ADVANCES IN GEODESY AND GEOINFORMATION
(formerly GEODESY AND CARTOGRAPHY)
Vol. 72, no. 1, article no. e34, 2023

https://doi.org/10.24425/agg.2022.141920

Original paper

The Helmert transformation: a proposal for the problem of post-transformation corrections

Tadeusz Gargula, Pelagia Gawronek*

University of Agriculture in Krakow, Krakow, Poland
e-mail: tadeusz.gargula@urk.edu.pl; ORCID: http://orcid.org/0000-0003-3109-5922
e-mail: pelagia.gawronek@urk.edu.pl; ORCID: http://orcid.org/0000-0001-7806-909X

*Corresponding author: Pelagia Gawronek, e-mail: pelagia.gawronek@urk.edu.pl

Received: 2022-08-11 / Accepted: 2022-11-05

Abstract: The geodesy literature seems to offer comprehensive insight into the planar Helmert transformation with Hausbrandt corrections. Specialist literature is mainly devoted to the issues of 3D transformation. The determination of the sought values, coordinates in the target system, requires two stages of computations. The classical approach yields 'new' coordinates of reference points in the target system that should not be changed in principle. This requires the Hausbrandt corrections. The paper proposes to simplify the two-stage process of planar transformation by assigning adjustment corrections to reference point coordinates in the source system. This approach requires solving the Helmert transformation by adjusting conditioned observations with unknowns. This yielded transformation results consistent with the classical method. The proposed algorithm avoided the issue of correcting the official coordinates of the control network and using additional (post-transformation) corrections for the transformed points. The proposed algorithm for solving the plane Helmert transformation for $n > 2$ reference points simplifies the computing stages compared to the classical approach. The assignment of adjustment corrections to coordinates of reference points in the source system helps avoid correcting coordinates of the reference points in the target system and additional corrections for transformed points. The main goal of any data adjustment process with the use of the least squares method is (by definition) obtaining unambiguous quantities that would strictly meet the mathematical relationships between them. Therefore, this work aims to show such a transformation adjusting procedure, so that no additional computational activities related to the correction of already aligned results are necessary.

Keywords: planar Helmert transformation, Hausbrandt post-transformation corrections, 2D transformation algorithm

1. Introduction

The Helmert transformation is a basic geometric transformation between coordinate systems used in basic land surveying tasks, engineering surveying, geodesy, and photogrammetry (Sjöberg, 2013; Zeng et al., 2018; Odziemczyk, 2020). Both 2D and 3D transformations involve solving non-linear systems of equations with the least-squares method where the rotation matrix is orthonormal. The 2D transformation is converted into a linear problem and can be solved using classical methods. The 3D transformation requires linearization and, usually, iterative algorithms (Zeng and Yi, 2011; Zeng et al., 2016; Odziemczyk, 2020) or analytical algorithms (Shen et al., 2006; Zeng, 2015) which deprives the rotation matrix of its orthonormality (Sjöberg, 2013; Ioannidou and Pantazis, 2020). The iterative algorithm utilize the numerical (iterative) computation technique to seek the transformation parameters. A good initial estimate of transformation parameters is usually required to start the iterative computation. In some cases the inital estimate is no needed, however it may cause much more computation time and burden (Zeng et al., 2019). The analytical algorithm is rare to see because the complexity of mathematical derivation. The analytical algorithms are fast and reliable because the parameters are recovered directly by the given exact formulae Therefore, over the years the literature offers much more focus on 3D transformations than 2D transformations (Zeng et al., 2019; Bektas, 2022). However, 2D transformation can be streamlined regarding computation steps as demonstrated here.

The problem of the Helmert transformation in plane usually occurs when converting local (source) system coordinates into the national-level 2D coordinate system in the national spatial frame of reference (Polish Journal of Laws (2020); Kadaj 2001). The solution to this problem leads to the determination of transformation parameters (translation vector, rotation angle, and scale factor) based on at least two reference points. When the reference point number is higher ($n > 2$), the transformation equation system becomes overdetermined (inconsistent). The problem can be solved with the least-squares adjustment method. An adjustment ensures the mathematical consistency of the system of equations but at the expense of the coordinates of the reference points in the target coordinate system, which are 'tainted' by adjustment corrections (Vanicek and Steeves, 1996). The coordinates of reference points in the target system are usually those of official control network points (detailed, third class, for example) that are 'well determined' and should not be 'corrected' through transformation alignment. Using the classical approach (Polish Journal of Laws (2020); Kadaj, 2002a), the problem is resolved by preserving the official coordinates unchanged in the target system and applying so-called Hausbrandt post-transformation corrections to the coordinates of the transformed points (Swieton, 2010; 2012).

