|}} \bigotimes_{x=1}^{q}\left(\vartheta_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g_b}}$$

$$= \left(\bigotimes_{\eta=1}^{q}\left(\vartheta_\eta\vec{\tilde{\Theta}}_\eta\right)\right)^{\frac{1}{q}}.$$

4. The SFPPMSM AOs

This part we shall present some new AOs based on PMSM operator and PA operator under SFSs.

4.1. The SFPPMSM operator

Definition 9 *Let* $\vec{\tilde{N}} = \{\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\}$ *be a set of SFNs, which are divided into e different partitions* $\vec{\tilde{Y}}_1, \vec{\tilde{Y}}_2, \ldots, \vec{\tilde{Y}}_e$ *with* $\vec{\tilde{Y}}_\eta \cap \vec{\tilde{Y}}_x = \emptyset$ *and* $\bigcup_{b=1}^{e}\vec{\tilde{Y}}_b = \vec{\tilde{N}}$, *then*

SFPPMSM operator is defined as following:

$$SFPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right) = \frac{1}{e}\left(\bigoplus_{b=1}^{e}\left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\bigotimes_{x=1}^{g_b}\left(\frac{q\left(1+T(\vec{\tilde{\Theta}}_{\eta_x})\right)}{\sum_{o=1}^{q}\left(1+T(\vec{\tilde{\Theta}}_o)\right)}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g_b}}\right), \tag{22}$$

where $|\vec{\tilde{Y}}_b|$ is the cardinality of $\vec{\tilde{Y}}_b$ $(b = 1, 2, \ldots, e)$ and $\sum_{b=1}^{e}|\vec{\tilde{Y}}_b| = q$, g_b is the parameter in the partition $\vec{\tilde{Y}}_b$ and $g_b = 1, 2, \cdots, |\vec{\tilde{Y}}_b|$. $(\eta_1, \eta_2, \ldots, \eta_{g_b})$ traverses all the g_b-tuple combination of $(1, 2, \ldots, |\vec{\tilde{Y}}_b|)$, and $C_{|\vec{\tilde{Y}}_b|}^{g_b}$ represents the binomial coefficient satisfying $C_{|\vec{\tilde{Y}}_b|}^{g_b} = \frac{|\vec{\tilde{Y}}_b|!}{g_b!(|\vec{\tilde{Y}}_b|-g_b)!}$. Meanwhile, $T(\vec{\tilde{\Theta}}_\eta) = \sum_{o=1,\eta\neq o}^{q} Sup(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o)$, $Sup(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o) = 1 - Dis(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o)$ and $Sup(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o)$ is the support for $\vec{\tilde{\Theta}}_\eta$ from $\vec{\tilde{\Theta}}_o$, satisfying the following conditions: (1) $Sup(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o) \in [0, 1]$; (2) $Sup(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o) = Sup(\vec{\tilde{\Theta}}_o, \vec{\tilde{\Theta}}_\eta)$; (3) $Sup(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o) \geqslant Sup(\vec{\tilde{\Theta}}_i, \vec{\tilde{\Theta}}_j)$, if $Dis(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o) \leqslant Dis(\vec{\tilde{\Theta}}_i, \vec{\tilde{\Theta}}_j)$, here $Dis(\vec{\tilde{\Theta}}_\eta, \vec{\tilde{\Theta}}_o)$ represents distance between $\vec{\tilde{\Theta}}_\eta$ and $\vec{\tilde{\Theta}}_o$ defined in Definition 4.

To simplify (22), let

$$\vec{\tilde{\omega}}_\eta = \frac{\left(1+T\left(\vec{\tilde{\Theta}}_\eta\right)\right)}{\sum_{o=1}^{q}\left(1+T(\vec{\tilde{\Theta}}_o)\right)}, \tag{23}$$

where $\vec{\tilde{\omega}}_\eta$ is called the power weight, and $\vec{\tilde{\omega}}_\eta \in [0, 1]$ with $\sum_{\eta=1}^{q}\vec{\tilde{\omega}}_\eta = 1$. Then (22) can be further expressed as:

$$SFPPMSM^{(g_1,g_2,\cdots g_e)}\left(\vec{\tilde{\Theta}}_1,\vec{\tilde{\Theta}}_2,\cdots,\vec{\tilde{\Theta}}_q\right) = \frac{1}{e}\left(\bigoplus_{b=1}^{e}\left(\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\bigoplus_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\bigotimes_{x=1}^{g_b}\left(q\vec{\tilde{\omega}}_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g_b}}\right). \tag{24}$$

Theorem 10 *Let* $\vec{\vec{\Theta}}_\eta = (\vec{\vec{\mu}}_\eta, \vec{\vec{\pi}}_\eta, \vec{\vec{\upsilon}}_\eta)$ $(\eta = 1, 2, \ldots, q)$ *be a set of SFNs with parameter vector* $(g_1, g_2, \ldots, g_e)$*, then the aggregated value by SFPPMSM operator is still a SFN, and*

$$SFPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2, \cdots, \vec{\vec{\Theta}}_q\right)$$

$$= \left(\left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\vec{\mu}}_{\eta_x}^2\right)^{q\vec{\vec{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)}\right)^{\frac{1}{e}},\right.$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\vec{\pi}}_{\eta_x}\right)^{2q\vec{\vec{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left.\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\vec{\upsilon}}_{\eta_x}\right)^{2q\vec{\vec{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right) \tag{25}$$

Proof. In accordance with operational rules of Definition 2, then

$$q\vec{\vec{\omega}}_{\eta_x}\vec{\vec{\Theta}}_{\eta_x} = \left(\sqrt{1-\left(1-\vec{\vec{\mu}}_{\eta_x}^2\right)^{q\vec{\vec{\omega}}_{\eta_x}}}, \left(\vec{\vec{\pi}}_{\eta_x}\right)^{q\vec{\vec{\omega}}_{\eta_x}}, \left(\vec{\vec{\upsilon}}_{\eta_x}\right)^{q\vec{\vec{\omega}}_{\eta_x}}\right),$$

$$\bigotimes_{x=1}^{g_b} q\vec{\vec{\omega}}_{\eta_x}\vec{\vec{\Theta}}_{\eta_x} = \left(\prod_{x=1}^{g_b}\sqrt{1-\left(1-\vec{\vec{\mu}}_{\eta_x}^2\right)^{q\vec{\vec{\omega}}_{\eta_x}}},\right.$$

$$\left.\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\vec{\pi}}_{\eta_x}\right)^{2q\vec{\vec{\omega}}_{\eta_x}}\right)}, \sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\vec{\upsilon}}_{\eta_x}\right)^{2q\vec{\vec{\omega}}_{\eta_x}}\right)}\right)$$

and

$$\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \bigotimes_{x=1}^{g_b} q\vec{\tilde{\omega}}_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}$$

$$= \left(\sqrt{1-\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\tilde{\mu}}_{\eta_x}^2\right)^{q\vec{\tilde{\omega}}_{\eta_x}}\right)\right)}, \right.$$

$$\left. \prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\pi}}_{\eta_x}\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}, \prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\upsilon}}_{\eta_x}\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)} \right).$$

Thereafter, we can get

$$\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}} \left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \bigotimes_{x=1}^{g_b} q\vec{\tilde{\omega}}_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x} \right)$$

$$= \left(\sqrt{1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\tilde{\mu}}_{\eta_x}^2\right)^{q\vec{\tilde{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}}, \right.$$

$$\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\pi}}_{\eta_x}\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}},$$

$$\left. \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|} \sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\upsilon}}_{\eta_x}\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}} \right)$$

and

$$\bigoplus_{b=1}^{e}\left(\left(\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}\left(\bigoplus_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}\bigotimes_{x=1}^{g_b} q\vec{\bar{\omega}}_{\eta_x}\vec{\bar{\Theta}}_{\eta_x}\right)\right)^{\frac{1}{g_b}}\right)$$

$$=\left(\sqrt{1-\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\bar{\mu}}_{\eta_x}^{2}\right)^{q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)},\right.$$

$$\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\pi}}_{\eta_x}\right)^{2q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}},$$

$$\left.\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\upsilon}}_{\eta_x}\right)^{2q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right).$$

Therefore,

$$SFPPMSM^{(g_1,g_2,\cdots g_e)}(\vec{\bar{\Theta}}_1,\vec{\bar{\Theta}}_2,\cdots,\vec{\bar{\Theta}}_q)$$

$$=\left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\bar{\mu}}_{\eta_x}^{2}\right)^{q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\pi}}_{\eta_x}\right)^{2q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left.\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\upsilon}}_{\eta_x}\right)^{2q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right).$$

Theorem 11 (Idempotence) *Let $\vec{\bar{\Theta}}_\eta = (\vec{\bar{\mu}}_\eta, \vec{\bar{\pi}}_\eta, \vec{\bar{\upsilon}}_\eta)$ ($\eta = 1, 2, \ldots, q$) be a set of SFN with parameter vector $(g_1, g_2, \ldots, g_e)$, if $\vec{\bar{\Theta}}_\eta = \vec{\bar{\Theta}} = \left(\vec{\bar{\mu}}, \vec{\bar{\pi}}, \vec{\bar{\upsilon}}\right)$ for all η, then*

$$SFPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q\right) = \vec{\bar{\Theta}}. \tag{26}$$

Proof. Since $\vec{\bar{\Theta}}_\eta = \vec{\bar{\Theta}} = \left(\vec{\bar{\mu}}, \vec{\bar{\pi}}, \vec{\bar{\upsilon}}\right)$, we have $Sup\left(\vec{\bar{\Theta}}_\eta, \vec{\bar{\Theta}}_o\right) = 1$ for all $\eta, o = 1, 2, \cdots, q$. Thereby $\vec{\bar{\omega}}_\eta = \dfrac{1}{q}$, $\eta = 1, 2, \ldots, q$, and

$$\begin{aligned}
&SFPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q\right) \\
&= \left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\left(1-\vec{\bar{\mu}}^{2g_b}\right)^{C^{g_b}_{|\vec{\bar{Y}}_b|}}\right)^{\frac{1}{C^{g_b}_{|\vec{\bar{Y}}_b|}}}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}}\right., \\
&\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\left(1-\left(1-\vec{\bar{\pi}}^2\right)^{g_b}\right)^{C^{g_b}_{|\vec{\bar{Y}}_b|}}\right)^{\frac{1}{C^{g_b}_{|\vec{\bar{Y}}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}} \\
&\left.\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\left(1-\left(1-\vec{\bar{\upsilon}}^2\right)^{g_b}\right)^{C^{g_b}_{|\vec{\bar{Y}}_b|}}\right)^{\frac{1}{C^{g_b}_{|\vec{\bar{Y}}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right) \\
&= \left(\sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(\vec{\bar{\mu}}^{2g_b}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}}, \left(\prod_{b=1}^{e}\sqrt{1-\left(\left(1-\vec{\bar{\pi}}^2\right)^{g_b}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right., \\
&\left.\left(\prod_{b=1}^{e}\sqrt{1-\left(\left(1-\vec{\bar{\upsilon}}^2\right)^{g_b}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right), \\
&= \left(\hat{\bar{\mu}}_\eta, \hat{\bar{\pi}}_\eta, \hat{\bar{\upsilon}}_\eta\right) = \hat{\bar{\Theta}}_\eta
\end{aligned}$$

which completes the proof. □

Theorem 12 (Boundness) *Let* $\vec{\tilde{\Theta}}_\eta = (\vec{\mu}_\eta, \vec{\pi}_\eta, \vec{\upsilon}_\eta)$ $(\eta = 1, 2, \ldots, q)$ *be a set of SFNs with parameter vector* $(g_1, g_2, \ldots, g_e)$*, if* $\vec{\tilde{\Theta}}^- = \min_\eta\{\vec{\tilde{\Theta}}_\eta\} = \left(\vec{\mu}^-, \vec{\pi}^-, \vec{\upsilon}^-\right)$ *and* $\vec{\tilde{\Theta}}^+ = \max_\eta\{\vec{\tilde{\Theta}}_\eta\} = (\vec{\mu}^+, \vec{\pi}^+, \vec{\upsilon}^+)$*, then*

$$\vec{\tilde{X}} \leqslant SFPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) \leqslant \vec{\tilde{Y}}, \tag{27}$$

where

$$\vec{\tilde{X}} = \left(\sqrt{1 - \left(\prod_{b=1}^{e}\left(1 - \left(1 - \left(\prod_{\substack{1\leqslant\eta_1<\cdots \\ <\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1 - \prod_{x=1}^{g_b}\left(1 - \left(1 - (\vec{\mu}^-)^2\right)^{q\vec{\omega}_{\eta x}}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{g_b}}\right)\right)}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1 - \left(1 - \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1 - \prod_{x=1}^{g_b}\left(1 - \left(\vec{\pi}^-\right)^{2q\vec{\omega}_{\eta x}}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1 - \left(1 - \left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1 - \prod_{x=1}^{g_b}\left(1 - \left(\vec{\upsilon}^-\right)^{2q\vec{\omega}_{\eta x}}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\Bigg)$$

and

$$\vec{\tilde{Y}} = \left(\sqrt{1 - \left(\prod_{b=1}^{e}\left(1 - \left(1 - \left(\prod_{\substack{1\leqslant\eta_1<\cdots \\ <\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1 - \prod_{x=1}^{g_b}\left(1 - \left(1 - (\vec{\mu}^+)^2\right)^{q\vec{\omega}_{\eta x}}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{\tilde{Y}}_b|}}}\right)^{\frac{1}{g_b}}\right)\right)}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\dots<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\pi}}^{+}\right)^{2q\vec{\bar{\omega}}_{\eta x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{1\leqslant\eta_1<\dots<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\upsilon}}^{+}\right)^{2q\vec{\bar{\omega}}_{\eta x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\Bigg).$$

