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Algorithm for solution of systems of singularly perturbed differential equations with a differential turning point

Valentyn SOBCHUK, **Iryna ZELENSKA**, and **Oleksandr LAPTIEV***

Taras Shevchenko National University of Kyiv, Ukraine

Abstract. The dynamic development of science requires constant improvement of approaches to modeling physical processes and phenomena. Practically all scientific problems can be described by systems of differential equations. Many scientific problems are described by systems of differential equations of a special class, which belong to the group of so-called singularly perturbed differential equations. Mathematical models of processes described by such differential equations contain a small parameter near the highest derivatives, and it was the presence of this small factor that led to the creation of a large mathematical theory. The work proposes a developed algorithm for constructing uniform asymptotics of solutions to systems of singularly perturbed differential equations.

Key words: small parameter; turning point; singular perturbations, asymptotics; Airy–Langer functions.

1. INTRODUCTION

The dynamic development of natural science in recent scientific history requires constant improvement of approaches to modeling physical processes and phenomena. Many problems of astronomy, quantum mechanics, solid medium mechanics, problems of hydrodynamic stability, problems with a classical oscillator, and optimization problems proposed by the Neuro-Tabu Search algorithm [1], etc. are described by systems of differential equations of a special class, which belong to the group of so-called singularly perturbed differential equations (SSPDE).

Mathematical models of processes described by such differential equations contain a small parameter near the highest derivatives, and it was the presence of this small factor that led to the creation of a great theory. An approach in which formally setting a small parameter equal to zero is unacceptable. After all, such a simplification of the mathematical model will lead to the study of a mathematical formalism that does not characterize the subtle properties of the initial system. Therefore, there is a need to develop methods for building solutions for such systems that fully take into account the peculiarities of the behavior of a real evolutionary system, the mathematical description of which contains a small parameter. It is also worth noting that such systems are often characterized by wave phenomena, where the action of an incoming and reflected wave occurs simultaneously at a certain point – such points are called turning points.

A wide range of scientific works is dedicated to the study and research of the issue of constructing a solution to problems with a turning point.

The foundations of this direction of research were laid in the works of Liouville and Green.

The paper presents the interpretation of fractional calculus for positive and negative orders of functions based on sample-measured values and their errors associated with digital signal processing. The question of constructing uniform asymptotics of the solution of systems of singularly perturbed differential equations with a differential turning point is touched upon, but not fully considered [2].

Currently, the Liouville equation and differential equations of the third and fourth orders with an algebraic turning point are well studied [3–5]. V. Vazov obtained results in the study of a system of differential equations of the second order with an algebraic turning point [6]. For such problems, there are several methods (Langer's method, Vazov's method, Dorodnitsyn's method of reference equations, Lomov's regularization method, and the method of essentially singular functions) for constructing a uniform asymptotic solution, which is also suitable at the turning point itself [3]. The generalization of the results to singularly perturbed differential equations with a differential turning point, and even more so to SSPDE of higher orders, leads to complex technical explanations. In their works, Lin, Nakano, and Nishimoto studied the solutions of the Orr–Sommerfeld equation [7]. Their studies concerned partial cases and were not generalized to the entire class of equations of this type [6, 8, 9]. In [10], the inverse problem of the synthesis of parameters of differential systems with a finite number of features and turning points of arbitrary order was studied. In [11], a differential system with a finite number of singularities and turning points of an arbitrary order was studied, the properties of spectral characteristics were determined, and the asymptotic forms of the solution of the system were found. In [12], a system of singularly perturbed differential equations of the first order with a zero characteristic number was studied, the existence

*e-mail: olaptiev@knu.ua

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of an exact solution with a step contrast structure was proved, and its uniform asymptotic expansion was constructed.

Singular boundary conditions for non-self-adjoint Sturm-Liouville operators with singularities and turning points are formulated in [13]. Here, the properties of the spectrum are also investigated and the completeness of the system of root functions for boundary value problems with singular boundary conditions is proved. By considering two configurations, one of which uses the short-wave limiting approximation, and the other the long-wave limiting approximation, in [14] asymptotic eigenvalues and eigenfunctions for the Orr–Sommerfeld equation in two-dimensional and three-dimensional incompressible fluid flows in the infinite domain and semi-infinite domain were obtained.

Several works by A.M. Samoilenko and I.H. Klyuchnyk proposed an asymptotic method of integrating systems of differential equations with simple and multiple turning points [12]. Later P.F. Samusenko proposed an algorithm for finding asymptotic solutions for a singularly perturbed differential-algebraic system with a simple turning point [13].

For a system of differential equations with a small parameter at the highest passing point and a turning point, a uniform asymptotic solution was constructed [14]. Orr–Sommerfeld is currently not received. Therefore, the scientific task of solving systems of singularly perturbed differential equations with a differential turning point is relevant.

The purpose of this paper is to construct uniform asymptotics of the solution of a system of equations on a certain segment provided that this system has a turning point.

