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SPECIAL SECTION

Some recent developments in inerter-based devices for vibration mitigation

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Abstract. Reducing the effect of unwanted vibrations is an important topic in many engineering applications. In this paper we describe some recent developments in the area of passive vibration mitigation. This is based on a new device called the *inerter* which can be exploited in a range of different contexts. In this paper we consider two recent examples; (i) where a flywheel inerter is combined with a hysteretic damper, and (ii) in which a pivoted bar inerter is developed for a machining application. In both cases, experimental test results show that the devices can outperform existing methods.

Key words: inerter; vibration; mitigation; damping

1. INTRODUCTION

Unwanted vibrations occur in a range of important engineering application areas. For example, in civil engineering, tall buildings and structures can suffer from vibrations caused by earthquakes, tsunamis or strong winds. In the most severe cases, the human and economic consequences can be devastating. One of the most long-standing state-of-the-art techniques engineers can use to guard against this type of problem is called the tuned-mass-damper[1]. It is based on an idea patented by Hermann Frahm in 1909, and has been used extensively in many engineering applications. One of the most well-known examples is the large tuned-mass-damper installed into the Taipai 101 building in Taiwan, which is shown in Fig. 1.

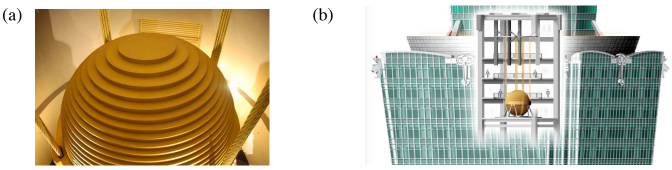

Fig. 1. The tuned-mass-damper (TMD) showing (a) a photograph of the 660-tonne mass from the Taipei 101 tuned-mass-damper, and (b) the mass is suspended on cables, across four storeys at the top of the building acting like a pendulum version of the TMD. A review of TMDs with a list of applications to buildings is reported in Gutierrez and Adeli [1]. Photo credits: Guillaume Paumier

Although many modifications and minor improvements to Frahm's idea have been developed over the past century, nothing fundamentally changed in the field of passive vibration devices until the advent of the *inerter*. The term inerter was first coined by Malcolm Smith [2] to represent a mechanical device that produced an inertia force from a relative acceleration. In fact, these types of devices had previously been known about for a range of other mechanical and civil engineering applications, but by different names – see [3] for a historical review.

In automotive and aerospace applications inerter-like devices have been used primarily as vibration isolators, for example in engine mounts of cars, and helicopters since the 1960s – see [3, 5] for details of this historical context and references. This included both mechanical and fluid based devices (for example hydramounts), which are used to try and minimise the amount of unwanted vibration that is transmitted to a passenger cabin – a technology that is still the state-of-the-art today.

In the early 2000s, Smith and co-workers also developed the inerter concept for automotive applications, particularly suspension systems, with McLaren Formula-1 to great success, and mechanical inerters are now available commercially for performance motorsport [6]. Two of the mechanical inerter concepts designed and tested by Papageorgiou and Smith [4] around this time are shown in Fig. 2.

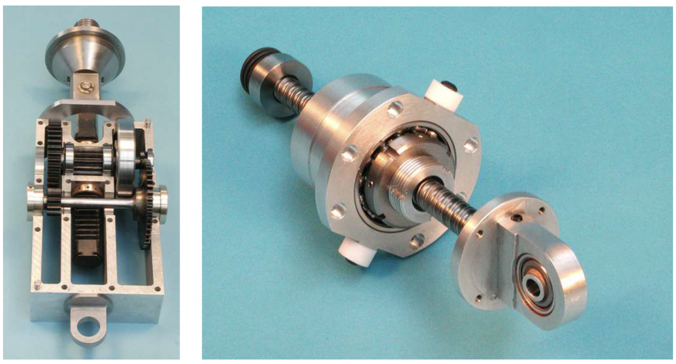

Fig. 2. Mechanical inerter concepts designed by Papageorgiou and Smith [4]. On the left is a rack and pinion inerter, and on the right is a ball-screw inerter. Photo credits: Papageorgiou and Smith [4]

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[1] It can also be called "tuned-vibration-absorber" or "tuned dynamic absorber".