Proof. Since

$$q\vec{\bar{\omega}}_{\eta_x}\vec{\bar{\Theta}}_{\eta_x}=\left(\sqrt{1-\left(1-\vec{\bar{\mu}}_{\eta_x}^{2}\right)^{q\vec{\bar{\omega}}_{\eta x}}},\ \left(\vec{\bar{\pi}}_{\eta_x}\right)^{q\vec{\bar{\omega}}_{\eta x}},\ \left(\vec{\bar{\upsilon}}_{\eta_x}\right)^{q\vec{\bar{\omega}}_{\eta x}}\right)$$
$$\geqslant\left(\sqrt{1-\left(1-(\mu^{-})^{2}\right)^{q\vec{\bar{\omega}}_{\eta x}}},\ \left(\vec{\bar{\pi}}^{-}\right)^{q\vec{\bar{\omega}}_{\eta x}},\ \left(\vec{\bar{\upsilon}}^{-}\right)^{q\vec{\bar{\omega}}_{\eta x}}\right).$$

So,

$$\bigotimes_{x=1}^{g_b}q\vec{\bar{\omega}}_{\eta_x}\vec{\bar{\Theta}}_{\eta_x}=\left(\prod_{x=1}^{g_b}\sqrt{1-\left(1-\vec{\bar{\mu}}_{\eta_x}^{2}\right)^{q\vec{\bar{\omega}}_{\eta x}}},\right.$$
$$\left.\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\pi}}_{\eta_x}\right)^{2q\vec{\bar{\omega}}_{\eta x}}\right)},\ \sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\upsilon}}_{\eta_x}\right)^{2q\vec{\bar{\omega}}_{\eta x}}\right)}\right)$$
$$\geqslant\left(\prod_{x=1}^{g_b}\sqrt{1-\left(1-(\vec{\bar{\mu}}^{-})^{2}\right)^{q\vec{\bar{\omega}}_{\eta x}}},\ \sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\pi}}^{-}\right)^{2q\vec{\bar{\omega}}_{\eta x}}\right)},\right.$$
$$\left.\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\upsilon}}^{-}\right)^{2q\vec{\bar{\omega}}_{\eta x}}\right)}\right)$$

then

$$\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \bigotimes_{x=1}^{g_b} q\vec{\tilde{\omega}}_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x} = \left(\sqrt{1-\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\tilde{\mu}}_{\eta_x}^2\right)^{q\vec{\tilde{\omega}}_{\eta_x}}\right)\right)},\right.$$

$$\left.\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\pi}}_{\eta_x}\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)},\ \prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\upsilon}}_{\eta_x}\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}\right)$$

$$\geqslant\left(\sqrt{1-\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-(\vec{\tilde{\mu}}^-)^2\right)^{q\vec{\tilde{\omega}}_{\eta_x}}\right)\right)},\right.$$

$$\left.\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\pi}}^-\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)},\ \prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\upsilon}}^-\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}\right).$$

Further, we can get

$$\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}} \bigotimes_{x=1}^{g_b} q\vec{\tilde{\omega}}_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}\right) = \left(\sqrt{1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\tilde{\mu}}_{\eta_x}^2\right)^{q\vec{\tilde{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}},\right.$$

$$\left.\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\pi}}_{\eta_x}\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}},\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\upsilon}}_{\eta_x}\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right)$$

$$\geqslant\left(\sqrt{1-\left(\prod_{1\leqslant\eta_1<\cdots<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-(\vec{\tilde{\mu}}^-)^2\right)^{q\vec{\tilde{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}},\right.$$

$$\left.\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\pi}}^-\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}},\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\sqrt{1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\tilde{\upsilon}}^-\right)^{2q\vec{\tilde{\omega}}_{\eta_x}}\right)}\right)^{\frac{1}{C_{|\vec{\tilde{Y}}_b|}^{g_b}}}\right).$$

Therefore,

$$SFPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\Theta}}_1,\vec{\bar{\Theta}}_2,\cdots,\vec{\bar{\Theta}}_q\right)$$

$$=\bigoplus_{b=1}^{e}\left(\left(\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\bigotimes_{x=1}^{g_b}q\vec{\bar{\omega}}_{\eta_x}\vec{\bar{\Theta}}_{\eta_x}\right)\right)^{\frac{1}{g_b}}\right)$$

$$=\left(\sqrt{1-\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\bar{\mu}}_{\eta_x}^2\right)^{q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)}\right),$$

$$\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\pi}}_{\eta_x}\right)^{2q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}},$$

$$\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\upsilon}}_{\eta_x}\right)^{2q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\Bigg)$$

$$\geqslant\left(\sqrt{1-\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-(\vec{\bar{\mu}}^-)^2\right)^{q\vec{\bar{\omega}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)}\right),$$

$$\prod_{b=1}^{e} \sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{Y}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\pi}^-\right)^{2q\vec{\omega}_{\eta x}}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}},$$

$$\prod_{b=1}^{e} \sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{Y}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\upsilon}^-\right)^{2q\vec{\omega}_{\eta x}}\right)\right)\right)^{\frac{1}{C^{g_b}_{|\vec{Y}_b|}}}\right)^{\frac{1}{g_b}}}\Bigg) = \vec{X}.$$

Similarly, it is easy to prove that $SFPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q\right) \leqslant \vec{Y}$. Hence, we can get $\vec{\bar{X}} \leqslant SFPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q\right) \leqslant \vec{\bar{Y}}$.

In the following we investigate some special circumstances for SFPPMSM operator.

(1) When all attributes belong to the same partition and the types of relationship among attributes are same, in other words, $e = 1$, $|\vec{Y}_1| = q$ and $g_1 = g = 1, 2, \ldots, q$, then SFPPMSM operator becomes spherical fuzzy power MSM (SFPMSM) operator as follows:

$$SFPPMSM^{g_1}\left(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q\right) = \left(\frac{1}{C^{g_1}_{|\vec{Y}_1|}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_1}\leqslant|\vec{Y}_1|}} \bigotimes_{x=1}^{g_1}\left(q\vec{\omega}_{\eta_x}\vec{\bar{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g_1}}$$

$$= \left(\frac{1}{C^{g}_{q}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g}\leqslant q}} \bigotimes_{x=1}^{g}\left(q\vec{\bar{\omega}}_{\eta_x}\vec{\bar{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g}} = SFPMSM^{g}\left(\vec{\bar{\Theta}}_1, \vec{\bar{\Theta}}_2, \cdots, \vec{\bar{\Theta}}_q\right).$$

(2) When $e = 1$ and $g = 1$, SFPPMSM operator becomes the spherical fuzzy power averaging operator as follows [49]:

$$SFPPMSM^{1}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) = \left(\frac{1}{C^{1}_{|\vec{\tilde{Y}}_1|}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_1}\leqslant|\vec{\tilde{Y}}_1|}} \bigotimes_{x=1}^{1}\left(q\vec{\tilde{\omega}}_{\eta_x}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)$$
$$= \bigoplus_{\eta=1}^{q}\vec{\tilde{\omega}}_{\eta}\vec{\tilde{\Theta}}_{\eta} = \left(\sqrt{1-\prod_{\eta=1}^{q}\left(1-\vec{\tilde{\mu}}_{\eta}^{2}\right)^{\vec{\tilde{\omega}}_{\eta}}}, \prod_{\eta=1}^{q}\vec{\tilde{\pi}}_{\eta}^{\vec{\tilde{\omega}}_{\eta}}, \prod_{\eta=1}^{q}\vec{\tilde{\upsilon}}_{\eta}^{\vec{\tilde{\omega}}_{\eta}}\right).$$

(3) When $e = 1$ and $g = 2$, SFPPMSM operator becomes the special spherical fuzzy power BM (SFPBM) operator as follows:

$$SFPPMSM^{2}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right) = \left(\frac{1}{q(q-1)}\bigoplus_{\substack{\eta,x=1\\ \eta\neq x}}^{q}\left(q\vec{\tilde{\omega}}_{\eta}\vec{\tilde{\Theta}}_{\eta}\otimes q\vec{\tilde{\omega}}_{x}\vec{\tilde{\Theta}}_{x}\right)\right)^{\frac{1}{2}}$$
$$= SFPBM^{1,1}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right).$$

4.2. The SFWPPMSM operator

Definition 10 *Let $\vec{\tilde{N}} = \{\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\}$ be a set of SFNs, which are divided into e different partitions $\vec{\tilde{Y}}_1, \vec{\tilde{Y}}_2, \cdots, \vec{\tilde{Y}}_e$ with $\vec{\tilde{Y}}_\eta \cap \vec{\tilde{Y}}_x = \emptyset$ and $\bigcup_{b=1}^{e}\vec{\tilde{Y}}_b = \vec{\tilde{N}}$. The $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_q)^T$ is the weight vector of $\vec{\tilde{\Theta}}_\eta$ $(\eta = 1, 2, \ldots, q)$, with $\vartheta_\eta \in [0, 1]$ and $\sum\limits_{\eta=1}^{q}\vartheta_\eta = 1$, then SFWPPMSM operator is defined as following:*

$$SFWPPMSM^{(g_1,g_2,\cdots,g_e)}\left(\vec{\tilde{\Theta}}_1, \vec{\tilde{\Theta}}_2, \cdots, \vec{\tilde{\Theta}}_q\right)$$
$$= \frac{1}{e}\left(\bigoplus_{b=1}^{e}\left(\frac{1}{C^{g_b}_{|\vec{\tilde{Y}}_b|}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\tilde{Y}}_b|}}\bigotimes_{x=1}^{g_b}\left(\frac{q\vartheta_{\eta_x}\left(1+T(\vec{\tilde{\Theta}}_{\eta_x})\right)}{\sum\limits_{o=1}^{q}\vartheta_o\left(1+T(\vec{\tilde{\Theta}}_o)\right)}\vec{\tilde{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g_b}}\right), \quad (28)$$

where $|\vec{\vec{Y}}_b|$ *is the cardinality of* $\vec{\vec{Y}}_b$ $(b = 1, 2, \cdots, e)$ *and* $\sum_{b=1}^{e} |\vec{\vec{Y}}_b| = q$, g_b *is the parameter in the partition* $\vec{\vec{Y}}_b$ *and* $g_b = 1, 2, \cdots, |\vec{\vec{Y}}_b|$. $(\eta_1, \eta_2, \ldots, \eta_{g_b})$ *traverses all the* g_b*-tuple combination of* $(1, 2, \ldots, |\vec{\vec{Y}}_b|)$, *and* $C_{|\vec{\vec{Y}}_b|}^{g_b}$ *represents the binomial coefficient satisfying* $C_{|\vec{\vec{Y}}_b|}^{g_b} = \frac{|\vec{\vec{Y}}_b|!}{g_b!(|\vec{\vec{Y}}_b| - g_b)!}$. *Meanwhile,* $T(\vec{\vec{\Theta}}_\eta) = \sum_{o=1,\eta\neq o}^{q} Sup\left(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o\right)$, $Sup\left(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o\right) = 1 - Dis\left(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o\right)$ *and* $Sup\left(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o\right)$ *is the support for* $\vec{\vec{\Theta}}_\eta$ *from* $\vec{\vec{\Theta}}_o$, *satisfying the following conditions: (1)* $Sup(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o) \in [0, 1]$; *(2)* $Sup(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o) = Sup(\vec{\vec{\Theta}}_o, \vec{\vec{\Theta}}_\eta)$; *(3)* $Sup(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o) \geqslant Sup(\vec{\vec{\Theta}}_i, \vec{\vec{\Theta}}_j)$, *if* $Dis(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o) \leqslant Dis(\vec{\vec{\Theta}}_i, \vec{\vec{\Theta}}_j)$, *here* $Dis(\vec{\vec{\Theta}}_\eta, \vec{\vec{\Theta}}_o)$ *represents distance between* $\vec{\vec{\Theta}}_\eta$ *and* $\vec{\vec{\Theta}}_o$ *defined in Definition 4.*