2. REGULARIZATION OF THE SYSTEM OF SINGULARLY PERTURBED EQUATIONS

Problems with turning points, which are reduced to systems of linear equations of the form: $\varepsilon \cdot x' = A(\xi,\varepsilon)\cdot x$, $\xi \in R$, which depend singularly on a small parameter ε, are classic complex problems of asymptotic analysis. Without going into details, turning points can be characterized as points $\xi = 0$, at which approximations using regular asymptotic expansions break down. A well-known example is the analysis of the eigenvalue problem for the one-dimensional Schrödinger equation. Note that the interest of researchers in this field in recent years is explained by the fact that this theory is important in applied mathematics, where the problem of turning points arises very often in problems that are mathematical models of various processes.

Consider a singularly perturbed differential equation

$$\varepsilon^2 y''' + xa(x)y' + b(x)y = h(x), \tag{1}$$

where ε is a small parameter, $a(x)$ and $b(x)$ is an unbounded differentiable function on the segment $[-l,0]$.

Let us write equation (1) in the form of a system. For this, we will introduce new variables

$$y = y_1 \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y_1}{\mathrm{d}x} = y_2\,,$$
$$\varepsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \varepsilon y_1'' = \varepsilon y_2' = y_3\,,$$
$$\varepsilon^2 y''' = \varepsilon\left(\varepsilon y'''\right) = \varepsilon\left(\varepsilon y_1'''\right) = \varepsilon\left(y_2''\right) = \varepsilon\left(\varepsilon y_1'\right),$$

$$\begin{cases} \varepsilon y_1' - \varepsilon y_2 = 0, \\ \varepsilon y_2' - y_3 = 0, \\ \varepsilon y_3' + xa(x)y_2 + b(x)y_1 = 0. \end{cases}$$

Then, we have a system of singularly perturbed differential equations (SPDE)

$$\varepsilon Y' - A(x)Y(x) = H(x), \tag{2}$$

where $\varepsilon \to +0$, $x \in [0;l]$, where $A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b & -a & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b = b(x)$, $a = x\cdot a(x)$.

$A_0(x) + \varepsilon A_1(x) = \begin{pmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & 1 \\ -b(x) & -a(x) & 0 \end{pmatrix}$ is a known matrix, $Y(x,\varepsilon) = \begin{pmatrix} y_1(x,\varepsilon) \\ y_2(x,\varepsilon) \\ y_3(x,\varepsilon) \end{pmatrix}$ is the desired vector function, and $H(x) = colomn\,(h_1(x), h_2(x), h_3(x))$ is the given vector function. The purpose of this paper is to construct the uniform solution asymptotics (UnSA) of system (2) on the segment $[-l,0]$ provided there is a turning point in this system $x = 0$.

To construct the UnSA system (2), we apply the modified method of essentially singular functions, which was developed for the Liouville equation and SPDE systems. In order to select all significantly singular functions (SSF) that arise in the solution of system (2) due to the singular point $\varepsilon = 0$, we introduce a regularizing variable $t = \varepsilon^{-p}\cdot\phi(x)$, where the indicator p and the function $\phi(x)$ must be defined.

To determine the "extended" function, we obtain an "extended" vector equation

$$L_\varepsilon \tilde{Y}(x,t,\varepsilon) \equiv \varepsilon^{1-p}\phi'(x)\frac{\partial\tilde{y}(x,t,\varepsilon)}{\partial x} + \varepsilon\frac{\partial\tilde{y}(x,t,\varepsilon)}{\partial x} - A(x)\tilde{Y}(x,t,\varepsilon) = H(x). \tag{3}$$

We construct the asymptotics of the solution of the extended equation (3) in the form of a series

$$\tilde{Y}(x,t,\varepsilon) = \sum_{k=1}^{2}\left[\alpha_k(x,\varepsilon)U_i(t) + \varepsilon^\gamma\beta_k(x,\varepsilon)U_i'(t)\right] + f_k(x,\varepsilon)\nu(t) + \varepsilon^\gamma g_k(x,\varepsilon)\nu'(t) + \omega_k(x,\varepsilon),$$

where $\theta(x,\varepsilon) \equiv \{\alpha_k(x,\varepsilon), \beta_k(x,\varepsilon), f_k(x,\varepsilon), g_k(x,\varepsilon), \omega_k(x,\varepsilon)\}$ is analytical functions of a relatively small parameter $\varepsilon > 0$ and infinitely differentiable in a variable $x \in [-l,0]$, $U_i(t)$, $i = 1,2$ is Airy–Langer functions, i.e. $Ai(t)$ and $Bi(t)$.

A significantly special function $v(t)$ is the solution to the following problem

$$v''(t) - v(t) = \pi^{-1},$$
$$v(0) = 3^{-2/3}\Gamma\left(\frac{1}{3}\right), \quad v'(0) = -3^{-1/3}\Gamma\left(\frac{2}{3}\right).$$

Let us determine the complete derivative in terms of a variable x from the vector function $\tilde{Y}(x,t,\varepsilon)$ and substitute its value in the extended vector equation.