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The development of the inerter concept for vibration suppression applications has been a topic of great interest in recent years. The inerter is typically combined with damper and spring elements to create an inerter device (*ID*) as shown schematically in Fig. 3a. The host structure has mass M (kg), damping c (kg/s) and stiffness k (N/m), and (in this example) is subject to a base motion of $r(t)$ (m). The vibration mitigation problem is to reduce (ideally minimise) the displacement $x(t)$ (m) by using the inerter device *ID*, which in Fig. 3a is placed in parallel to the host system spring and damper.

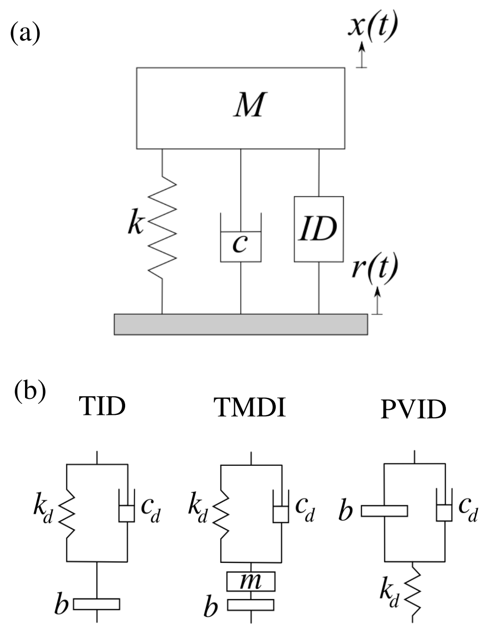

Fig. 3. Tuned inerter device layout variants showing: (a) The host system with mass M, stiffness k, and damping c, and inerter-device denoted *ID*, the base displacement input is $r(t)$ and the system displacement response is $x(t)$. (b) Three variants of inerter device; tuned-inerter-damper (TID); tuned-mass-damper-inerter(TMDI); parallel-viscous-inerter-damper (PVID), also sometimes called a tuned-viscous-mass-damper (TVMD). The inerter element has inertance b, the *ID* has mass m, stiffness k_d and damping c_d

Depending on the exact context, devices can be designed to be vibration isolators or absorbers. Whichever case is required, the design of an inerter device involves choosing an appropriate *arrangement* of elements (inerter, mass, damper and stiffness) and then selecting, or *tuning*, the associated parameter values to give the required vibration mitigation.

Thus far, there are three device configurations that have emerged as the most important for applications, as shown schematically in Fig. 3b. The first to be introduced was the tuned-viscous-mass-damper (TVMD, also called the PVID), described in detail in Ikago *et al.* [7]. The device consists of a parallel-connected inerter-viscous damper in series with a spring element, and at least one version of this device has been put into service in a real structure in Japan [8].

The second type of device is the tuned-inerter-damper (TID) which was proposed by Lazar *et al.* [9]. This device consists of a parallel connected spring and viscous damper in series with an inerter – an arrangement that is similar to that of a tuned-mass-damper with the mass element being replaced by an inerter. The third device proposed is by Marian and Giaralis [10] and is called a tuned-mass-damper-inerter (TMDI). This device consists of a traditional tuned-mass-damper with an inerter attached to the mass element. Note that the inerter element has inertance b (in kg), and the inerter device other parameters are denoted as mass m (kg), stiffness k_d (N/m) and damping c_d (kg/s).

All these devices have been proven to have similar or better performance to the traditional tuned-mass-damper (TMD) in terms of vibration mitigation tasks, such as reducing the displacement amplitudes around the targeted resonance. In addition there are additional benefits compared to the TMD, e.g. the large reduction in mass ratio needed to achieve optimum performance.

Much more detail of these, and other device characteristics can be found in the related literature. For example, there are now a series of review papers describing different aspects of the inerter – see for example [11–15] and references therein. Many recent innovations have been proposed such as clutched inerter systems [16–20], nonlinear inerter systems [21–25], fluid inerters [26–28] and rocking block inerter [17, 29, 30].

In this paper, we will describe two recent developments carried out at the University of Sheffield. Firstly, we will describe, the inerter device built with hysteretic (e.g. structural) rather than viscous damping [31]. This device was constructed using gel-damper elements so that the damping behaviour was hysteretic rather than viscous. As a result, it is now possible to verify what type of behaviour occurs in practice when hysteretic rather than viscous damping is present in the system This device has been demonstrated in a vibration mitigation experiment, and sample results are shown in Section 2 below.