To simplify (28), let

$$\vec{\vec{\varpi}}_\eta = \frac{\vartheta_\eta \left(1 + T(\vec{\vec{\Theta}}_\eta)\right)}{\sum_{o=1}^{q} \vartheta_o \left(1 + T(\vec{\vec{\Theta}}_o)\right)}. \tag{29}$$

Then (28) can be further expressed as

$$SFWPPMSM^{(g_1,g_2,\ldots,g_e)}\left(\vec{\vec{\Theta}}_1, \vec{\vec{\Theta}}_2, \cdots, \vec{\vec{\Theta}}_q\right)$$
$$= \frac{1}{e}\left(\bigoplus_{b=1}^{e}\left(\frac{1}{C_{|\vec{\vec{Y}}_b|}^{g_b}}\left(\bigoplus_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\vec{Y}}_b|}} \bigotimes_{x=1}^{g_b}\left(q\vec{\vec{\varpi}}_{\eta_x}\vec{\vec{\Theta}}_{\eta_x}\right)\right)\right)^{\frac{1}{g_b}}\right) \tag{30}$$

Theorem 13 *Let* $\vec{\vec{\Theta}}_\eta = (\vec{\vec{\mu}}_\eta, \vec{\vec{\pi}}_\eta, \vec{\vec{v}}_\eta)$ $(\eta = 1, 2, \ldots, q)$ *be a set of SFNs with parameter vector* $(g_1, g_2, \ldots, g_e)$, *and* $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_q)^T$ *is the weight vector of* $\vec{\vec{\Theta}}_\eta(\eta = 1, 2, \ldots, q)$, *with* $\vartheta_\eta \geqslant 0$, $\sum_{\eta=1}^{q} \vartheta_\eta = 1$, *then the aggregated value by*

SFWPPMSM operator is still a SFN, and

$$SFWPPMSM^{(g_1,g_2,\ldots g_e)}\left(\vec{\bar{\Theta}}_1,\vec{\bar{\Theta}}_2,\cdots,\vec{\bar{\Theta}}_q\right)$$

$$= \left\langle \sqrt{1-\left(\prod_{b=1}^{e}\left(1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(1-\vec{\bar{\mu}}_{\eta_x}^2\right)^{q\vec{\bar{\varpi}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}\right)\right)^{\frac{1}{e}}}\right.,$$

$$\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\pi}}_{\eta_x}\right)^{2q\vec{\bar{\varpi}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}},$$

$$\left.\left(\prod_{b=1}^{e}\sqrt{1-\left(1-\left(\prod_{\substack{1\leqslant\eta_1<\cdots\\<\eta_{g_b}\leqslant|\vec{\bar{Y}}_b|}}\left(1-\prod_{x=1}^{g_b}\left(1-\left(\vec{\bar{\upsilon}}_{\eta_x}\right)^{2q\vec{\bar{\varpi}}_{\eta_x}}\right)\right)\right)^{\frac{1}{C_{|\vec{\bar{Y}}_b|}^{g_b}}}\right)^{\frac{1}{g_b}}}\right)^{\frac{1}{e}}\right\rangle. \tag{31}$$

The proof is similar to Theorem 10.

5. A novel method for MAGDM based on SFWPPMSM operator

For this section, we shall establish a novel approach for MAGDM with SFNs by SFWPPMSM operator.

Let $\vec{\bar{\wp}} = \{\vec{\bar{\wp}}_1, \vec{\bar{\wp}}_2, \cdots, \vec{\bar{\wp}}_p\}$ be the set of p alternatives and $\vec{\bar{\aleph}} = \{\vec{\bar{\aleph}}_1, \vec{\bar{\aleph}}_2, \cdots, \vec{\bar{\aleph}}_q\}$ is the corresponding q attributes with attribute weights $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_q)^T$, $\vartheta_k \geqslant 0$, $\sum_{k=1}^{q}\vartheta_k = 1$, and $\vec{\bar{\mathfrak{R}}} = \{\vec{\bar{\mathfrak{R}}}_1, \vec{\bar{\mathfrak{R}}}_2, \cdots, \vec{\bar{\mathfrak{R}}}_z\}$ stands for z DMs, where $\vec{\bar{\delta}} = (\vec{\bar{\delta}}_1, \vec{\bar{\delta}}_2, \cdots, \vec{\bar{\delta}}_z)^T$ stands for the weight vector of z experts with

$\vec{\vec{\delta}}_h \geqslant 0$ and $\sum_{h=1}^{z} \vec{\vec{\delta}}_h = 1$. Assume that attributes are divided into e distinct partitions $\vec{\vec{Y}}_1, \vec{\vec{Y}}_2, \cdots, \vec{\vec{Y}}_e$, and any g_b attributes have an interrelationship in partition $\vec{\vec{Y}}_b$ but attributes in distinct partitions are irrelevant.

Moreover, considering the uncertainty of evaluation environment in MAGDM problems, SFNs are employed to express the preference information of DMs. Let $\vec{\vec{\Xi}}^h = \left(\vec{\vec{\Xi}}^h_{fk}\right)_{p\times q}$ represents a spherical fuzzy decision matrix, where $\vec{\vec{\Xi}}^h_{fk} = (\vec{\mu}^h_{fk}, \vec{\pi}^h_{fk}, \vec{\upsilon}^h_{fk})$ $(h = 1, 2, \cdots, z)$ denotes SFN of the h-th DM about the f-th alternative under the k-th attribute.

Next, we shall give the general steps of the established approach to MAGDM issues.

Step 1. Acquire spherical fuzzy decision matrices by DMs.

Step 2. Normalize decision matrix $\vec{\vec{\Xi}}^h$ $(h = 1, 2, \ldots, z)$ depending on (32):

$$\hat{\hat{W}}^h = \left(\hat{\hat{w}}^h_{fk}\right)_{p\times q}, \quad f = 1, 2, \cdots, p; \quad k = 1, 2, \cdots, q; \quad h = 1, 2, \cdots, z,$$

$$\begin{aligned} \hat{\hat{w}}^h_{fk} &= \left(\hat{\hat{\mu}}^h_{fk}, \hat{\hat{\pi}}^h_{fk}, \hat{\hat{\upsilon}}^h_{fk}\right) \\ &= \begin{cases} \vec{\vec{\Xi}}^h_{fk} = \vec{\mu}^h_{fk},\ \vec{\pi}^h_{fk},\ \vec{\upsilon}^h_{fk}, & \text{for benefit attribute} \\ (\vec{\vec{\Xi}}^h_{fk})^c = \left(\vec{\upsilon}^h_{fk},\ \vec{\pi}^h_{fk},\ \vec{\mu}^h_{fk}\right), & \text{for } \cos t \text{ attribute.} \end{cases} \end{aligned} \tag{32}$$

Step 3. Determine support by (33):

$$\begin{gathered} Sup\left(\hat{\hat{w}}^h_{fk}, \hat{\hat{w}}^i_{fk}\right) = 1 - Dis\left(\hat{\hat{w}}^h_{fk}, \hat{\hat{w}}^i_{fk}\right), \\ f = 1, 2, \ldots, p; \quad k = 1, 2, \ldots, q; \quad h, i = 1, 2, \ldots, z, \quad h \neq i. \end{gathered} \tag{33}$$

Here, $Sup\left(\hat{\hat{w}}^h_{fk}, \hat{\hat{w}}^i_{fk}\right)$ meets the support conditions, and $Dis\left(\hat{\hat{w}}^h_{fk}, \hat{\hat{w}}^i_{fk}\right)$ indicates the distance measure computed by (9).

Step 4. Calculate the support degree $T(\hat{\hat{w}}^h_{fk})$ of the SFN $\hat{\hat{w}}^h_{fk}$ to the other SFNs $\hat{\hat{w}}^i_{fk}$ $(h, i = 1, 2, \cdots, z, h \neq i)$ with (34):

$$T\left(\hat{\hat{w}}^h_{fk}\right) = \sum_{i=1, i\neq h}^{z} Sup\left(\hat{\hat{w}}^h_{fk}, \hat{\hat{w}}^i_{fk}\right). \tag{34}$$

Step 5. Compute power weights $\vec{\bar{\varpi}}^h_{fk}$ by weight $\vec{\bar{\delta}}_h$ $(h = 1, 2, \cdots, z)$ for DM associated with the SFN $\hat{\hat{w}}^h_{fk}$ given as follow:

$$\vec{\bar{\varpi}}^h_{fk} = \frac{\vec{\bar{\delta}}_h \left(1 + T\left(\hat{\hat{w}}^h_{fk}\right)\right)}{\sum\limits_{h=1}^{z} \vec{\bar{\delta}}_h \left(1 + T\left(\hat{\hat{w}}^h_{fk}\right)\right)} \quad (h = 1, 2, \ldots, z) \tag{35}$$

and $\vec{\bar{\varpi}}^h_{fk} \geqslant 0$, $\sum\limits_{h=1}^{z} \vec{\bar{\varpi}}^h_{fk} = 1$.

Step 6. Get the comprehensive decision matrix $\tilde{\bar{W}} = \left(\tilde{\bar{w}}_{fk}\right)_{p\times q}$.

Since there are no partition among DMs, all the individual decision matrices $\hat{\hat{W}}^h = (\hat{\hat{w}}^h_{fk})_{p\times q}$ are aggregated into comprehensive matrix $\tilde{\bar{W}} = \left(\tilde{\bar{w}}_{fk}\right)_{p\times q}$ by using SFWPPMSM operator with $e = 1$, and

$$\begin{aligned} \tilde{\bar{w}}_{fk} &= SFWPPMSM^{(g_1, g_2, \cdots, g_e)}\left(\hat{\hat{w}}^1_{fk}, \hat{\hat{w}}^2_{fk}, \cdots, \hat{\hat{w}}^z_{fk}\right) \\ &= SFWPMSM^{g}\left(\hat{\hat{w}}^1_{fk}, \hat{\hat{w}}^2_{fk}, \cdots, \hat{\hat{w}}^z_{fk}\right). \end{aligned} \tag{36}$$

Step 7. Calculate the support $Sup(\tilde{\bar{w}}_{fk}, \tilde{\bar{w}}_{fr})$ by (37):

$$\begin{gathered} Sup\left(\tilde{\bar{w}}_{fk}, \tilde{\bar{w}}_{fr}\right) = 1 - Dis\left(\tilde{\bar{w}}_{fk}, \tilde{\bar{w}}_{fr}\right), \\ (f = 1, 2, \ldots, p; \quad k, r = 1, 2, \ldots, q, \quad k \neq r), \end{gathered} \tag{37}$$

where, $Dis(\tilde{\bar{w}}_{fk}, \tilde{\bar{w}}_{fr})$ is computed by Definition 4.