$$\tilde{Y}(x,t,\varepsilon) = \alpha(x,\varepsilon)\cdot U_i(t) + \varepsilon^{\gamma}\beta(x,\varepsilon)\cdot U_i'(t),$$
$$\tilde{Y}(x,t) = \begin{pmatrix} \varepsilon^{s_1}\alpha_1(x,\varepsilon) \\ \varepsilon^{s_2}\alpha_2(x,\varepsilon) \\ \varepsilon^{s_3}\alpha_3(x,\varepsilon) \end{pmatrix} U_i(t) + \varepsilon^{\gamma} \begin{pmatrix} \varepsilon^{k_1}\beta_1(x,\varepsilon) \\ \varepsilon^{k_2}\beta_2(x,\varepsilon) \\ \varepsilon^{k_3}\beta_3(x,\varepsilon) \end{pmatrix} U_i'(t),$$

$$\begin{aligned}
&\tilde{L}_{\varepsilon}\left(\alpha(x,\varepsilon)U(t) + \varepsilon^{\gamma}\beta(x,\varepsilon)U'(t)\right) \\
&\quad = \varepsilon^{1-\mu}\alpha(x,\varepsilon)\phi'(x)U'(t) \\
&\qquad + \varepsilon^{1-2\mu}\beta(x,\varepsilon)\phi'(x)\phi(x)U(t) \\
&\qquad - A(x)\alpha(x,\varepsilon)U(t) - A(x)\beta(x,\varepsilon)U'(t) \\
&\qquad + \varepsilon\alpha'(x,\varepsilon)U(t) + \varepsilon\beta(x,\varepsilon)U'(t) = H(x).
\end{aligned} \tag{4}$$

We equate the coefficients for SSF in the resulting system. We will have the following vector equations, $i = 1,2$; $k = \overline{1,3}$.

$$\begin{aligned}
U_i'(t): &\quad \varepsilon^{1-\mu}\alpha(x)\phi'(x) - A(x)\varepsilon^{\gamma}\beta(x) + \varepsilon\beta(x) = 0, \\
U_i(t): &\quad \varepsilon^{1+\gamma-2\mu}\beta(x)\phi(x)\phi'(x) - A(x)\alpha(x) + \varepsilon\alpha(x) = 0.
\end{aligned} \tag{5}$$

Let us write these vector equations in the form of an algebraic system of equations

$$\begin{cases}
\varepsilon^{1-p+s_1}\alpha_1(x)\phi'(x) = 0, \\
\varepsilon^{1-p+s_2}\alpha_2(x)\phi'(x) - \varepsilon^{\gamma+k_3}\beta_3(x) = 0, \\
\varepsilon^{1-p+s_3}\phi'(x) + b\varepsilon^{\gamma+k_1}\beta_1(x) + a\varepsilon^{\gamma+k_2}\beta_2(x) = 0, \\
\varepsilon^{1+\gamma-2p+k_1}\beta_1(x)\phi(x)\phi'(x) = 0, \\
\varepsilon^{1+\gamma-2p+k_2}\beta_2(x)\phi(x)\phi'(x) - \varepsilon^{s_3}\alpha_3(x) = 0, \\
\varepsilon^{1+\gamma-2p+k_3}\beta_3(x)\phi(x)\phi'(x) + b\varepsilon^{s_1}\alpha_1(x) \\
\qquad + a\varepsilon^{s_2}\alpha_2(x) = 0.
\end{cases} \tag{6}$$

We will require that the resulting algebraic systems of equations (6) are regularly perturbed with respect to a small parameter $\varepsilon > 0$. To do this, we equate the so far undefined exponents of the degree k_1, k_2, k_3, s_1, s_2, s_3, p and γ in each equation and we obtain the following system of power exponents

$$\begin{cases}
1 - p + s_1 = 0, \\
1 - p + s_2 = \gamma + k_3, \\
1 - p + s_3 = \gamma + k_1 = \gamma + k_2, \\
1 + \gamma - 2p + k_1 = 0, \\
1 + \gamma - 2p + k_2 = s_3 \\
1 + \gamma - 2p + k_3 = s_1 = s_2.
\end{cases} \tag{7}$$

To solve this system, we get a result

$$p = \frac{2}{3}, \quad \gamma = \frac{1}{3}, \quad k_1 = k_2 = s_3 = 0, \quad s_1 = s_2 = k_3 = -\frac{1}{3}.$$

Proposition 1. For the existence of solutions for system (4), it is necessary that the condition be fulfilled

$$p = \frac{2}{3}, \quad \gamma = \frac{1}{3}.$$

Taking (7) into account, vector equations (6) can be written in the form

$$\begin{cases}
\phi'(x)\alpha_{i1}(x,\varepsilon) = \mu^3\left[\beta_{i2}(x,\varepsilon) - \beta_{i1}'(x,\varepsilon)\right], \\
\phi'(x)\alpha_{i2}(x,\varepsilon) - \beta_{i3}(x,\varepsilon) = -\mu^3\beta_{i2}'(x,\varepsilon), \\
\phi'(x)\alpha_{i3}(x,\varepsilon) + b(x)\beta_{i1}(x,\varepsilon) + a(x)\beta_{i2}(x,\varepsilon) \\
\qquad = -\mu^3\beta_{i3}'(x,\varepsilon), \\
\phi(x)\phi'(x)\beta_{i1}(x,\varepsilon) = \mu^3\left[\alpha_{i1}'(x,\varepsilon) - \alpha_{i2}(x,\varepsilon)\right], \\
\phi(x)\phi'(x)\beta_{i2}(x,\varepsilon) + \alpha_{i3}(x,\varepsilon) = \mu^3\alpha_{i2}'(x,\varepsilon), \\
\phi(x)\phi'(x)\beta_{i3}(x,\varepsilon) - b(x)\alpha_{i1}(x,\varepsilon) - a(x)\alpha_{i2}(x,\varepsilon) \\
\qquad = \mu^3\alpha_{i3}'(x,\varepsilon),
\end{cases} \tag{8}$$

where $\mu = \varepsilon^{\frac{1}{3}}$.