Secondly, we will give insights into a new type of inerter device for suppression of machine tool vibration and chatter [32]. This device was designed using “living hinges” instead of more traditional mechanical hinge joints. In addition the device was constructed in a supporting structure so that it could be mounted directly on a test-piece rather than between two parts of the structure. We show some of the early results obtained from that device in Section 3.

2. HYSTERETIC DAMPING IN INERTER DEVICES

It was noted above that inerter devices are designed based on a specific arrangement of elements (e.g. as shown in the three cases in Fig. 3b) and then “tuning” the element parameters (inertance b, mass m, device stiffness k_d and device damping c_d). Tuning parameters either theoretically (in simple cases) or using numerical methods is relatively straightforward, but building devices that replicate the required parameter values in practice is often very difficult.

In particular, obtaining the correct damping values is often a difficult thing to achieve in practice. This is further complicated by the fact that viscous damping (typically used in theory and numerical design) is based on a specific set of assumptions which are often not present in physical devices. Some physical devices have dampers that behave in an approximately viscous way, and others have damping effects that are closer to hysteretic (sometimes also called structural) damping behaviours.

For these reasons, a research study was undertaken at the University of Sheffield to understand the behaviour of hysteretic damping in inerter devices. Once the behaviour was understood, it was anticipated that design methods for inerter devices with this type of damping could be developed.

The device developed was a tuned-mass-hysteretic-damper-inerter (TMhDI) which is a modification of the TMDI described above. Linear hysteretic damping was assumed for the TMhDI, and this was modelled numerically using a complex stiffness model that was solved using a numerical time-integration method. The model was then used to develop a design technique for choosing the loss factor associated with the hysteretic damper in the inerter device [33]. To build hysteretic dampers in practice silicone gel (Magic Power Gel, from Raytech) was used – see [31] and references therein.

In order to try and make a comparison between viscous and hysteretic damping in the experiments a system of viscous damping was realised by using eddy current dampers (ECDs also sometimes called magnetic dampers). The ECD dampers allowed qualitative and quantitative comparisons to be made, and full details are given in [31].

To describe the main points of this study, the 3-storey structure shown in Fig. 4 was considered. The experimental structure (Fig. 4b) is a 3-storey shear building constructed in the Laboratory for Verification and Validation at the University of Sheffield. For the purposes of dynamic testing, it was attached

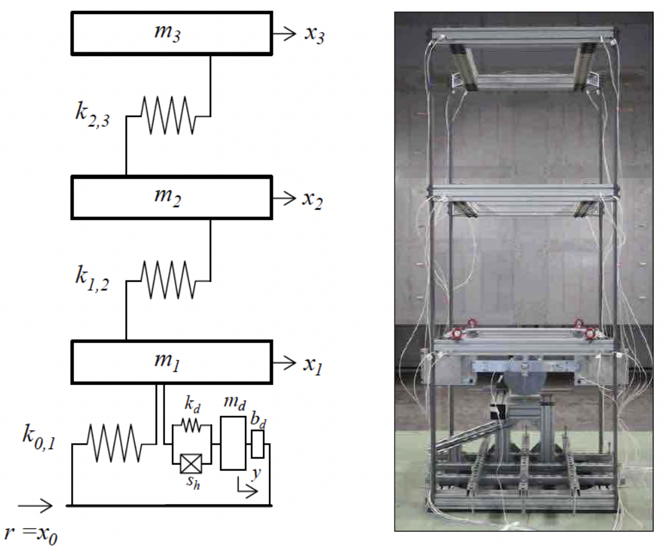

Fig. 4. Three-storey structure inerter experiments showing: (a) lumped-mass model (b) 3-storey experimental test structure with tuned-viscous-mass-hysteretic-damper (TMhDI) fitted between the base and first storey. The masses of each storey are m_1, m_2 and m_3, and the stiffness between the storeys are $k_{0,1}$, $k_{1,2}$, $k_{2,3}$. The base structure is assumed to be undamped. The inerter-device parameters are mass m_d, inertance b_d, stiffness k_d and hysteretic damping s_h. The displacement of the inerter-device mass is denoted by y. Full details are given in [31]

to the multi-axis shaker table (MAST) system in a chamber as shown in Fig. 4b. The shake table dimension is 3.2 m by 2.2 m with a test frequency range of 5–70 Hz.