Step 8. Determine the support degree $T(\tilde{\bar{w}}_{fk})$ of the SFN $\tilde{\bar{w}}_{fk}$ to the other SFNs $\tilde{\bar{w}}_{fr}$ $(r = 1, 2, \ldots, q, \ r \neq k)$ with (38):

$$T\left(\tilde{\bar{w}}_{fk}\right) = \sum_{r=1, r\neq k}^{q} Sup\left(\tilde{\bar{w}}_{fk}, \tilde{\bar{w}}_{fr}\right). \tag{38}$$

Step 9. Obtain the power weights ℓ_{fk} $(f = 1, 2, \cdots, p, k = 1, 2, \cdots, q)$ associated with the SFN $\tilde{\bar{w}}_{fk}$ by attribute weight ϑ_k, and

$$\ell_{fk} = \frac{\vartheta_k \left(1 + T\left(\tilde{\bar{w}}_{fk}\right)\right)}{\sum\limits_{k=1}^{q} \vartheta_k \left(1 + T\left(\tilde{\bar{w}}_{fk}\right)\right)}. \tag{39}$$

Step 10. Aggregate all SFNs $\widetilde{\overline{w}}_{fk}$ to obtain the total assessment value $\widetilde{\overline{w}}_f$ of each alternative by using the SFWPPMSM operator,

$$\widetilde{\overline{w}}_f = SFWPPMSM(\widetilde{\overline{w}}_{f1}, \widetilde{\overline{w}}_{f2}, \ldots, \widetilde{\overline{w}}_{fq}), \quad (f = 1, 2, \ldots, p). \tag{40}$$

Step 11. Determine the score $Sf\left(\widetilde{\overline{w}}_f\right)$ $(f = 1, 2, \ldots, p)$ by Definition 3.

Step 12. Sort all alternatives based on $Sf\left(\widetilde{\overline{w}}_f\right)$ $(f = 1, 2, \ldots, p)$. The largest score value is the best alternative. If $Sf\left(\widetilde{\overline{w}}_f\right) = Sf\left(\widetilde{\overline{w}}_t\right)$ $(f, t = 1, 2, \ldots, p,$ $t \neq f)$, then sort alternatives $\vec{\vec{\mathfrak{R}}}_f$ and $\vec{\vec{\mathfrak{R}}}_t$ according to the accuracy degrees $Af\left(\widetilde{\overline{w}}_f\right)$ and $Af\left(\widetilde{\overline{w}}_t\right)$, the more the accuracy degree is , the alternative will be.

6. Numerical example and discussion

6.1. Numerical example

In this part, we certify the practicability and feasibility of the established method with an example of hydroelectric power plant construction projects.

Example 1 Suppose a company invites three experienced experts $(\vec{\vec{\mathfrak{R}}}_1, \vec{\vec{\mathfrak{R}}}_2, \vec{\vec{\mathfrak{R}}}_3)$ to evaluate the following five projects ($\vec{\vec{\wp}}_1, \vec{\vec{\wp}}_2, \vec{\vec{\wp}}_3, \vec{\vec{\wp}}_4, \vec{\vec{\wp}}_5$) according to five attributes: (1) workforce quantity $\vec{\vec{\aleph}}_1$, (2) power generation capacity $\vec{\vec{\aleph}}_2$, (3) construction cost $\vec{\vec{\aleph}}_3$, (4) environmental damage impact $\vec{\vec{\aleph}}_4$, (5) security level $\vec{\vec{\aleph}}_5$ to build a hydroelectric power plant, where $\vec{\vec{\aleph}}_1$, $\vec{\vec{\aleph}}_3$ and $\vec{\vec{\aleph}}_4$ are cost attributes. $(0.35, 0.4, 0.25)^T$ is the weight vector of three experts $\vec{\vec{\mathfrak{R}}}_1$, $\vec{\vec{\mathfrak{R}}}_2$, $\vec{\vec{\mathfrak{R}}}_3$, and the weights of attributes $\vec{\vec{\aleph}}_1, \vec{\vec{\aleph}}_2, \vec{\vec{\aleph}}_3, \vec{\vec{\aleph}}_4, \vec{\vec{\aleph}}_5$ are 0.16, 0.25, 0.21, 0.20 and 0.18 separately. Assume that five attributes are divided into two parts: $\vec{\vec{Y}}_1 = \{\vec{\vec{\aleph}}_1, \vec{\vec{\aleph}}_3\}$ and $\vec{\vec{Y}}_2 = \{\vec{\vec{\aleph}}_2, \vec{\vec{\aleph}}_4, \vec{\vec{\aleph}}_5\}$, that is, workforce quantity and construction cost are classified as part $\vec{\vec{Y}}_1$ and power generation capacity, environmental damage impact and security level are classified as part $\vec{\vec{Y}}_2$. The evaluation information obtained by experts using SFNs is presented in Tables 1–3. Next, we use the established approach to deal with this problem.

Step 1. Evaluation matrices of DMs are given in Tables 1–3, so Step 1 has been completed.

Step 2. Normalize the attributes by (32) (see Tables 4–6).

Table 1: The evaluation information from $\vec{\vec{\mathfrak{R}}}_1$

Alternatives	$\vec{\vec{\aleph}}_1$	$\vec{\vec{\aleph}}_2$	$\vec{\vec{\aleph}}_3$	$\vec{\vec{\aleph}}_4$	$\vec{\vec{\aleph}}_5$
$\vec{\vec{\wp}}_1$	(0.3,0.4,0.5)	(0.3,0.6,0.4)	(0.6,0.3,0.3)	(0.5,0.5,0.5)	(0.2,0.3,0.5)
$\vec{\vec{\wp}}_2$	(0.6,0.4,0.5)	(0.3,0.2,0.1)	(0.3,0.3,0.7)	(0.3,0.2,0.3)	(0.3,0.6,0.4)
$\vec{\vec{\wp}}_3$	(0.1,0.2,0.5)	(0.7,0.2,0.3)	(0.1,0.2,0.8)	(0.2,0.6,0.5)	(0.8,0.1,0.4)
$\vec{\vec{\wp}}_4$	(0.2,0.5,0.6)	(0.9,0.1,0.1)	(0.4,0.5,0.2)	(0.4,0.7,0.4)	(0.4,0.5,0.5)
$\vec{\vec{\wp}}_5$	(0.6,0.5,0.1)	(0.6,0.4,0.2)	(0.1,0.2,0.8)	(0.5,0.1,0.4)	(0.3,0.2,0.7)

Table 2: The evaluation information from $\vec{\vec{\mathfrak{R}}}_2$

Alternatives	$\vec{\vec{\aleph}}_1$	$\vec{\vec{\aleph}}_2$	$\vec{\vec{\aleph}}_3$	$\vec{\vec{\aleph}}_4$	$\vec{\vec{\aleph}}_5$
$\vec{\vec{\wp}}_1$	(0.3,0.4,0.6)	(0.4,0.6,0.4)	(0.5,0.2,0.8)	(0.3,0.4,0.5)	(0.5,0.4,0.3)
$\vec{\vec{\wp}}_2$	(0.6,0.5,0.5)	(0.3,0.1,0.8)	(0.4,0.8,0.4)	(0.8,0.1,0.2)	(0.9,0.3,0.1)
$\vec{\vec{\wp}}_3$	(0.3,0.2,0.5)	(0.7,0.6,0.2)	(0.1,0.7,0.3)	(0.2,0.6,0.7)	(0.7,0.6,0.3)
$\vec{\vec{\wp}}_4$	(0.5,0.5,0.6)	(0.3,0.8,0.2)	(0.4,0.7,0.5)	(0.3,0.8,0.4)	(0.4,0.3,0.6)
$\vec{\vec{\wp}}_5$	(0.2,0.8,0.1)	(0.5,0.6,0.4)	(0.4,0.2,0.3)	(0.1,0.1,0.8)	(0.3,0.6,0.4)

Table 3: The evaluation information from $\vec{\vec{\mathfrak{R}}}_3$

Alternatives	$\vec{\vec{\aleph}}_1$	$\vec{\vec{\aleph}}_2$	$\vec{\vec{\aleph}}_3$	$\vec{\vec{\aleph}}_4$	$\vec{\vec{\aleph}}_5$
$\vec{\vec{\wp}}_1$	(0.6,0.7,0.1)	(0.3,0.1,0.7)	(0.4,0.6,0.3)	(0.3,0.7,0.4)	(0.3,0.4,0.6)
$\vec{\vec{\wp}}_2$	(0.2,0.3,0.5)	(0.4,0.3,0.5)	(0.6,0.4,0.2)	(0.5,0.2,0.6)	(0.2,0.7,0.4)
$\vec{\vec{\wp}}_3$	(0.2,0.3,0.9)	(0.7,0.2,0.3)	(0.2,0.3,0.7)	(0.4,0.3,0.5)	(0.3,0.4,0.2)
$\vec{\vec{\wp}}_4$	(0.2,0.1,0.5)	(0.3,0.3,0.7)	(0.2,0.5,0.1)	(0.2,0.1,0.3)	(0.5,0.5,0.6)
$\vec{\vec{\wp}}_5$	(0.1,0.4,0.2)	(0.6,0.6,0.4)	(0.4,0.6,0.4)	(0.1,0.5,0.2)	(0.3,0.1,0.7)

Step 3. Determine support $Sup\left(\hat{\vec{w}}^{h}_{fk}, \hat{\vec{w}}^{i}_{fk}\right)$ by (33). For simplicity, let $Sup\left(\hat{\vec{w}}^{h}_{fk}, \hat{\vec{w}}^{i}_{fk}\right) = \hat{\vec{S}}^{hi}_{fk}$, then we have

$$\hat{\vec{S}}^{12}_{11} = \hat{\vec{S}}^{21}_{11} = 0.96333, \quad \hat{\vec{S}}^{12}_{12} = \hat{\vec{S}}^{21}_{12} = 0.97667, \quad \hat{\vec{S}}^{12}_{13} = \hat{\vec{S}}^{21}_{13} = 0.76333,$$

$$\hat{\vec{S}}^{12}_{14} = \hat{\vec{S}}^{21}_{14} = 0.91667, \quad \hat{\vec{S}}^{12}_{15} = \hat{\vec{S}}^{21}_{51} = 0.85333, \quad \hat{\vec{S}}^{12}_{21} = \hat{\vec{S}}^{21}_{21} = 0.97000,$$

$$\hat{\vec{S}}^{12}_{22} = \hat{\vec{S}}^{21}_{22} = 0.78000, \quad \hat{\vec{S}}^{12}_{23} = \hat{\vec{S}}^{21}_{23} = 0.68333, \quad \hat{\vec{S}}^{12}_{24} = \hat{\vec{S}}^{21}_{24} = 0.79000,$$

Table 4: Normalized evaluation information of $\vec{\vec{\mathfrak{R}}}_1$

Alternatives	$\vec{\vec{\aleph}}_1$	$\vec{\vec{\aleph}}_2$	$\vec{\vec{\aleph}}_3$	$\vec{\vec{\aleph}}_4$	$\vec{\vec{\aleph}}_5$
$\vec{\vec{\wp}}_1$	(0.5,0.4,0.3)	(0.3,0.6,0.4)	(0.3,0.3,0.6)	(0.5,0.5,0.5)	(0.2,0.3,0.5)
$\vec{\vec{\wp}}_2$	(0.5,0.4,0.6)	(0.3,0.2,0.1)	(0.7,0.3,0.3)	(0.3,0.2,0.3)	(0.3,0.6,0.4)
$\vec{\vec{\wp}}_3$	(0.5,0.2,0.1)	(0.7,0.2,0.3)	(0.8,0.2,0.1)	(0.5,0.6,0.2)	(0.8,0.1,0.4)
$\vec{\vec{\wp}}_4$	(0.6,0.5,0.2)	(0.9,0.1,0.1)	(0.2,0.5,0.4)	(0.4,0.7,0.4)	(0.4,0.5,0.5)
$\vec{\vec{\wp}}_5$	(0.1,0.5,0.6)	(0.6,0.4,0.2)	(0.8,0.2,0.1)	(0.4,0.1,0.5)	(0.3,0.2,0.7)