The necessity to adjust the transformation is the inconsistency of the geometric reference point set system in the target system with its counterpart in the source system. According to the Authors the two systems should be reconciled by correcting the coordinates in the source (local) system because (presumably) they are burdened with much greater uncertainty of point positioning (based on surveying) than the official coordinates of the control (target system). This research gap was been filled by the transformation adjustment algorithm proposed here involves the assignment of adjustment corrections to reference point coordinates in the source coordinate system to avoid the issue of correcting official coordinates of control network points and additional (post-transformation) corrections of the transformed points.

2. The classical Helmert transformation adjustment algorithm using post-transformation corrections

Plane rectangular coordinates are usually transformed between systems according to the Helmert (similarity) model. The initial transformation equations (Wysocki, 2011) define a strict relation between the location (coordinates) of a point in the source system $(x,\ y)$ and its coordinates in the target system $(X,\ Y)$:

$$\begin{cases} X = X_0 + x \cdot C + y \cdot S, \\ Y = Y_0 + y \cdot C - x \cdot S, \end{cases} \tag{1}$$

where:

$$\begin{cases} C = k \cdot \cos\alpha, \\ S = k \cdot \sin\alpha, \end{cases} \tag{2}$$

C, S – the transformation coefficients, $(X_0,\ Y_0)$ – the translation (coordinates of the origin of the source system in the target system), α – the rotation angle between the source and target systems, k – the transformation scale factor.

The four parameters of the transformation $(X_0,\ Y_0,\ \alpha,\ k)$ can be determined only when there are at least two points of reference (n). When there are more reference points $(n > 2)$, their coordinates need to be adjusted. The functional model of the adjustment is based on transformation Equations (1) specified for each i-th reference point. The coordinates in both coordinate systems are reduced to a certain reference point (the same in both systems: x_0, y_0 and X_0, Y_0) that is the centre of gravity of the geometric system of the reference points (Kadaj, 2002b):

$$\begin{cases} X_i - X_0 = (x_i - x_0) \cdot C + (y_i - y_0) \cdot S, \\ Y_i - Y_0 = (y_i - y_0) \cdot C - (x_i - x_0) \cdot S, \end{cases} \quad i = 1, 2, 3, \ldots, n, \tag{3}$$

$$x_0 = \frac{1}{n}\sum_{i=1}^{n} x_i\,, \quad y_0 = \frac{1}{n}\sum_{i=1}^{n} y_i\,, \quad X_0 = \frac{1}{n}\sum_{i=1}^{n} X_i\,, \quad Y_0 = \frac{1}{n}\sum_{i=1}^{n} Y_i\,. \tag{4}$$

This way, Eq. (3), Eq. (4), two parameters are eliminated from the system of transformation equations: the coordinates of the translation vector $(X_0,\ Y_0)$.

Let us introduce auxiliary symbols for coordinate increments:

$$A_i = (X_i - X_0)\,, \quad B_i = (Y_i - Y_0)\,, \quad a_i = (x_i - x_0)\,, \quad b_i = (y_i - y_0)\,, \tag{5}$$

Eq. (3) becomes:

$$\begin{cases} A_i = a_i \cdot C + b_i \cdot S, \\ B_i = b_i \cdot C - a_i \cdot S. \end{cases} \tag{6}$$

In the classical Helmert transformation, adjustment corrections (V_X, V_Y) are assigned to coordinates of each i-th reference point in the target system:

$$\begin{cases} A_i + V_{X_i} = a_i \cdot C + b_i \cdot S, \\ B_i + V_{Y_i} = b_i \cdot C - a_i \cdot S. \end{cases} \tag{7}$$

It yields an overdetermined system of correction equations with two parameters (coefficients C and S):

$$\begin{cases} V_{X_i} = a_i \cdot C + b_i \cdot S - A_i, \\ V_{Y_i} = b_i \cdot C - a_i \cdot S - B_i. \end{cases} \tag{8}$$