The inerter device is positioned at the base of the 3-storey shear building (also visible in Fig. 4b). The device uses a flywheel inerter due to its simplicity and ease of tuning b to the selected value. This is achieved by changing the flywheel support position (which also acts as one of the terminals of the inerter) and thus changing the distance between the two terminals giving a controlled way of tuning the inertance value.

Two gel dampers were constructed by having an aluminium plate that moved in and out of a box filled with the silicone gel. Therefore, as the plate moved it is resisted by the shear motion of the gel creating hysteretic damper effect. This effect was tested and characterised so that the parameters could be tuned appropriately to achieve the required vibration mitigation. Lastly the device stiffness was tuned by selecting appropriate stiffness for the aluminium support frames.

The experiments were performed in the horizontal x-axis only. A simple lumped-mass-model, shown in Fig. 4a, was used to compare the experimental results with numerical simulations. In the case considered here the vibration mitigation problem was to reduce the vibration response of the top storey (displacement x_3 shown in Fig. 4a) induced by the harmonic ground motion $r(t)$.

The results are plotted in terms of the base-to-top-storey transmissibility X_3/R, where X_3 is the harmonic amplitude of x_3 and R is the corresponding quantity for $r(t)$. In Fig. 5 the results are plotted for both the uncontrolled structure (Fig. 5a) and the structure equipped with inerter device (Fig. 5b). A steady-state harmonic input was used to excite the shake table across the selected frequency range. For each frequency, the steady-state response of the top storey was measured from which the transmissibility could be computed.

From Fig. 5a it can be seen that the correlation between the experimental results and the simulated model output is very good. Therefore, the mass and stiffness properties of the host structure are considered to be accurate enough for the purposes of the investigation. In particular they can be used for selection and tuning of parameters for the inerter devices. This was then used to create four variants of the numerical model.

The model variants used were TID and TMDI as defined above, both with viscous damping. In addition TIhD and TMhDI models were simulated where instead of viscous damping, hysteretic damping was used instead. In each case of the four models the device needs to be 'tuned' (e.g. optimised) based on specific criteria, which is typically taken to be reducing the resonant vibrations of the first resonance peak. For simple structures with harmonic inputs, this optimisation can be achieved analytically based on tuning rules – see for example Hu *et al*. [34]. For more complex structures and/or inputs then other optimisation type techniques need to be used – see for example De Domenico *et al*. [27]. The use of a (non-causal) complex stiffness to represent the hysteretic damping in the TIhD and TMhDI models presents a specific set of difficulties, and in this case the approach developed by Deastra *et al*. [33] was applied.

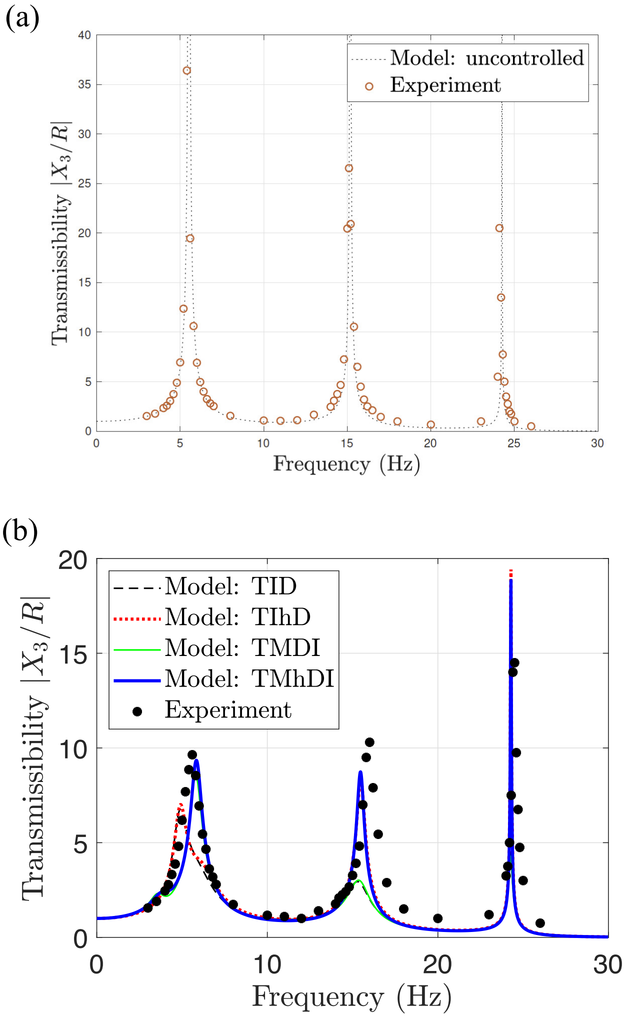

Fig. 5. Results of the three-storey structure with the inerter-based device tuned to suppress the first resonance showing: (a) The uncontrolled structure top storey transmissibility $|X_3/R|$ where X_3 and R are the displacement amplitudes of x_3 and r respectively. (b) The structure top storey transmissibility when equipped with different inerter-devices. For full details see [31, 35]