Table 5: Normalized evaluation information of $\vec{\vec{\mathfrak{R}}}_2$

Alternatives	$\vec{\vec{\aleph}}_1$	$\vec{\vec{\aleph}}_2$	$\vec{\vec{\aleph}}_3$	$\vec{\vec{\aleph}}_4$	$\vec{\vec{\aleph}}_5$
$\vec{\vec{\wp}}_1$	(0.6,0.4,0.3)	(0.4,0.6,0.4)	(0.8,0.2,0.5)	(0.5,0.4,0.3)	(0.5,0.4,0.3)
$\vec{\vec{\wp}}_2$	(0.5,0.5,0.6)	(0.3,0.1,0.8)	(0.4,0.8,0.4)	(0.2,0.1,0.8)	(0.9,0.3,0.1)
$\vec{\vec{\wp}}_3$	(0.5,0.2,0.3)	(0.7,0.6,0.2)	(0.3,0.7,0.1)	(0.7,0.6,0.2)	(0.7,0.6,0.3)
$\vec{\vec{\wp}}_4$	(0.6,0.5,0.5)	(0.3,0.8,0.2)	(0.5,0.7,0.4)	(0.4,0.8,0.3)	(0.4,0.3,0.6)
$\vec{\vec{\wp}}_5$	(0.1,0.8,0.2)	(0.5,0.6,0.4)	(0.3,0.2,0.4)	(0.8,0.1,0.1)	(0.3,0.6,0.4)

Table 6: Normalized evaluation information of $\vec{\vec{\mathfrak{R}}}_3$

Alternatives	$\vec{\vec{\aleph}}_1$	$\vec{\vec{\aleph}}_2$	$\vec{\vec{\aleph}}_3$	$\vec{\vec{\aleph}}_4$	$\vec{\vec{\aleph}}_5$
$\vec{\vec{\wp}}_1$	(0.1,0.7,0.6)	(0.3,0.1,0.7)	(0.3,0.6,0.4)	(0.4,0.7,0.3)	(0.3,0.4,0.6)
$\vec{\vec{\wp}}_2$	(0.5,0.3,0.2)	(0.4,0.3,0.5)	(0.2,0.4,0.6)	(0.6,0.2,0.5)	(0.2,0.7,0.4)
$\vec{\vec{\wp}}_3$	(0.9,0.3,0.2)	(0.7,0.2,0.3)	(0.7,0.3,0.2)	(0.5,0.3,0.4)	(0.3,0.4,0.2)
$\vec{\vec{\wp}}_4$	(0.5,0.1,0.2)	(0.3,0.3,0.7)	(0.1,0.5,0.2)	(0.3,0.1,0.2)	(0.5,0.5,0.6)
$\vec{\vec{\wp}}_5$	(0.2,0.4,0.1)	(0.6,0.6,0.4)	(0.4,0.6,0.4)	(0.2,0.5,0.1)	(0.3,0.1,0.7)

$$\hat{S}_{25}^{12} = \hat{S}_{25}^{21} = 0.62000, \quad \hat{S}_{31}^{12} = \hat{S}_{31}^{21} = 0.97333, \quad \hat{S}_{32}^{12} = \hat{S}_{32}^{21} = 0.87667,$$
$$\hat{S}_{33}^{12} = \hat{S}_{33}^{21} = 0.66667, \quad \hat{S}_{34}^{12} = \hat{S}_{34}^{21} = 0.92000, \quad \hat{S}_{35}^{12} = \hat{S}_{35}^{21} = 0.81000,$$
$$\hat{S}_{41}^{12} = \hat{S}_{41}^{21} = 0.93000, \quad \hat{S}_{42}^{12} = \hat{S}_{42}^{21} = 0.54000, \quad \hat{S}_{43}^{12} = \hat{S}_{43}^{21} = 0.85000,$$
$$\hat{S}_{44}^{12} = \hat{S}_{44}^{21} = 0.92667, \quad \hat{S}_{45}^{12} = \hat{S}_{45}^{21} = 0.91000, \quad \hat{S}_{51}^{12} = \hat{S}_{51}^{21} = 0.76333,$$
$$\hat{S}_{52}^{12} = \hat{S}_{52}^{21} = 0.85667, \quad \hat{S}_{53}^{12} = \hat{S}_{53}^{21} = 0.76667, \quad \hat{S}_{54}^{12} = \hat{S}_{54}^{21} = 0.76000,$$
$$\hat{S}_{55}^{12} = \hat{S}_{55}^{21} = 0.78333, \quad \hat{S}_{11}^{13} = \hat{S}_{11}^{31} = 0.72000, \quad \hat{S}_{12}^{13} = \hat{S}_{12}^{31} = 0.77333,$$

$$\hat{S}^{13}_{13} = \hat{S}^{31}_{13} = 0.84333, \quad \hat{S}^{13}_{14} = \hat{S}^{31}_{14} = 0.83667, \quad \hat{S}^{13}_{15} = \hat{S}^{31}_{15} = 0.92333,$$
$$\hat{S}^{13}_{21} = \hat{S}^{31}_{21} = 0.87000, \quad \hat{S}^{13}_{22} = \hat{S}^{31}_{22} = 0.88000, \quad \hat{S}^{13}_{23} = \hat{S}^{31}_{23} = 0.73667,$$
$$\hat{S}^{13}_{24} = \hat{S}^{31}_{24} = 0.85667, \quad \hat{S}^{13}_{25} = \hat{S}^{31}_{25} = 0.94000, \quad \hat{S}^{13}_{31} = \hat{S}^{31}_{31} = 0.78667,$$
$$\hat{S}^{13}_{32} = \hat{S}^{31}_{32} = 1.00000, \quad \hat{S}^{13}_{33} = \hat{S}^{31}_{33} = 0.92333, \quad \hat{S}^{13}_{34} = \hat{S}^{31}_{34} = 0.87000,$$
$$\hat{S}^{13}_{35} = \hat{S}^{31}_{35} = 0.72667, \quad \hat{S}^{13}_{41} = \hat{S}^{31}_{41} = 0.88333, \quad \hat{S}^{13}_{42} = \hat{S}^{31}_{42} = 0.57333,$$
$$\hat{S}^{13}_{43} = \hat{S}^{31}_{43} = 0.95000, \quad \hat{S}^{13}_{44} = \hat{S}^{31}_{44} = 0.77667, \quad \hat{S}^{13}_{45} = \hat{S}^{31}_{45} = 0.93333,$$
$$\hat{S}^{13}_{51} = \hat{S}^{31}_{51} = 0.84333, \quad \hat{S}^{13}_{52} = \hat{S}^{31}_{52} = 0.89333, \quad \hat{S}^{13}_{53} = \hat{S}^{31}_{53} = 0.68333,$$
$$\hat{S}^{13}_{54} = \hat{S}^{31}_{54} = 0.80000, \quad \hat{S}^{13}_{55} = \hat{S}^{31}_{55} = 0.99000, \quad \hat{S}^{23}_{11} = \hat{S}^{32}_{11} = 0.68333,$$
$$\hat{S}^{23}_{12} = \hat{S}^{32}_{12} = 0.75000, \quad \hat{S}^{23}_{13} = \hat{S}^{32}_{13} = 0.68000, \quad \hat{S}^{23}_{14} = \hat{S}^{32}_{14} = 0.86000,$$
$$\hat{S}^{23}_{15} = \hat{S}^{32}_{15} = 0.85667, \quad \hat{S}^{23}_{21} = \hat{S}^{32}_{21} = 0.84000, \quad \hat{S}^{23}_{22} = \hat{S}^{32}_{22} = 0.82000,$$
$$\hat{S}^{23}_{23} = \hat{S}^{32}_{23} = 0.73333, \quad \hat{S}^{23}_{24} = \hat{S}^{32}_{24} = 0.75333, \quad \hat{S}^{23}_{25} = \hat{S}^{32}_{25} = 0.56000,$$
$$\hat{S}^{23}_{31} = \hat{S}^{32}_{31} = 0.78000, \quad \hat{S}^{23}_{32} = \hat{S}^{32}_{32} = 0.87667, \quad \hat{S}^{23}_{33} = \hat{S}^{32}_{33} = 0.72333,$$
$$\hat{S}^{23}_{34} = \hat{S}^{32}_{34} = 0.79000, \quad \hat{S}^{23}_{35} = \hat{S}^{32}_{35} = 0.78333, \quad \hat{S}^{23}_{41} = \hat{S}^{32}_{41} = 0.81333,$$
$$\hat{S}^{23}_{42} = \hat{S}^{32}_{42} = 0.66667, \quad \hat{S}^{23}_{43} = \hat{S}^{32}_{43} = 0.80000, \quad \hat{S}^{23}_{44} = \hat{S}^{32}_{44} = 0.75000,$$
$$\hat{S}^{23}_{45} = \hat{S}^{32}_{45} = 0.91667, \quad \hat{S}^{23}_{51} = \hat{S}^{32}_{51} = 0.82000, \quad \hat{S}^{23}_{52} = \hat{S}^{32}_{52} = 0.96333,$$
$$\hat{S}^{23}_{53} = \hat{S}^{32}_{53} = 0.87000, \quad \hat{S}^{23}_{54} = \hat{S}^{32}_{54} = 0.72000, \quad \hat{S}^{23}_{55} = \hat{S}^{32}_{55} = 0.77333.$$

Step 4. Calculate the support degree $T(\hat{w}^h_{fk})$ with (34). For simplicity, let $T(\hat{w}^h_{fk}) = \hat{T}^h_{fk}$, then we have

$$\hat{T}^1_{11} = 1.68333, \ \hat{T}^1_{12} = 1.75000, \ \hat{T}^1_{13} = 1.60667, \ \hat{T}^1_{14} = 1.75333, \ \hat{T}^1_{15} = 1.77667,$$
$$\hat{T}^1_{21} = 1.84000, \ \hat{T}^1_{22} = 1.66000, \ \hat{T}^1_{23} = 1.42000, \ \hat{T}^1_{24} = 1.64667, \ \hat{T}^1_{25} = 1.56000,$$
$$\hat{T}^1_{31} = 1.76000, \ \hat{T}^1_{32} = 1.87667, \ \hat{T}^1_{33} = 1.59000, \ \hat{T}^1_{34} = 1.79000, \ \hat{T}^1_{35} = 1.53667,$$
$$\hat{T}^1_{41} = 1.81333, \ \hat{T}^1_{42} = 1.11333, \ \hat{T}^1_{43} = 1.80000, \ \hat{T}^1_{44} = 1.70333, \ \hat{T}^1_{45} = 1.84333,$$
$$\hat{T}^1_{51} = 1.60667, \ \hat{T}^1_{52} = 1.75000, \ \hat{T}^1_{53} = 1.45000, \ \hat{T}^1_{54} = 1.56000, \ \hat{T}^1_{55} = 1.77333,$$

$\hat{T}^2_{11} = 1.64667,\ \hat{T}^2_{12} = 1.72667,\ \hat{T}^2_{13} = 1.44333,\ \hat{T}^2_{14} = 1.77667,\ \hat{T}^2_{15} = 1.71000,$

$\hat{T}^2_{21} = 1.81000,\ \hat{T}^2_{22} = 1.60000,\ \hat{T}^2_{23} = 1.41667,\ \hat{T}^2_{24} = 1.54333,\ \hat{T}^2_{25} = 1.18000,$

$\hat{T}^2_{31} = 1.75333,\ \hat{T}^2_{32} = 1.75333,\ \hat{T}^2_{33} = 1.39000,\ \hat{T}^2_{34} = 1.71000,\ \hat{T}^2_{35} = 1.59333,$

$\hat{T}^2_{41} = 1.74333,\ \hat{T}^2_{42} = 1.20667,\ \hat{T}^2_{43} = 1.65000,\ \hat{T}^2_{44} = 1.67667,\ \hat{T}^2_{45} = 1.82667,$

$\hat{T}^2_{51} = 1.58333,\ \hat{T}^2_{52} = 1.82000,\ \hat{T}^2_{53} = 1.63667,\ \hat{T}^2_{54} = 1.48000,\ \hat{T}^2_{55} = 1.55667,$

$\hat{T}^3_{11} = 1.40333,\ \hat{T}^3_{12} = 1.52333,\ \hat{T}^3_{13} = 1.52333,\ \hat{T}^3_{14} = 1.69667,\ \hat{T}^3_{15} = 1.78000,$

$\hat{T}^3_{21} = 1.71000,\ \hat{T}^3_{22} = 1.70000,\ \hat{T}^3_{23} = 1.47000,\ \hat{T}^3_{24} = 1.61000,\ \hat{T}^3_{25} = 1.50000,$

$\hat{T}^3_{31} = 1.56667,\ \hat{T}^3_{32} = 1.87667,\ \hat{T}^3_{33} = 1.64667,\ \hat{T}^3_{34} = 1.66000,\ \hat{T}^3_{35} = 1.51000,$

$\hat{T}^3_{41} = 1.69667,\ \hat{T}^3_{42} = 1.24000,\ \hat{T}^3_{43} = 1.75000,\ \hat{T}^3_{24} = 1.52667,\ \hat{T}^3_{25} = 1.85000,$

$\hat{T}^3_{51} = 1.66333,\ \hat{T}^3_{52} = 1.85667,\ \hat{T}^3_{53} = 1.55333,\ \hat{T}^3_{54} = 1.52000,\ \hat{T}^3_{55} = 1.76333.$

Step 5. Compute power weights $\vec{\varpi}^h_{fk}$ $(h = 1,2,3,\ f,k = 1,\ldots,5)$ by utilizing (35).