After substituting the obtained indicators in (6), a regularly disturbed system was finally obtained, which is described by expression (8).

3. CONSTRUCTION OF FORMAL SOLUTIONS OF A HOMOGENEOUS EXTENDED SYSTEM

We construct solutions of the homogeneous extended system (8) in the form of a series

$$\alpha_{ik}(x,\varepsilon) = \sum_{r=0}^{+\infty}\varepsilon^r\alpha_{ikr}(x), \quad \beta_{ik}(x,\varepsilon) = \sum_{r=0}^{+\infty}\varepsilon^r\beta_{ikr}(x). \tag{9}$$

Let us substitute rows (9) in (8). Then to define the vector functions

$$\alpha_{ikr}(x) = colomn\left(\alpha_{i1r}(x), \alpha_{i2r}(x), \alpha_{i3r}(x)\right),$$
$$\beta_{ikr}(x) = colomn\left(\beta_{i1r}(x), \beta_{i2r}(x), \beta_{i3r}(x)\right)$$

we obtain the following recurrent systems of equations

$$\Phi(x)Z_{ik0}(x) = 0, \quad \Phi(x)Z_{ikr}(x) = -Z_{ik(r-1)}'(x), \quad r \geq 1, \tag{10}$$

where $Z_{ikr}(x) = colomn \begin{pmatrix} \alpha_{i1r}(x), \alpha_{i2r}(x), \alpha_{i3r}(x), \\ \beta_{i1r}(x), \beta_{i2r}(x), \beta_{i3r}(x) \end{pmatrix}$,

$$\Phi(x) = \begin{pmatrix} \phi'(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi'(x) & 0 & 0 & 0 & -1 \\ 0 & 0 & \phi'(x) & b & a & 0 \\ 0 & 0 & 0 & \phi(x)\phi'(x) & 0 & 0 \\ 0 & 0 & -1 & 0 & \phi(x)\phi'(x) & 0 \\ b & a & 0 & 0 & 0 & \phi(x)\phi'(x) \end{pmatrix}. \tag{11}$$

Let us calculate the determinant of matrix (11). As a result, we get

$$\det \varphi(x) = \left(a^2(x) + 2a(x)\phi(x)\phi'^2(x) + \phi^2(x)\phi'^4(x)\right)\phi(x)\phi'^2(x).$$

Currently, the regularizing function $\phi(x)$ is not defined. Therefore, we define it as a solution to the problem

$$a^2(x) + 2a(x)\phi(x)\phi'^2(x) + \phi^2(x)\phi'^4(x) = 0,$$
$$\left(a(x) + \phi(x)\phi'^2(x)\right)^2 = 0,$$
$$\phi(x)\phi'^2(x) = -a(x).$$

So,

$$\left[\phi'(x)\right]^2 \phi(x) = -a \equiv \det A(x), \quad \phi(0) = 0. \tag{12}$$

Before continuing the research and writing the solution to problem (8), it is necessary to understand the sign on the right side of the equation. When the condition is met $\det A(x) = -a(x) > 0$ the solution will be a function

$$\phi(x) \cdot \phi(x)'^2 = -a(x),$$
$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 \cdot \phi(x) = -a(x),$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot \sqrt{\phi(x)} = \sqrt{-a(x)},$$
$$\int \phi^{1/2}(x)\,\mathrm{d}\phi = \int_0^x \sqrt{-a(x)}\,\mathrm{d}x.$$
$$\frac{2}{3}\phi^{3/2}(x) = \int_0^x \sqrt{-a(x)}\,\mathrm{d}x,$$
$$\phi^{3/2}(x) = \frac{3}{2}\int_0^x \sqrt{-a(x)}\,\mathrm{d}x,$$
$$\phi(x) = \left(\frac{3}{2}\int_0^x \sqrt{-a(x)}\,\mathrm{d}x\right)^{2/3}.$$

Because $\det \Phi(x) \equiv 0$, then there exists a non-trivial solution of system (6) of the form

$$Z_{k0}(x) = colomn \begin{pmatrix} 0, & \frac{1}{\phi'(x)}\beta_{i30}(x), & -\frac{a(x)}{\phi'(x)}\beta_{i20}(x), \\ 0, & \beta_{i20}(x), & \beta_{i30}(x) \end{pmatrix}, \tag{13}$$

where $\beta_{ik0}(x)$, $i = 1,2$, $k = \overline{1,3}$ is up to a certain point, free, sufficiently smooth functions at $x \in [-0; l]$.

From the obvious appearance of matrix (11) and solution (13), it is clear that system (8) can be divided into two systems

$$\begin{cases} \phi'(x)\alpha_{i2r}(x) - \beta_{i3r}(x) = \alpha'_{i2(r-1)}(x), \\ a(x)\alpha_{i2r}(x) + \phi(x)\phi'(x)\beta_{i3r}(x) = \beta_{i3(r-1)}(x) \end{cases} \tag{14}$$

and

$$\begin{cases} \phi'(x)\alpha_{i3r}(x) + a(x)\beta_{i2r}(x) = \alpha'_{i3(r-1)}(x), \\ -\alpha_{i3r}(x) + \phi(x)\phi'(x)\beta'_{i2r} = \beta'_{i2(r-1)}. \end{cases} \tag{15}$$

These systems are independent of each other. But when constructing the asymptotics of the solution of the extended equation, they participate simultaneously at each iterative step.