The results of the numerical model variants compared to one of the experimental results is shown in Fig. 5b. It can be seen from the results shown in Fig. 5b that the TMhDI is the closest match to the experimental results obtained from the system with the gel dampers.

An interesting qualitative feature shown in Fig. 5b is that the amplitudes of the second and third resonance peaks are considerably larger for the case of hysteretic damping. This is because, unlike the viscous damping models, hysteretic damping is not frequency dependent. As a result, care should be exercised if using viscous models, as this may predict lower resonant amplitudes than are obtainable in practice.

3. CHATTER SUPPRESSION INERTER DEVICE

Chatter phenomena in machining and manufacturing processes is a longstanding problem requiring vibration mitigation. In this study we considered how to use an inerter device to try and suppress chatter during a milling operation. One interesting aspect of this that is not present in the previous example, is the requirement for stable operation. As a result, the vibration mitigation strategy needs to ensure stable operation, which is usually informed by the real part of the eigenvalues of the response of the system, plotted on a stability chart.

The inerter element was based on a pivoted bar device which has some similarities to previous dynamic anti-vibration mounts – see [3] for further details. Instead of mechanical pivots, living hinges were used, based on the inerter designed by [5]. Fine tuning of the inerter to obtain the optimal inertance of 0.052 kg for $k_{\text{notch}} = 9000$ N/m was achieved by adding equal additional small masses to the end of the inerter bar. The mass of some of the elements acted as an unwanted 'parasitic mass' which could affect the results in some circumstances.

The damping element was provided using the same gel dampers as were used in the previous example. The stiffness elements were designed based on the living hinge (notches) analysis – see [32] for complete details.

A schematic of the device mounted on the workpiece that was used for the experimental characterisation tests is shown in Fig. 6a. Here it can be seen that the inerter device is mounted on top of the *workpiece* (e.g. the piece of material to be machined), and the *fixture* that holds the workpiece to the ground has some flexibility. The system is excited with a *shaker* at-

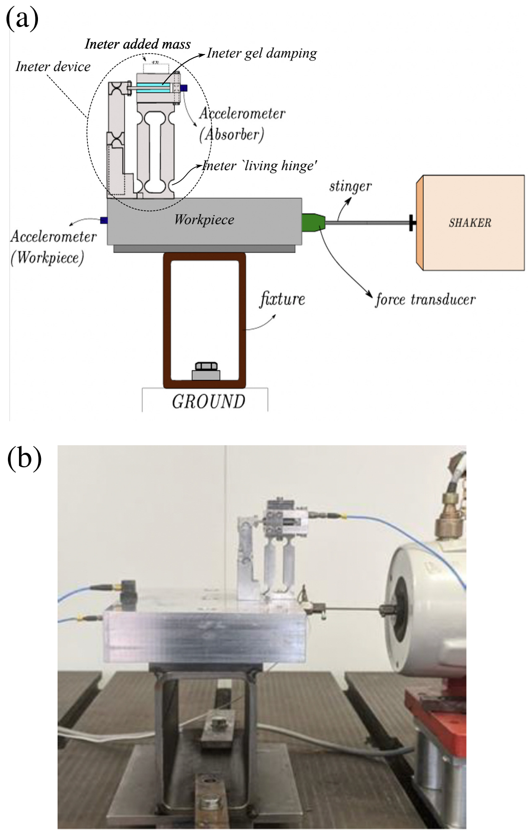

Fig. 6. Experimental setup of the host structure with the prototype inerter device showing: (a) a schematic of the experimental test with the prototype inerter device mounted on the aluminium block which setup, and shaker attached (b) an image of the experimental setup. Full details are given in [32]

tached to the workpiece via a *stinger*. Two accelerometers and a force transducer are used to measure the vibration behaviour of the system.