$\vec{\varpi}^1_{11} = 0.36140,\ \vec{\varpi}^1_{12} = 0.35861,\ \vec{\varpi}^1_{13} = 0.36197,\ \vec{\varpi}^1_{14} = 0.35062,\ \vec{\varpi}^1_{15} = 0.35329,$

$\vec{\varpi}^1_{21} = 0.35557,\ \vec{\varpi}^1_{22} = 0.35185,\ \vec{\varpi}^1_{23} = 0.34839,\ \vec{\varpi}^1_{24} = 0.35681,\ \vec{\varpi}^1_{25} = 0.37443,$

$\vec{\varpi}^1_{31} = 0.35659,\ \vec{\varpi}^1_{32} = 0.35611,\ \vec{\varpi}^1_{33} = 0.35913,\ \vec{\varpi}^1_{34} = 0.35828,\ \vec{\varpi}^1_{35} = 0.34781,$

$\vec{\varpi}^1_{41} = 0.35726,\ \vec{\varpi}^1_{42} = 0.33893,\ \vec{\varpi}^1_{43} = 0.35930,\ \vec{\varpi}^1_{44} = 0.35725,\ \vec{\varpi}^1_{45} = 0.35062,$

$\vec{\varpi}^1_{51} = 0.34935,\ \vec{\varpi}^1_{52} = 0.34318,\ \vec{\varpi}^1_{53} = 0.33621,\ \vec{\varpi}^1_{54} = 0.35584,\ \vec{\varpi}^1_{55} = 0.36163,$

$\vec{\varpi}^2_{11} = 0.40739,\ \vec{\varpi}^2_{12} = 0.40636,\ \vec{\varpi}^2_{13} = 0.38775,\ \vec{\varpi}^2_{14} = 0.40410,\ \vec{\varpi}^2_{15} = 0.39406,$

$\vec{\varpi}^2_{21} = 0.40207,\ \vec{\varpi}^2_{22} = 0.39305,\ \vec{\varpi}^2_{23} = 0.39761,\ \vec{\varpi}^2_{24} = 0.39186,\ \vec{\varpi}^2_{25} = 0.36440,$

$\vec{\varpi}^2_{31} = 0.40655,\ \vec{\varpi}^2_{32} = 0.38953,\ \vec{\varpi}^2_{33} = 0.37874,\ \vec{\varpi}^2_{34} = 0.39773,\ \vec{\varpi}^2_{35} = 0.40637,$

$\vec{\varpi}^2_{41} = 0.39814,\ \vec{\varpi}^2_{42} = 0.40446,\ \vec{\varpi}^2_{43} = 0.38863,\ \vec{\varpi}^2_{44} = 0.40425,\ \vec{\varpi}^2_{45} = 0.39836,$

$\vec{\varpi}^2_{51} = 0.39569,\ \vec{\varpi}^2_{52} = 0.40219,\ \vec{\varpi}^2_{53} = 0.41351,\ \vec{\varpi}^2_{54} = 0.39396,\ \vec{\varpi}^2_{55} = 0.38100,$

$\vec{\varpi}^3_{11} = 0.23121,\ \vec{\varpi}^3_{12} = 0.23503,\ \vec{\varpi}^3_{13} = 0.25028,\ \vec{\varpi}^3_{14} = 0.24529,\ \vec{\varpi}^3_{15} = 0.25265,$

$\vec{\varpi}^3_{21} = 0.24235,\ \vec{\varpi}^3_{22} = 0.25510,\ \vec{\varpi}^3_{23} = 0.25399,\ \vec{\varpi}^3_{24} = 0.25133,\ \vec{\varpi}^3_{25} = 0.26118,$

$\vec{\varpi}^3_{31} = 0.23686,\ \vec{\varpi}^3_{32} = 0.25436,\ \vec{\varpi}^3_{33} = 0.26213,\ \vec{\varpi}^3_{34} = 0.24399,\ \vec{\varpi}^3_{35} = 0.24582,$

$\vec{\bar{\varpi}}^3_{41} = 0.24460$, $\vec{\bar{\varpi}}^3_{42} = 0.25661$, $\vec{\bar{\varpi}}^3_{43} = 0.25206$, $\vec{\bar{\varpi}}^3_{44} = 0.23850$, $\vec{\bar{\varpi}}^3_{45} = 0.25103$,
$\vec{\bar{\varpi}}^3_{51} = 0.25496$, $\vec{\bar{\varpi}}^3_{52} = 0.25464$, $\vec{\bar{\varpi}}^3_{53} = 0.25028$, $\vec{\bar{\varpi}}^3_{54} = 0.25020$, $\vec{\bar{\varpi}}^3_{55} = 0.25737$.

Step 6. Get the comprehensive decision matrix $\widetilde{\bar{W}} = \left(\widetilde{\bar{w}}_{fk}\right)_{p\times q}$ by (36) (suppose $g = 1$), the results are shown in Table 7.

Table 7: Comprehensive decision matrix

Alterna-tives	$\vec{\aleph}_1$	$\vec{\aleph}_2$	$\vec{\aleph}_3$	$\vec{\aleph}_4$	$\vec{\aleph}_5$
$\vec{\wp}_1$	(0.50,0.46,0.35)	(0.35,0.39,0.46)	(0.60,0.30,0.51)	(0.48,0.50,0.36)	(0.38,0.36,0.43)
$\vec{\wp}_2$	(0.50,0.41,0.46)	(0.33,0.17,0.34)	(0.52,0.48,0.40)	(0.39,0.15,0.50)	(0.69,0.49,0.24)
$\vec{\wp}_3$	(0.68,0.22,0.18)	(0.70,0.31,0.26)	(0.66,0.36,0.12)	(0.60,0.51,0.24)	(0.69,0.29,0.30)
$\vec{\wp}_4$	(0.58,0.34,0.29)	(0.68,0.31,0.22)	(0.35,0.57,0.34)	(0.38,0.46,0.30)	(0.43,0.41,0.56)
$\vec{\wp}_5$	(0.13,0.57,0.25)	(0.56,0.52,0.32)	(0.59,0.26,0.25)	(0.61,0.15,0.18)	(0.30,0.25,0.57)

Step 7. Calculate the support $Sup(\widetilde{\bar{w}}_{fk}, \widetilde{\bar{w}}_{fr})$ by (37), for convenience, let $Sup(\widetilde{\bar{w}}_{fk}, \widetilde{\bar{w}}_{fr}) = \bar{S}^{kr}_f$, then

$\bar{S}^{12}_1=\bar{S}^{21}_1=0.91083$, $\bar{S}^{12}_2=\bar{S}^{21}_2=0.87519$, $\bar{S}^{12}_3=\bar{S}^{21}_3=0.96362$, $\bar{S}^{12}_4=\bar{S}^{21}_4=0.93838$,
$\bar{S}^{12}_5=\bar{S}^{21}_5=0.86997$, $\bar{S}^{13}_1=\bar{S}^{31}_1=0.88002$, $\bar{S}^{13}_2=\bar{S}^{31}_2=0.95638$, $\bar{S}^{13}_3=\bar{S}^{31}_3=0.96084$,
$\bar{S}^{13}_4=\bar{S}^{31}_4=0.84851$, $\bar{S}^{13}_5=\bar{S}^{31}_5=0.80466$, $\bar{S}^{14}_1=\bar{S}^{41}_1=0.97828$, $\bar{S}^{14}_2=\bar{S}^{41}_2=0.90551$,
$\bar{S}^{14}_3=\bar{S}^{41}_3=0.88935$, $\bar{S}^{14}_4=\bar{S}^{41}_4=0.89967$, $\bar{S}^{14}_5=\bar{S}^{41}_5=0.76981$, $\bar{S}^{15}_1=\bar{S}^{51}_1=0.91814$,
$\bar{S}^{15}_2=\bar{S}^{51}_2=0.84981$, $\bar{S}^{15}_3=\bar{S}^{51}_3=0.96222$, $\bar{S}^{15}_4=\bar{S}^{51}_4=0.85413$, $\bar{S}^{15}_5=\bar{S}^{51}_5=0.80313$,
$\bar{S}^{23}_1=\bar{S}^{32}_1=0.88173$, $\bar{S}^{23}_2=\bar{S}^{32}_2=0.56523$, $\bar{S}^{23}_3=\bar{S}^{32}_3=0.95491$, $\bar{S}^{23}_4=\bar{S}^{32}_4=0.78689$,
$\bar{S}^{23}_5=\bar{S}^{32}_5=0.91041$, $\bar{S}^{24}_1=\bar{S}^{42}_1=0.90658$, $\bar{S}^{24}_2=\bar{S}^{42}_2=0.93977$, $\bar{S}^{24}_3=\bar{S}^{42}_3=0.89819$,
$\bar{S}^{24}_4=\bar{S}^{42}_4=0.83805$, $\bar{S}^{24}_5=\bar{S}^{42}_5=0.87391$, $\bar{S}^{25}_1=\bar{S}^{52}_1=0.97636$, $\bar{S}^{25}_2=\bar{S}^{52}_2=0.78821$,
$\bar{S}^{25}_3=\bar{S}^{52}_3=0.98510$, $\bar{S}^{25}_4=\bar{S}^{52}_4=0.79251$, $\bar{S}^{25}_5=\bar{S}^{52}_5=0.78127$, $\bar{S}^{34}_1=\bar{S}^{43}_1=0.86147$,
$\bar{S}^{34}_2=\bar{S}^{43}_2=0.86188$, $\bar{S}^{34}_3=\bar{S}^{43}_3=0.91547$, $\bar{S}^{34}_4=\bar{S}^{43}_4=0.94885$, $\bar{S}^{34}_5=\bar{S}^{43}_5=0.96350$,
$\bar{S}^{35}_1=\bar{S}^{53}_1=0.88880$, $\bar{S}^{35}_2=\bar{S}^{53}_2=0.89343$, $\bar{S}^{35}_3=\bar{S}^{53}_3=0.94729$, $\bar{S}^{35}_4=\bar{S}^{53}_4=0.85839$,
$\bar{S}^{35}_5=\bar{S}^{53}_5=0.82716$, $\bar{S}^{45}_1=\bar{S}^{54}_1=0.91389$, $\bar{S}^{45}_2=\bar{S}^{54}_2=0.75531$, $\bar{S}^{45}_3=\bar{S}^{54}_3=0.89058$,
$\bar{S}^{45}_4=\bar{S}^{54}_4=0.89507$, $\bar{S}^{45}_5=\bar{S}^{54}_5=0.79377$.

Step 8. Determine the support degree $T(\widetilde{\overline{w}}_{fk})$ $(f, k = 1, 2, \cdots, 5)$ by (38) (see Table 8).
Step 9. Obtain the power weights ℓ_{fk} $(f, k = 1, 2, \cdots, 5)$ based on (39) (see Table 9).