Let us solve the given system. First, consider these systems for $r = 1$. Taking into account the obtained solution (13), we obtain:

$$\begin{cases} \phi'(x)\alpha_{i21}(x) - \beta_{i31}(x) = \alpha'_{i20}(x) \equiv \left(\frac{1}{\phi'}\beta_{i30}\right)', \\ a(x)\alpha_{i21}(x) + \phi(x)\phi'(x)\beta_{i31}(x) = \beta_{i30}(x) \end{cases} \tag{16}$$

and

$$\begin{cases} \phi'(x)\alpha_{i31}(x) + a(x)\beta_{i21}(x) = \left(-\frac{a}{\phi'}\beta_{i20}\right)', \\ -\alpha_{i31}(x) + \phi(x)\phi'(x)\beta'_{i21} = \beta'_{i20}. \end{cases} \tag{17}$$

Regulating function $\phi(x)$ chosen as a solution to problem (12) is chosen as a solution to problem (12). Therefore, the determinant of this system is identically equal to zero, i.e. $\Delta(x) = [\phi'(x)]^2 \phi(x) + a(x) \equiv 0$. Therefore, in the general case, there are no heterogeneous systems of algebraic equations (16) and (17). Therefore, it is necessary to investigate in more detail the right parts of these systems. According to the Kronecker-Capelli theorem, for a non-trivial solution of systems (16) and (17) to exist, it is necessary and sufficient that the ranks of the extended matrices coincide with the corresponding ranks of the matrices of these systems. To fulfill these conditions, we use the arbitrariness of the functions $\beta_{ik0}(x)$

$$\frac{\phi'(x)}{a(x)} \equiv \frac{1}{\phi(x)\phi'(x)} = \frac{\left(\frac{1}{\phi'(x)}\beta_{i30}(x)\right)'}{\beta'_{i30}(x)}, \quad i = 1,2, \tag{18}$$

$$\frac{\phi'(x)}{-1} \equiv \frac{a(x)}{\phi(x)\phi'(x)} = \frac{\left(-\frac{a(x)}{\phi'(x)}\beta_{i20}(x)\right)'}{\beta'_{i20}(x)}, \quad i = 1,2. \tag{19}$$

From (18) with fixed $i = 1,2$, we obtain the differential equation

$$(1 - \phi(x))\,\beta'_{i30}(x) - \frac{a(x)}{\phi'(x)} \cdot \left(\frac{1}{\phi'(x)}\right)' \beta_{i30}(x) = 0. \tag{20}$$

From (19) with fixed $i = 1,2$, we obtain the differential equation

$$(\phi(x)-1)\,\beta'_{i20} - \frac{1}{\phi'(x)}\left(-\frac{a(x)}{\phi'(x)}\right)'\beta_{i20}\cdot\phi'(x) = 0. \tag{21}$$

In (20) and (21) the functions will be smooth solutions $\beta_{ik0}(x) = \beta^0_{ik0}\cdot\tilde{\beta}_{ik0}(x)$, $k=\overline{1,3}$, $i=1,2$, where $\beta^0_{ik0}(x)$ is arbitrary steels, $\tilde{\beta}_{ik0}(x)$ is partial, smooth enough for everyone $x\in[-l;0]$, solutions of homogeneous equations (11).

By analogy, smooth solutions will be functions $\beta_{i10}(x) = \beta^0_{i10}\cdot\tilde{\beta}_{i10}(x)$, $i=1,2$, where $\beta^0_{1k0}(x)$ is arbitrary steels, $\tilde{\beta}_{1k0}(x)$ is partial, smooth enough for everyone $x\in[-l;0]$, solutions of homogeneous equations (14).

With this definition of vector functions $Z_{k0}(x)$ there are solutions of heterogeneous systems of algebraic equations (20) and (21) of the form

$$Z_{kr}(x) = colomn\left(\begin{array}{l} 0,\ \dfrac{\beta_{i30}(x)-\alpha'_{i2(r-1)}(x)}{\phi'(x)},\\ \dfrac{-a(x)\beta_{i20}(x)-\alpha''_{i3(r-1)}(x)}{\phi'(x)},\\ 0,\ \beta_{i20}(x),\ \beta_{i30}(x)\end{array}\right), \tag{22}$$

where $\beta_{ikr}(x)$, $i=1,2$, $k=\overline{1,3}$ are arbitrary until a certain time, sufficiently smooth for all $x\in[-l;0]$.

Continuing to solve the iterative systems of algebraic equations (16) and (17) for $r>1$, it is possible to show by the method of mathematical induction that these systems of equations are asymptotically correct in this sense. If the shunting of the solutions of the systems of equations (16) and (17) at $r=\overline{0;q}$, then each of these systems for $r=\overline{0;q-1}$, is determined with an accuracy of two arbitrary scalar factors $\beta^0_{ik0}(x)$, which form an arbitrary vector $\beta^0_{ikr} = colomn\left(\beta^0_{i2r},\beta^0_{i3r}\right)$.