The inerter device itself (also denoted the *absorber*) consists of three vertical beams (two directly attached and one to the side connected via the gel damper) and a mass at the top, with living hinges used as connections. An added mass was used to fine-tune the inertia properties of the inerter device. The gel damper was developed in exactly the same way as the device discussed in Section 2, except on a smaller scale.

The photograph shown in Fig. 6b shows the experimental test set-up in the laboratory. Note that the host structure is a single-degree-of-freedom system. The mass, M is the aluminium workpiece, and the stiffness and damping are provided by the fixture that attached the mass to the ground. The vibration input is provided by a shaker that is attached to the host structure using a stinger, as shown in both Fig. 6a and b.

The tuned inerter device parameters were initially obtained by neglecting the notch stiffness ($k_{\text{notch}} = 0$). In practice, the stiffness in the notches prevents the inerter from generating constant inertance in the resonance region. The notch stiffness was theoretically calculated to be approximately 5000 N/m, but the estimate from testing was closer to 9000 N/m, a discrepancy believed to be caused by manufacturing errors and possibly stress stiffening.

As an initial exploration of the dynamic behaviour close to the primary resonance, a series of impact hammer tests were performed. The hammer tests were used to compute the magnitude of the frequency response function (FRF) of the host structure. The results obtained from the modal tests with the impulse hammer in comparison to a series of models are shown in Fig. 7. The inertial effect of the parasitic mass was neglected for this case.

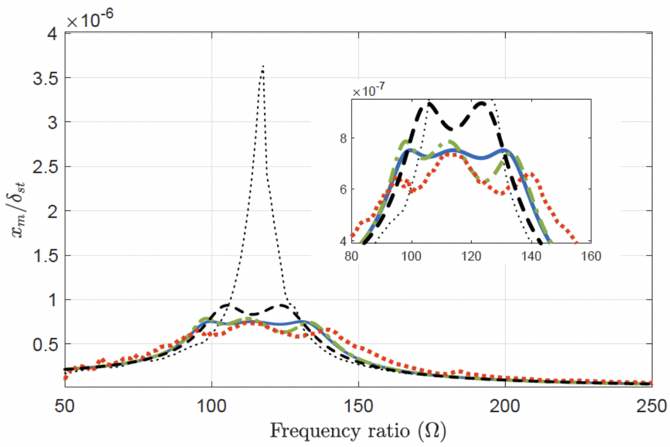

Fig. 7. Experimental results (red dashed line) of the host structure with the prototype inerter device with a mass ratio of 0.045 and a parasitic mass ratio of 0.054 in comparison with the experimental result of the uncontrolled host structure compared to numerical simulations: The uncontrolled structure (thin black dashed line); tuned-mass-damped (thick black dashed line); inerter device with $k_{\text{notch}} = 0$ (blue line); inerter device with $k_{\text{notch}} = 9000$ N/m (green dashed line). Full details are given in [32]

Figure 7 demonstrates that the prototype achieved a 79.7% vibration suppression effect. This is calculated based on the decrease in peak amplitude value of 36.34×10^{-7} m/N to 7.37×10^{-7} m/N. Comparing the inerter device to the (numerically computed) tuned-mass-damper performance (numerically obtained) there is an improvement of approximately 20%, which can be observed in the figure.

4. CONCLUSIONS

In this paper we have briefly discussed the background to the development of inerter devices for passive vibration mitigation. After describing the background, we described two recent examples of research activities from the University of Sheffield.

In the first example, the focus was on the type of damping that physical inerter devices may have in practice. As a result the investigation included modelling and experiments of a hysteretic damping system in combination with a flywheel inerter. Good agreement was found between the model and experimental results. Furthermore, the difference in behaviour between viscous and hysteretic damping was elucidated.

In the second research topic, a pivoted bar inerter device was designed to mitigate vibrations from a machining application. The device was designed to operate within a frame so that it was self-contained, and could be mounted onto a workpiece without needing to be between two specific points on the structure. To the author's knowledge, this was the first experimentally validated inerter device that can be mounted as a localised addition similar to a classical TMD.

As in the first example, close agreement was found between the model and experimental results. In addition, the results showed that the inerter device could performing slightly better than a classical TMD.

Another important feature was that the inertance could be fine tuned by using additional small masses. This is useful because, although the design parameters are optimised, assumptions such as neglecting the notch stiffness, results in a non-optimal performance of the physical device.

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