Table 8: The support degree $(T(\widetilde{\overline{w}}_{fk}))_{5\times5}$

Alternatives	$\vec{\aleph}_1$	$\vec{\aleph}_2$	$\vec{\aleph}_3$	$\vec{\aleph}_4$	$\vec{\aleph}_5$
$\vec{\wp}_1$	3.68726	3.67550	3.51203	3.66022	3.69718
$\vec{\wp}_2$	3.58689	3.46840	3.57692	3.46247	3.28677
$\vec{\wp}_3$	3.77603	3.80182	3.77851	3.59359	3.78519
$\vec{\wp}_4$	3.54069	3.35583	3.44264	3.58163	3.40011
$\vec{\wp}_5$	3.24757	3.43555	3.50573	3.40098	3.20533

Table 9: Power weight matrix $\ell = (\ell_{fk})_{5\times5}$

Alternatives	$\vec{\aleph}_1$	$\vec{\aleph}_2$	$\vec{\aleph}_3$	$\vec{\aleph}_4$	$\vec{\aleph}_5$
$\vec{\wp}_1$	0.16149	0.25170	0.20404	0.20070	0.18207
$\vec{\wp}_2$	0.16395	0.24956	0.21472	0.19938	0.17238
$\vec{\wp}_3$	0.16094	0.25283	0.21134	0.19349	0.18140
$\vec{\wp}_4$	0.16301	0.24434	0.20933	0.20560	0.17771
$\vec{\wp}_5$	0.15545	0.25364	0.21643	0.20133	0.17314

Step 10. Obtain the total assessment value $\widetilde{\overline{w}}_f$ of each alternative over all attributes by (40) (suppose $g_1 = g_2 = 2$), and

$$\widetilde{\overline{w}}_1 = (0.47288, 0.42042, 0.43136), \quad \widetilde{\overline{w}}_2 = (0.48109, 0.35864, 0.40782),$$
$$\widetilde{\overline{w}}_3 = (0.66132, 0.34163, 0.22583), \quad \widetilde{\overline{w}}_4 = (0.47092, 0.43934, 0.35584),$$
$$\widetilde{\overline{w}}_5 = (0.41707, 0.39792, 0.31896).$$

Step 11. Determine the score $Sf(\widetilde{\overline{w}}_f)$ $(f = 1, 2, \ldots, 5)$ by Definition 4.

$$Sf(\widetilde{\overline{w}}_1) = 0.54037, \quad Sf(\widetilde{\overline{w}}_2) = 0.57154, \quad Sf(\widetilde{\overline{w}}_3) = 0.69795,$$
$$Sf(\widetilde{\overline{w}}_4) = 0.55858, \quad Sf(\widetilde{\overline{w}}_5) = 0.56673.$$

Step 12. Rank all alternatives to obtain the optimal one based on $Sf(\widetilde{\overline{w}}_f)$ $(f = 1, 2, \ldots, 5)$.

$$\vec{\wp}_3 > \vec{\wp}_2 > \vec{\wp}_5 > \vec{\wp}_4 > \vec{\wp}_1 .$$

Through the above calculation, it can be concluded that $\vec{\wp}_3$ is the best hydropower station construction project.

6.2. The influence of parameters on results

Next, we take different values of parameters g_1, g_2 to study their influence on evaluation results, as shown in Table 10.

Table 10: The evaluation results of SFWPPMSM operator under different parameter values

Methods	Score values	Rankings
$g_1 = 1,\ g_2 = 1$	$Sf(\vec{\wp}_1) = 0.55383, Sf(\vec{\wp}_2) = 0.60043,$ $Sf(\vec{\wp}_3) = 0.71310, Sf(\vec{\wp}_4) = 0.59065,$ $Sf(\vec{\wp}_5) = 0.63206$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1$
$g_1 = 1,\ g_2 = 2$	$Sf(\vec{\wp}_1) = 0.55038, Sf(\vec{\wp}_2) = 0.57474,$ $Sf(\vec{\wp}_3) = 0.70491, Sf(\vec{\wp}_4) = 0.56418,$ $Sf(\vec{\wp}_5) = 0.60919$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1$
$g_1 = 1,\ g_2 = 3$	$Sf(\vec{\wp}_1) = 0.54884, Sf(\vec{\wp}_2) = 0.55643,$ $Sf(\vec{\wp}_3) = 0.70114, Sf(\vec{\wp}_4) = 0.55337,$ $Sf(\vec{\wp}_5) = 0.59202$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1$
$g_1 = 2,\ g_2 = 1$	$Sf(\vec{\wp}_1) = 0.54394, Sf(\vec{\wp}_2) = 0.59736,$ $Sf(\vec{\wp}_3) = 0.70631, Sf(\vec{\wp}_4) = 0.58545,$ $Sf(\vec{\wp}_5) = 0.59226$	$\vec{\wp}_3 > \vec{\wp}_2 > \vec{\wp}_5 > \vec{\wp}_4 > \vec{\wp}_1$
$g_1 = 2,\ g_2 = 2$	$Sf(\vec{\wp}_1) = 0.54037, Sf(\vec{\wp}_2) = 0.57154,$ $Sf(\vec{\wp}_3) = 0.69795, Sf(\vec{\wp}_4) = 0.55858,$ $Sf(\vec{\wp}_5) = 0.56673$	$\vec{\wp}_3 > \vec{\wp}_2 > \vec{\wp}_5 > \vec{\wp}_4 > \vec{\wp}_1$
$g_1 = 2,\ g_2 = 3$	$Sf(\vec{\wp}_1) = 0.53878, Sf(\vec{\wp}_2) = 0.55316,$ $Sf(\vec{\wp}_3) = 0.69411, Sf(\vec{\wp}_4) = 0.54763,$ $Sf(\vec{\wp}_5) = 0.54704$	$\vec{\wp}_3 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_5 > \vec{\wp}_1$

As can be seen from Table 10, for different values of parameters g_1, g_2, the optimal alternative $\vec{\wp}_3$ and the worst alternative $\vec{\wp}_1$ stay the same, whereas the orders of alternatives $\vec{\wp}_2$, $\vec{\wp}_4$, $\vec{\wp}_5$ have changed for different values of g_1, g_2. The reason for the difference is that the relational structure of attributes has changed about different parameter values. The variation of parameters can capture any types of interrelationships among attributes in the same partition. According to the characteristics of SFWPPMSM operator, parameters g_1, g_2 can reflect DMs' risk attitude. In the actual decision making, DMs can set appropriate parameter values according to their own risk attitude. When DMs pursue the risk type, they can assign large parameter values g_1, g_2 within the allowable range. Otherwise, smaller parameter values g_1, g_2 will be assigned if DMs is risk-averse. Thus, DMs can choose suitable parameter values to make decisions according to the actual meaning of attributes and personal preference. In addition, it is observed that the more interrelationships among attributes are considered in the same partition, the smaller the scores will be.

6.3. Comparative analysis

6.3.1. The effectiveness of the proposed method

In this part, we deal with aforementioned example by existing methods under SFSs to illustrate the effectiveness of the proposed method. The results are shown in Table 11.

Table 11: Score values and orders of different methods in Example 1

Methods	Score values	Orders
SFNWAA [32]	$Sf(\vec{\wp}_1) = 0.55301, Sf(\vec{\wp}_2) = 0.61742, Sf(\vec{\wp}_3) = 0.70941, Sf(\vec{\wp}_4) = 0.60241, Sf(\vec{\wp}_5) = 0.63770$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1$
SFNWGA [32]	$Sf(\vec{\wp}_1) = 0.50745, Sf(\vec{\wp}_2) = 0.51778, Sf(\vec{\wp}_3) = 0.67136, Sf(\vec{\wp}_4) = 0.52333, Sf(\vec{\wp}_5) = 0.54144$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_4 > \vec{\wp}_2 > \vec{\wp}_1$
SWAM [33]	$Sf(\vec{\wp}_1) = 0.52845, Sf(\vec{\wp}_2) = 0.56242, Sf(\vec{\wp}_3) = 0.66445, Sf(\vec{\wp}_4) = 0.55246, Sf(\vec{\wp}_5) = 0.59051$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1$
SFPWA [49]	$Sf(\vec{\wp}_1) = 0.55188, Sf(\vec{\wp}_2) = 0.61204, Sf(\vec{\wp}_3) = 0.71066, Sf(\vec{\wp}_4) = 0.59888, Sf(\vec{\wp}_5) = 0.63808$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1$
SFPWG [49]	$Sf(\vec{\wp}_1) = 0.50747, Sf(\vec{\wp}_2) = 0.51631, Sf(\vec{\wp}_3) = 0.67274, Sf(\vec{\wp}_4) = 0.52162, Sf(\vec{\wp}_5) = 0.54272$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_4 > \vec{\wp}_2 > \vec{\wp}_1$
SFWBM [39] (suppose $p = 2, q = 1$)	$Sf(\vec{\wp}_1) = 0.51479, Sf(\vec{\wp}_2) = 0.53582, Sf(\vec{\wp}_3) = 0.67116, Sf(\vec{\wp}_4) = 0.53004, Sf(\vec{\wp}_5) = 0.57264.$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1$
SFGWMSM [50] (suppose $k = 2$, $\lambda_1 = 0.5, \lambda_2 = 0.5$)	$Sf(\vec{\wp}_1) = 0.47811, Sf(\vec{\wp}_2) = 0.50954, Sf(\vec{\wp}_3) = 0.66314, Sf(\vec{\wp}_4) = 0.50889, Sf(\vec{\wp}_5) = 0.52400.$	$\vec{\wp}_3 > \vec{\wp}_5 > \vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1$
SFWPPMSM (suppose $g_1 = g_2 = 2$)	$Sf(\vec{\wp}_1) = 0.54037, Sf(\vec{\wp}_2) = 0.57154, Sf(\vec{\wp}_3) = 0.69795, Sf(\vec{\wp}_4) = 0.55858, Sf(\vec{\wp}_5) = 0.56673.$	$\vec{\wp}_3 > \vec{\wp}_2 > \vec{\wp}_5 > \vec{\wp}_4 > \vec{\wp}_1$

It is known from Table 11 that the sorting of the presented method is different from SFNWAA, SFNWGA, SWAM, SFPWA, SFPWG, SFWBM and SFWGMSM, whereas the selection of optimal and worst alternatives of all methods is consistent. The above comparative analysis proves the effectiveness of the established method.

6.3.2. The superiority of the proposed method

To further elaborate the superiority of the established approach, a novel instance by more complex case is given below:

Example 2 An organization plans to build a new office block among $\vec{\wp}_1$, $\vec{\wp}_2$, $\vec{\wp}_3$, $\vec{\wp}_4$ and $\vec{\wp}_5$ alternatives by considering the following five attributes: (1) construction cost $\vec{\aleph}_1$, (2) traffic convenience $\vec{\aleph}_2$, (3) surrounding environment $\vec{\aleph}_3$, (4) building quality $\vec{\aleph}_4$, (5) building area $\vec{\aleph}_5$, where $\vec{\aleph}_1$ is the cost attribute, and $(0.2, 0.18, 0.21, 0.23, 0.18)^T$ represents the weight vector of attributes. Based on the characteristics of attributes, the five attributes are divided into two parts: $\vec{Y}_1 = \{\vec{\aleph}_1, \vec{\aleph}_4, \vec{\aleph}_5\}$ and $\vec{Y}_2 = \{\vec{\aleph}_2, \vec{\aleph}_3\}$. Moreover, there is a correlation between any three attributes in $\vec{Y}_1$, and there is a correlation between any two attributes in $\vec{Y}_2$, in other words, $g_1 = 3$, $g_2 = 2$. There are three experts who evaluate the above five alternatives according to attributes $\vec{\aleph}_1$, $\vec{\aleph}_2$, $\vec{\aleph}_3$, $\vec{\aleph}_4$, $\vec{\aleph}_5$ using SFNs, the evaluation results are presented in Tables 12–14. In addition, the weights of three experts are 0.30, 0.35 and 0.35, respectively.