The matrix form of system of Eq. (6) for n reference points is:

$$\mathbf{V} = \mathbf{A} \cdot \mathbf{X} - \mathbf{L}, \tag{9}$$

where:

$$\begin{bmatrix} V_{X_i} \\ V_{Y_i} \\ \dots \end{bmatrix}_{2n,1} = \begin{bmatrix} a_i & b_i \\ b_i & -a_i \\ \dots & \dots \end{bmatrix}_{2n,2} \cdot \begin{bmatrix} C \\ S \end{bmatrix} - \begin{bmatrix} A_i \\ B_i \\ \dots \end{bmatrix}_{2n,1} . \tag{10}$$

System of Eq. (9) conforms to the least squares condition ($\mathbf{V^T} \cdot \mathbf{V} = \min$), when parameters C and S are determined the following way:

$$\hat{\mathbf{X}} = \left(\mathbf{A}^{\mathrm{T}} \cdot \mathbf{A}\right)^{-1} \cdot \mathbf{A}^{\mathrm{T}} \cdot \mathbf{L} = \begin{bmatrix} C \\ S \end{bmatrix}. \tag{11}$$

Next, using Eq. (2), we can calculate unique values of parameters k and α:

$$k = \sqrt{C^2 + S^2}, \tag{12}$$

$$\alpha = \operatorname{atan}\left(\frac{S}{C}\right). \tag{13}$$

Having substituted the calculated unknowns vector Eq. (11) into Eq. (9), we obtain corrections (V_{Xi}, V_{Yi}) to correct coordinates of the reference points in the target system:

$$\begin{cases} \bar{X}_i = X_i + V_{X_i}, \\ \bar{Y}_i = Y_i + V_{Y_i}. \end{cases} \tag{14}$$

Transformation coefficients C and S Eq. (11) facilitate the conversion of transformed points' coordinates from the source system into the target system using Eq. (3):

$$\begin{cases} X_j = \left(x_j - x_0\right) \cdot C + \left(y_j - y_0\right) \cdot S + X_0, \\ Y_j = \left(y_j - y_0\right) \cdot C - \left(x_j - x_0\right) \cdot S + Y_0, \end{cases} \qquad j = 1, 2, 3, \dots, t, \tag{15}$$

where: j – the index of the transformed point, t – the number of transformed points.

The coordinates of the reference points converted using Eq. (15) should be equal to the corrected coordinates Eq. (14), which demonstrates the consistency of the adjusted system of reference points:

$$\begin{cases} \bar{X}_i = (x_i - x_0) \cdot C + (y_i - y_0) \cdot S + X_0 \,, \\ \bar{Y}_i = (y_i - y_0) \cdot C - (x_i - x_0) \cdot S + Y_0 \,. \end{cases} \tag{16}$$

Accuracy is assessed by calculating mean square errors of the coordinate transformation M_X, M_Y and the general transformation error M_T:

$$M_X = \pm\sqrt{\frac{\sum V_{X_i}^2}{n}}\,, \quad M_Y = \pm\sqrt{\frac{\sum V_{Y_i}^2}{n}}\,, \tag{17}$$

$$M_T = \pm\sqrt{M_X^2 + M_Y^2}\,, \tag{18}$$

The Hausbrandt post-transformation correction consists in restoring values of the coordinates of the reference points in the target system Eq. (16) to the state before the adjustment (Hausbrandt, 1971; Beluch, 2009; Swieton, 2012):

$$\begin{cases} X_i = \bar{X}_i - V_{X_i} \,, \\ Y_i = \bar{Y}_i - V_{Y_i} \,. \end{cases} \tag{19}$$

Next, the adjustment corrections V_{Xi}, V_{Yi} Eq. (9) are used to calculate the post-transformation corrections $(V_{Xj},\ V_{Yj})$ for the coordinates of the transformed points Eq. (15):

$$V_{Xj} = \frac{\sum\left(V_{X_i} \cdot \frac{1}{d_{ij}^2}\right)}{\sum\left(\frac{1}{d_{ij}^2}\right)}\,, \quad V_{Yj} = \frac{\sum\left(V_{Y_i} \cdot \frac{1}{d_{ij}^2}\right)}{\sum\left(\frac{1}{d_{ij}^2}\right)}\,, \tag{20}$$

where: d_{ij} – the distance of the j-th transformed point to the i-th reference point.