4. CONSTRUCTION OF PARTIAL SOLUTIONS OF A HETEROGENEOUS EXTENDED SYSTEM

The construction of the asymptotics of partial solutions of the inhomogeneous extended equation (2) is provided by the following systems

$$\begin{aligned} \nu'(t):\ & \phi'(x)f_k(x,\varepsilon)-\left[A_0(x)+\mu^3A_1\right]\\ &\times g_k(x,\varepsilon) = -\mu^3 g'_k(x,\varepsilon),\\ \nu(t):\ & \phi(x)\phi'(x)g_k(x,\varepsilon)\phi(x)\phi'(x)\\ &-\left[A_0(x)+\mu^3A_1\right]f_k(x,\varepsilon) = -\mu^3 f'(x,\varepsilon), \end{aligned} \tag{23}$$

$$\varepsilon\omega'_k(x,\varepsilon)-A(x)\omega_k(x,\varepsilon)+\varepsilon\phi'(x)\cdot g_k(x,\varepsilon)=H(x). \tag{24}$$

Let us first examine system (23). Its structure is similar to system (6). That is, based on the results of the previous point, we could immediately write down the formal solutions of this system in the form of power series of a small parameter $\varepsilon>0$. However, in this case, we would not obtain the desired solutions of the recurrent system (24), which we will write in the form ($\mu=\varepsilon^{1/3}$) taking into account uniquely defined indicators

$$-A(x)\omega(x,\varepsilon)+\varepsilon^2\phi'(x)\cdot g(x,\varepsilon)+\mu^3\omega'(x,\varepsilon)=H(x). \tag{25}$$

From the above, to ensure the existence of a smooth solution of system (25), we construct the asymptotics of the solution of system (23) in the form of the series

$$f_k(x,\varepsilon)=\sum_{r=-2}^{+\infty}\mu^r f_{kr}(x),\quad g(x,\varepsilon)=\sum_{r=-2}^{+\infty}\mu^r g_{kr}(x). \tag{26}$$

To define vector functions $f_{kr}(x)$ and $g_{kr}(x)$ we obtain recurrent systems of equations

$$\begin{aligned} &\Phi(x)Y^{\text{part}}_{kr}(x)=0,\quad r=-2;-1;0,\\ &\Phi(x)Y^{\text{part}}_{kr}(x)=-Y^{part'}_{k(r-3)}(x),\quad r\ge 1. \end{aligned} \tag{27}$$

where $\Phi(x)$ is matrix (12) and
$Y^{\text{part}}_r(x) = colomn\left(f_{1r}(x),f_{2r}(x),f_{3r}(x),g_{1r}(x),g_{2r}(x),g_{3r}(x)\right)$ is unknown vector function.

Since $\det\Phi(x)\equiv 0$, then there exists a non-trivial solution of homogeneous systems (23) for $r=-2,-1,0$, in the form

$$Y^{\text{part}}_r(x)=colomn\left(\begin{array}{c} 0,\ \dfrac{1}{\phi'(x)}g_{3r}(x),\\ -\dfrac{a}{\phi'(x)}g_{2r}(x),0,g_{2r}(x),g_{3r}(x)\end{array}\right), \tag{28}$$

where $g_{ir}(x)$ is until a certain time is free, sufficiently smooth functions at $x\in[-l,0]$.

To determine the functions $f_r(x)$ and $g_r(x)$ at $r\ge 1$ we obtain the following two recurrent systems of equations

$$\begin{cases}\phi'(x)f_{2r}(x)-g_{3r}(x)=f'_{2(r-3)}(x),\\ a(x)f_{2r}(x)+\phi(x)\phi'(x)g_{3r}(x)=g_{3(r-3)}(x),\end{cases} \tag{29}$$

and

$$\begin{cases}\phi'(x)f_{3r}(x)+a(x)g_{2r}(x)=f'_{3(r-3)}(x),\\ -f_{3kr}(x)+\phi(x)\phi'(x)g'_{2r}=g'_{2(r-3)}.\end{cases} \tag{30}$$

The determinants of these systems are identically equal to zero. Therefore, we will again conduct a study of the right parts of these systems, taking into account previous studies.

Then there are solutions of inhomogeneous systems of equations (29) and (30) at $r=\overline{1,3}$ appearance

$$Y^{\text{part}}_r = colomn\left(\begin{array}{l} 0,\ \dfrac{g_{3r}(x)-f'_{2(r-3)}(x)}{\phi'(x)},\\ \dfrac{-a(x)g_{2r}(x)-f'_{3(r-3)}(x)}{\phi'(x)},\ 0,\ g_{2r}(x),g_{3r}(x)\end{array}\right).$$

Continuing to solve the iterative systems of algebraic equations (29) and (30) at $r>3$, it is possible to show by the method of mathematical induction that the systems of equations (29) and (30) are asymptotically correct.