Table 12: The evaluation information from $\vec{\mathfrak{R}}_1$

Alternatives	$\vec{\aleph}_1$	$\vec{\aleph}_2$	$\vec{\aleph}_3$	$\vec{\aleph}_4$	$\vec{\aleph}_5$
$\vec{\wp}_1$	(0.4,0.5,0.5)	(0.6,0.6,0.3)	(0.8,0.3,0.5)	(0.6,0.6,0.1)	(0.5,0.1,0.4)
$\vec{\wp}_2$	(0.2,0.2,0.5)	(0.7,0.1,0.4)	(0.3,0.1,0.6)	(0.9,0.1,0.4)	(0.4,0.2,0.6)
$\vec{\wp}_3$	(0.6,0.3,0.5)	(0.4,0.1,0.2)	(0.8,0.2,0.4)	(0.5,0.1,0.7)	(0.4,0.5,0.2)
$\vec{\wp}_4$	(0.7,0.4,0.5)	(0.9,0.3,0.2)	(0.6,0.2,0.3)	(0.4,0.4,0.7)	(0.6,0.4,0.4)
$\vec{\wp}_5$	(0.2,0.6,0.4)	(0.7,0.1,0.3)	(0.7,0.2,0.3)	(0.6,0.1,0.3)	(0.7,0.2,0.6)

We use the aforementioned methods to settle Example 2. The score values and rankings of alternatives with different approaches are shown in Table 15.

It is observed by Table 15 that there is a striking difference between the rankings of the presented approach and the existing approaches. The main reason for the difference is that the proposed approach can model interrelationships among any multiple attributes in same part while other methods cannot. And more detailed explanation is presented below:

Table 13: The evaluation information from $\vec{\vec{\mathfrak{K}}}_2$

Alternatives	$\vec{\aleph}_1$	$\vec{\aleph}_2$	$\vec{\aleph}_3$	$\vec{\aleph}_4$	$\vec{\aleph}_5$
$\vec{\wp}_1$	(0.6,0.4,0.6)	(0.3,0.6,0.4)	(0.8,0.2,0.3)	(0.8,0.3,0.4)	(0.8,0.3,0.2)
$\vec{\wp}_2$	(0.2,0.6,0.6)	(0.6,0.5,0.6)	(0.4,0.4,0.8)	(0.3,0.1,0.4)	(0.8,0.2,0.2)
$\vec{\wp}_3$	(0.2,0.2,0.4)	(0.2,0.6,0.4)	(0.2,0.8,0.2)	(0.3,0.3,0.8)	(0.8,0.4,0.2)
$\vec{\wp}_4$	(0.3,0.4,0.6)	(0.4,0.5,0.5)	(0.4,0.7,0.5)	(0.3,0.4,0.8)	(0.7,0.1,0.5)
$\vec{\wp}_5$	(0.7,0.4,0.2)	(0.2,0.1,0.4)	(0.3,0.2,0.4)	(0.7,0.3,0.2)	(0.7,0.3,0.6)

Table 14: The evaluation information from $\vec{\vec{\mathfrak{K}}}_3$

Alternatives	$\vec{\aleph}_1$	$\vec{\aleph}_2$	$\vec{\aleph}_3$	$\vec{\aleph}_4$	$\vec{\aleph}_5$
$\vec{\wp}_1$	(0.9,0.1,0.4)	(0.6,0.1,0.3)	(0.2,0.6,0.5)	(0.3,0.2,0.6)	(0.3,0.5,0.6)
$\vec{\wp}_2$	(0.3,0.4,0.5)	(0.4,0.3,0.2)	(0.1,0.5,0.4)	(0.5,0.1,0.5)	(0.2,0.7,0.4)
$\vec{\wp}_3$	(0.7,0.4,0.5)	(0.5,0.4,0.2)	(0.3,0.3,0.5)	(0.3,0.1,0.1)	(0.7,0.2,0.5)
$\vec{\wp}_4$	(0.1,0.2,0.5)	(0.8,0.1,0.2)	(0.1,0.3,0.8)	(0.4,0.1,0.3)	(0.6,0.3,0.6)
$\vec{\wp}_5$	(0.9,0.2,0.2)	(0.6,0.2,0.1)	(0.4,0.3,0.8)	(0.3,0.5,0.4)	(0.2,0.3,0.7)

1. Compared with the proposed method, SFNWAA, SFNWGA and SWAM mainly aggregate information by algebraic product and algebraic sum, which cannot automatically adjust parameters in the evaluation process according to DMs' risk preferences, nor can they reflect the relationship between the arguments. In addition, they also fail to capture interrelationships among any multiple attributes in the same partition. However, the proposed method makes up for all the shortcomings of the aforementioned three methods, so the presented approach is more robust and reasonable to dealing with MAGDM issues.

2. On the basis of algebraic product and algebraic sum operational rules, the methods of Garg, Ullah, Mahmood, Hassan and Jan [49] use PA operator to diminish the impact of negative data on evaluation results by computing the support degree between arguments. However, the presented approach not only utilizes the advantage of PA operator but also makes full use of PMSM operator to capture interrelationships among any multiple attributes in the same partition. Thus, the established method is more scientific than the methods of Garg, Ullah, Mahmood, Hassan and Jan [49].

Table 15: Score values and orders of different methods in Example 2

Methods	Score values	Rankings
SFNWAA [32]	$Sf(\vec{\wp}_1) = 0.64027, Sf(\vec{\wp}_2) = 0.65039, Sf(\vec{\wp}_3) = 0.63662, Sf(\vec{\wp}_4) = 0.63281, Sf(\vec{\wp}_5) = 0.62952.$	$\vec{\wp}_2 > \vec{\wp}_1 > \vec{\wp}_3 > \vec{\wp}_4 > \vec{\wp}_5$
SFNWGA [32]	$Sf(\vec{\wp}_1) = 0.55843, Sf(\vec{\wp}_2) = 0.56921, Sf(\vec{\wp}_3) = 0.54762, Sf(\vec{\wp}_4) = 0.54258, Sf(\vec{\wp}_5) = 0.53048.$	$\vec{\wp}_2 > \vec{\wp}_1 > \vec{\wp}_3 > \vec{\wp}_4 > \vec{\wp}_5$
SWAM [33]	$Sf(\vec{\wp}_1) = 0.60414, Sf(\vec{\wp}_2) = 0.60793, Sf(\vec{\wp}_3) = 0.59450, Sf(\vec{\wp}_4) = 0.60070, Sf(\vec{\wp}_5) = 0.60699.$	$\vec{\wp}_2 > \vec{\wp}_5 > \vec{\wp}_1 > \vec{\wp}_4 > \vec{\wp}_3$
SFPWA [49]	$Sf(\vec{\wp}_1) = 0.64245, Sf(\vec{\wp}_2) = 0.65062, Sf(\vec{\wp}_3) = 0.63556, Sf(\vec{\wp}_4) = 0.63288, Sf(\vec{\wp}_5) = 0.63157.$	$\vec{\wp}_2 > \vec{\wp}_1 > \vec{\wp}_3 > \vec{\wp}_4 > \vec{\wp}_5$
SFPWG [49]	$Sf(\vec{\wp}_1) = 0.56251, Sf(\vec{\wp}_2) = 0.57137, Sf(\vec{\wp}_3) = 0.54731, Sf(\vec{\wp}_4) = 0.54396, Sf(\vec{\wp}_5) = 0.53334.$	$\vec{\wp}_2 > \vec{\wp}_1 > \vec{\wp}_3 > \vec{\wp}_4 > \vec{\wp}_5$
SFWBM [39] (suppose $p = 2, q = 1$)	$Sf(\vec{\wp}_1) = 0.58431, Sf(\vec{\wp}_2) = 0.59114, Sf(\vec{\wp}_3) = 0.57196, Sf(\vec{\wp}_4) = 0.57990, Sf(\vec{\wp}_5) = 0.58331$	$\vec{\wp}_2 > \vec{\wp}_1 > \vec{\wp}_5 > \vec{\wp}_4 > \vec{\wp}_3$
SFGWMSM [50] (suppose $k = 2$, $\lambda_1 = 0.5, \lambda_2 = 0.5$)	$Sf(\vec{\wp}_1) = 0.55914, Sf(\vec{\wp}_2) = 0.56617, Sf(\vec{\wp}_3) = 0.55007, Sf(\vec{\wp}_4) = 0.56056, Sf(\vec{\wp}_5) = 0.55445.$	$\vec{\wp}_2 > \vec{\wp}_4 > \vec{\wp}_1 > \vec{\wp}_5 > \vec{\wp}_3$
SFWPPMSM (suppose $g_1 = 3$, $g_2 = 2$)	$Sf(\vec{\wp}_1) = 0.62361, Sf(\vec{\wp}_2) = 0.60650, Sf(\vec{\wp}_3) = 0.61057, Sf(\vec{\wp}_4) = 0.60754, Sf(\vec{\wp}_5) = 0.60711$	$\vec{\wp}_1 > \vec{\wp}_3 > \vec{\wp}_4 > \vec{\wp}_5 > \vec{\wp}_2$

3. During the evaluation procedure, SFWBM can reflect the correlation of any two attributes, and SFGMSM can consider some correlations of different attributes. However, the presented approach can not only capture the interrelationships among any multiple attributes in the same part, but also minish the impact of extreme values on results, so the presented approach is more effective than the methods of Farrokhizadeh, Seyfi Shishavan, Donyatalab, Kutlu Gündoğdu and Kahraman [39] and Liu, Zhu and Wang [50] for dealing with uncertain problems.

In addition, the main characteristics of the presented method and the existing methods are displayed in Table 16.

Table 16: Comparison of the characteristics of different methods

Methods	Whether can capture inter-relationship of two attributes	Whether can capture inter-relationships of among attributes	Whether can reduce the impact of extreme data	Whether can aggregate information flexibly	Whether can capture the interrelation-ships among any attributes in the same part
SFNWAA [32]	No	No	No	No	No
SFNWGA [32]	No	No	No	No	No
SWAM [33]	No	No	No	No	No
SFPWA [49]	No	No	Yes	No	No
SFPWG [49]	No	No	Yes	No	No
SFWBM [39]	Yes	No	No	Yes	No
SFGWMSM [50]	Yes	Yes	No	Yes	No
SFWPPMSM	Yes	Yes	Yes	Yes	Yes

7. Conclusions

In this article, we propose a novel spherical fuzzy MAGDM method based on SFWPPMSM operator. Firstly, we extend PMSM operator to SFSs and develop SFPMSM as well as SFWPMSM operators. In the meantime, some desirable properties and special cases of these two operators are investigated. Considering the advantage of PA operator, we integrate PA operator and PMSM operator under SFSs to further develop SFPPMSM operator as well as SFWPPMSM operator and investigate their corresponding properties and some special cases. Then a new MAGDM method on the basis of SFWPPMSM operator is proposed. Finally, the feasibility and superiority of the presented approach are proved by comparing existing methods. The proposed method can not only effectively reduce the impact of negative data on assessment results by calculating the support degree between arguments but also reflect interrelationships among any multiple attributes in the same partition, so the established approach is more comprehensive and rational to MAGDM issues.

However, there are some limitations for the proposed method. On one hand, this paper only considers the case where the attribute values are SFNs, but many complex decision environments may appear in the actual decision making. Therefore, in the subsequent research, we will expand the proposed method to other fuzzy environments to solve more uncertain problems. On the other hand, the proposed method in this paper is developed from algebraic t-norm and algebraic t-conorm on the basis of the assumption that MD, AD and N-MD are independent of each other. However, there is a certain interaction between membership degrees in many practical problems. In addition, Dombi t-norm and Dombi t-conorm, Hamacher t-norm and Hamacher t-conorm are better than algebraic t-norm and algebraic t-conorm in information fusion, because they can adjust parameters according to DMs' preference, which make the decision-making process more flexible. And Hamacher t-norm and Hamacher t-conorm are effective extensions of algebraic t-norm and algebraic t-conorm. Therefore, based on the research in this paper, we will consider more complex decision scenarios in a follow-up study and try to develop some better operators by combining the proposed AOs in this paper and interactive algorithm, Dombi t-norm and Dombi t-conorm, Hamache t-norm and Hamache t-conorm to solve practical problems more effectively.

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