3. Proposed algorithm for adjusting reference points in the source system in the Helmert transformation

In this case, the adjustment corrections $(V_x,\ V_y)$ are assigned to the coordinates of the reference points in the source system $(x_i,\ y_i)$ that are represented in transformation Eq. (6) by increments a_i, b_i in accordance with Eq. (5). Therefore, the functional model of the adjustment consists of conditional equations that can be expressed (for each i-th reference point) as:

$$\begin{cases} \left(a_i + V_{x_i}\right) \cdot C + \left(b_i + V_{y_i}\right) \cdot S = A_i \,, \\ \left(b_i + V_{y_i}\right) \cdot C - \left(a_i + V_{x_i}\right) \cdot S = B_i \,. \end{cases} \tag{21}$$

As is apparent, the conditional Eq. (21) also have parameters, the C and S transformation coefficients. Therefore, it is a system of non-linear conditional equations with unknowns. After expansion into a Taylor series (regarding corrections V_x, V_y and parameters C, S), Eq. (21) yield:

$$\begin{cases} a_i \cdot \delta C + b_i \cdot \delta S = -C_0 \cdot V_{x_i} - S_0 \cdot V_{y_i} + \omega_{x_i} , \\ b_i \cdot \delta C - a_i \cdot \delta S = -C_0 \cdot V_{y_i} + S_0 \cdot V_{x_i} + \omega_{y_i} , \end{cases} \tag{22}$$

where:

$$\begin{cases} \omega_{x_i} = -a_i \cdot C_0 - b_i \cdot S_0 + A_i , \\ \omega_{y_i} = -b_i \cdot C_0 + a_i \cdot S_0 + B_i . \end{cases} \tag{23}$$

System Eq. (22) can be represented generally as a matrix (for n reference points):

$$\mathbf{A} \cdot \boldsymbol{\delta}\mathbf{X} = \mathbf{C} \cdot \mathbf{V} + \mathbf{W}, \tag{24}$$

where:

$$\mathbf{A} = \left[\begin{array}{cc} a_1 & b_1 \\ b_1 & -a_1 \\ \hdashline a_2 & b_2 \\ b_2 & -a_2 \\ \hdashline \cdots & \cdots \end{array}\right]_{2n,2}, \quad \boldsymbol{\delta} X = \begin{bmatrix} \delta C \\ \delta S \end{bmatrix}, \quad \mathbf{C} = \left[\begin{array}{cc:c:cc} -C_0 & -S_0 & \cdots & 0 & 0 \\ S_0 & -C_0 & \cdots & 0 & 0 \\ \hdashline \cdots & \cdots & \ddots & \cdots & \cdots \\ \hdashline 0 & 0 & \cdots & -C_0 & -S_0 \\ 0 & 0 & \cdots & S_0 & -C_0 \end{array}\right]_{2n,2n}, \tag{25}$$

$$\mathbf{V} = \left[\left\{ V_{x_i};\ V_{y_i} \right\};\ i = 1, 2, \ldots, n \right]^T , \quad \mathbf{W} = \left[\left\{ \omega_{x_i};\ \omega_{y_i} \right\};\ i = 1, 2, \ldots, n \right]^T .$$

We will use the pair of transformation Eq. (6) for any reference point (such as $i = 1$) to calculate the approximate values of C_0, S_0:

$$\begin{cases} a_1 \cdot C_0 + b_1 \cdot S_0 = A_1 , \\ b_1 \cdot C_0 - a_1 \cdot S_0 = B_1 , \end{cases} \tag{26}$$

in matrix form:

$$\mathbf{A}_0 \cdot \mathbf{X}_0 = \mathbf{L}_0 , \quad \begin{bmatrix} a_1 & b_1 \\ b_1 & -a_1 \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}. \tag{27}$$

Having solved system Eq. (27), we obtain:

$$\mathbf{X}_0 = \mathbf{A}_0^{-1} \cdot \mathbf{L}_0 = \begin{bmatrix} C_0 \\ S_0 \end{bmatrix}. \tag{28}$$