Thus, it remains for us to investigate the solutions of system (2). We construct the asymptotics of the solution of this system in the form of a series

$$\omega_k(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \omega_{kr}(x). \tag{31}$$

To define vector functions $\omega_{kr}(x)$, we obtain recurrent systems of equations

$$\begin{aligned} -A(x)\omega_{k0}(x) &= H(x) - \phi'(x) g_{k(-2)}(x), \\ -A(x)\omega_{kr}(x) &= -\phi'(x) g_{k(r-2)}(x), \quad r = 1;2, \\ -A(x)\omega_{kr}(x) &= -\phi'(x) g_{k(r-2)}(x) - \omega'_{k(r-3)}(x), \quad r \geq 3, \end{aligned} \tag{32}$$

where $\omega_{kr}(x) = colomn\,(\omega_{1r}(x), \omega_{2r}(x), \omega_{2r}(x))$ is the unknown vector function.

Let us examine equation (32) at $r = 0$. To do this, we first calculate its right-hand side. To do this, we first calculate its right-hand side. For clarity, we will write it in scalar form. We will have

$$\begin{cases} 0 = h_1(x) - \phi'(x) g_{1(-2)}(x) \\ \qquad \equiv h_1(x) - \phi'(x) \times g^0_{1(-2)} \tilde{g}_{1(-2)}(x), \\ -\omega_{30}(x) = h_2(x) - \phi'(x) g_{2(-2)}(x) \\ \qquad \equiv h_2(x) - \phi'(x) g^0_{2(-2)} \tilde{g}_{2(-2)}(x), \\ b\omega_{10}(x) + a\omega_{20}(x) = h_3(x) - \phi'(x) g_{3(-2)} \\ \qquad \equiv h_3(x) - \phi'(x) g^0_{3(-2)} \tilde{g}_{3(-2)}(x). \end{cases} \tag{33}$$

In this case, we will have the following solutions for the system. Provided that $\omega_{10}(x) \equiv 0$ and $h_1(x) \equiv 0$, let us define $\omega_{20}(x)$ and $\omega_{30}(x)$

$$\begin{aligned} \omega_{20}(x) &= \frac{h_3(x) - \phi'(x) g_{3(-2)}(x)}{a}, \\ \omega_{30}(x) &= \phi'(x) g_{2(-2)}(x) - h_2(x). \end{aligned}$$

Let us study system (33) for $r = 1,2$. Provided that $\omega_{10}(x) \equiv 0$, we will have

$$\omega_{2r}(x) = \frac{-\phi'(x) g_{3r}(x)}{a}, \qquad \omega_{3r}(x) = \phi'(x) g_{2r}(x).$$

For $r \geq 3$ in an expanded form, we will have a system

$$\begin{cases} 0 = -\phi'(x) g^0_{1(r-2)} \tilde{g}_{1(r-2)}(x) - \omega'_{1(r-3)}(x), \\ -\omega_{30}(x) = -\phi'(x) g^0_{2(r-2)} \tilde{g}_{2(r-2)}(x) - \omega'_{2(r-3)}(x), \\ b\omega_{10}(x) + a\omega_{20}(x) = -\phi'(x) g^0_{3(r-2)} \\ \qquad \times \tilde{g}_{3(r-2)}(x) - \omega'_{3(r-3)}(x). \end{cases} \tag{34}$$

System (33) will have sufficiently smooth solutions if arbitrary constants are taken as

$$g^0_{1(r-2)} = \frac{-\omega'_{1(r-3)}(0)}{\phi'(0)\tilde{g}_{1(r-2)}(0)}, \qquad g^0_{1(r-2)} = 0, \quad r \geq 3.$$

Then for $r \geq 3$ we obtain smooth solutions

$$\begin{aligned} \omega_{1r}(x) &\equiv 0, \\ \omega_{2r}(x) &= -\frac{\phi'(x) g_{3(r-2)}(x) + \omega'_{3(r-3)}(x)}{a}, \\ \omega_{3r}(x) &= \phi'(x) g_{2(r-2)}(x) + \omega'_{2(r-3)}(x). \end{aligned} \tag{35}$$

The solution of the extended equation (2) is constructed in the form of formal series (10) and (26) at $t = \varepsilon^{-2/3} \cdot \phi(x)$, i.e. a series

$$\begin{aligned} &\tilde{Y}(x, \varepsilon^{-2/3} \cdot \phi(x), \varepsilon) \\ &= \sum_{r=0}^{+\infty} \varepsilon^r \times \left[\sum_{k=1}^{2} \left[\begin{matrix} \alpha_{ikr}(x) Ui\left(\varepsilon^{-2/3} \cdot \phi(x)\right) \\ + \varepsilon^{1/3} \beta_{ikr}(x) \dfrac{\mathrm{d}U_i(\varepsilon^{-2/3} \cdot \phi(x))}{\mathrm{d}(\varepsilon^{-2/3} \cdot \phi(x))} \end{matrix} \right] \right] \\ &+ \sum_{r=0}^{+\infty} \varepsilon^r \left[\begin{matrix} f_{kr}(x) \nu\left(\varepsilon^{-2/3} \cdot \phi(x)\right) \\ + \varepsilon^{1/3} g_{kr}(x) \dfrac{\mathrm{d}\nu(\varepsilon^{-2/3} \cdot \phi(x))}{\mathrm{d}(\varepsilon^{-2/3} \cdot \phi(x))} \end{matrix} \right] \\ &+ \sum_{r=0}^{+\infty} \varepsilon^r \omega_{kr}(x). \end{aligned} \tag{36}$$

Theorem 1. Let $a(x)$, $b(x)$, $H(x) \in C^\infty[-l;0]$ and conditions (2) are fulfilled. Then on to the segment $[-l;0]$, it is possible to construct the general solution of the singularly perturbed system of differential equations (2) in the form of an asymptotic series (36), the coefficients of which are fairly smooth functions on the segment $[-l;0]$.

In accordance with the formulated theorem and the described method of constructing uniform asymptotics of the solution for systems of singularly perturbed differential equations with a differential turning point, we will write down the sequence of actions step by step.

Step 1. *An extension of the singularly perturbed problem.*

In a singularly perturbed problem with a turning point next to the independent variable x a new vector variable is introduced $t = \varepsilon^{-p} \cdot \phi(x)$. Then instead of the desired vector function $Y(x,\varepsilon)$ a new "extended vector function" is being studied $\tilde{Y}(x,t,\varepsilon)$. Moreover, the expansion is conducted in such a way that the condition as in the regularization method is fulfilled

$$\tilde{Y}(x,t,\varepsilon)\big|_{t=\varepsilon^{-p}\cdot\phi(x)} \equiv Y(x,\varepsilon).$$

p and $\phi(x)$ determined for each specific case. There is a transition from a problem with one variable to a problem with two variables.

Step 2. *The space of resonance-free solutions.*

For regularization, a specific space of functions is introduced, this space is called the space of resonance-free solutions, and for each specific problem, this space has its own specificity.

$$\tilde{Y}(x,t,\varepsilon) = \sum_{k=1}^{2} D_k(x,t,\varepsilon) + B(x,t,\varepsilon) + \omega(x,\varepsilon),$$

$$D_k(x,t,\varepsilon) = \begin{pmatrix} \alpha_{k1}(x,\varepsilon) \\ \alpha_{k2}(x,\varepsilon) \\ \alpha_{k3}(x,\varepsilon) \end{pmatrix} U_k(t) + \mu \begin{pmatrix} \beta_{k1}(x,\varepsilon) \\ \beta_{k2}(x,\varepsilon) \\ \beta_{k3}(x,\varepsilon) \end{pmatrix} U_k'(t),$$

$$B(x,t,\varepsilon) = \begin{pmatrix} f_1(x,\varepsilon) \\ f_2(x,\varepsilon) \\ f_3(x,\varepsilon) \end{pmatrix} v(t) + \mu \begin{pmatrix} g_1(x,\varepsilon) \\ g_2(x,\varepsilon) \\ g_3(x,\varepsilon) \end{pmatrix} v'(t).$$

Step 3. *Regularization of a singularly perturbed problem.*

The extended problem is studied in a space without resonant solutions and is reduced to an equation in which a small $\varepsilon > 0$ parameter regularly enters.

Step 4. *The formalism of constructing a solution to the problem.*

Since the extended problem is regularly perturbed with respect to a small parameter, we will look for the solution to the problem in the form of a series

$$\tilde{Y}(x,t,\mu) = \sum_{r=-2}^{\infty} \mu^r Y_r(x,\varepsilon),$$

where $\mu = \sqrt[3]{\varepsilon}$.

We start the construction of the asymptotic series with negative powers of a small parameter in order to obtain uniform asymptotics of the solution of the SSPDE. The right part will have a break of the second kind at the turning point. Therefore, in the general case, it will not belong to the set of values of the main extended operator L_ε [15]. By substituting series (36) into problem (3), in order to determine the coefficients of this series, we will obtain a certain system of recurrent equations with point-like initial conditions or kraft conditions.

Step 5. *Construction of formal solutions of a homogeneous extended system.*

The recurrent equations obtained in the previous point for determining the coefficients of the series (36) are partial differential equations with point boundary conditions. We will show that this system of equations is asymptotically correct. At this stage, the theory of the existence of an iterative equation of the form is developed

$$\Phi(x) \cdot Z_{kr}(x) = F \cdot Z_{kr}(x),$$

where $\Phi(x)$ is system matrix (10), $Z_{kr}(x)$ is the column vector is composed of analytic functions $\theta(x,\varepsilon) \equiv \{\alpha_k(x,\varepsilon), \beta_k(x,\varepsilon)\}$. At the same time, the first terms of the asymptotic solution of the homogeneous problem under study are constructed.

Step 6. *Construction of formal solutions of a heterogeneous extended system.*

In this section, the solution of the inhomogeneous problem is constructed using a recurrent equation

$$\Phi(x) \cdot Z_{kr}(x) = F \cdot Z_{kr}(x),$$

where $\Phi(x)$ is system matrix (27), $Z_{kr}(x)$ is the column vector composed of analytic functions $\theta_2 = \{f_k(x,\varepsilon),\ g_k(x,\varepsilon),\ \omega_k(x,\varepsilon)\}$.

5. CONCLUSIONS

An algorithm for constructing the uniform asymptotics of the solution of systems of singularly perturbed differential equations with a differential turning point was developed. These results take into account the peculiarities of the behavior of the evolutionary system, the mathematical description of which contains a small parameter, and may be useful to researchers studying real natural processes that are described using the proposed mathematical apparatus.

In the future, research on the construction of uniform asymptotics for the solution of systems of singularly perturbed differential equations with a differential turning point of other types will be continued and practical applications of this theory will be considered.

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