In the next step, the pseudo-corrections vector $\underline{\mathbf{v}}$ is substituted into the matrix Eq. (24):

$$\mathbf{A} \cdot \boldsymbol{\delta}\mathbf{X} = \mathbf{W} + \underline{\mathbf{v}}, \tag{29}$$

where:

$$\underline{\mathbf{v}} = \mathbf{C} \cdot \mathbf{V}. \tag{30}$$

Under the least squares condition:

$$\underline{\mathbf{v}}^{\mathrm{T}} \cdot \mathbf{Q}_{\underline{\mathbf{v}}} \cdot \underline{\mathbf{v}} = \min, \tag{31}$$

system Eq. (29) is solved:

$$\boldsymbol{\delta}\mathbf{X} = \left(\mathbf{A}^{\mathrm{T}} \cdot \mathbf{Q}_{\underline{\mathbf{v}}} \cdot \mathbf{A}\right)^{-1} \cdot \mathbf{A}^{\mathrm{T}} \cdot \mathbf{Q}_{\underline{\mathbf{v}}} \cdot \mathbf{W}. \tag{32}$$

Matrix $\mathbf{Q}_{\underline{\mathbf{v}}}$ is obtained under the variance-covariance propagation law (Wisniewski, 2005; Gargula, 2011), following the form of Eq. (30):

$$\mathbf{Q}_{\underline{\mathbf{v}}} = \mathbf{C} \cdot \mathbf{Q}_{\mathbf{V}} \cdot \mathbf{C}^{\mathrm{T}}, \tag{33}$$

where:

$$\mathbf{Q}_{\mathbf{V}} = \mathbf{P}^{-1} = \operatorname{diag}\left\{\frac{1}{p_{x_i}};\ \frac{1}{p_{y_i}};\quad i = 1, 2, 3, \ldots, n\right\}, \tag{34}$$

$\mathbf{P}$ – the weighting matrix.

Weighing of reference point coordinates in the source system remains to be decided: for an i-th reference point, weights p_x, p_y can be assumed (for example) as inversely proportional to increment moduli a_i, b_i (see Eq. (5) and Eq. (4)), for example:

$$p_{x_i} = \frac{1}{|a_i|}, \qquad p_{y_i} = \frac{1}{|b_i|}, \tag{35}$$

or to the distance from this point to the centre of gravity x_0, y_0.

The vector of unknowns $\boldsymbol{\delta}\mathbf{X}$, calculated according to (32), helps determine the pseudo-corrections vector $\underline{\mathbf{v}}$ (based on Eq. (29)):

$$\underline{\mathbf{v}} = \mathbf{A} \cdot \boldsymbol{\delta}\mathbf{X} - \mathbf{W}. \tag{36}$$

The adjustment corrections vector $\mathbf{V}$ is determined (in the general case of a system of conditional equations – see e.g. Baran, 1999; Gargula, 2009) from the equation:

$$\mathbf{V} = \mathbf{Q}_{\mathbf{V}} \cdot \mathbf{C}^{\mathrm{T}} \cdot \mathbf{Q}_{\underline{\mathbf{v}}}^{-1} \cdot \underline{\mathbf{v}}. \tag{37}$$

Still, as in this case, the system of Eq. (30) is uniquely determinable (the number of corrections V_x, V_y is equal to the number of equations) and vector $\mathbf{V}$ can be obtained in the following way (assuming an inverse of matrix $\mathbf{C}$):

$$\mathbf{V} = \mathbf{C}^{-1} \cdot \underline{\mathbf{v}}. \tag{38}$$

The final results of the transformation adjustment using the proposed method are unique transformation parameters (C, S) and corrected coordinates of the reference points in the source system:

$$\mathbf{X} = \mathbf{X}_0 + \boldsymbol{\delta}\mathbf{X} = \begin{bmatrix} C_0 \\ S_0 \end{bmatrix} + \begin{bmatrix} \delta C \\ \delta S \end{bmatrix}, \tag{39}$$

$$\begin{cases} \bar{x}_i = x_i + V_{x_i} \,, \\ \bar{y}_i = y_i + V_{y_i} \,. \end{cases} \tag{40}$$

The adjusted values (Eq. (39) and Eq. (40)) should strictly satisfy the initial system of transformation Eq. (1) including intermediate steps Eq. (4) and Eq. (5).

Such adjusted transformation parameters Eq. (39) provide a basis for converting the coordinates of the transformed points from the source system into the target system according to Eq. (16). The reference points, converted for control according to the same Eq. (16), are assigned zero corrections in the target system (remain unchanged as originally assumed).

The mean-square errors of the transformation of coordinates x, y and the general transformation error can be calculated from adjustment corrections for the reference points in the source system (V_x, V_y) with equations similar to Eq. (17) and Eq. (18).

The complete calculation procedure (taking into account the new algorithm) is shown in the diagram (Fig. 1).

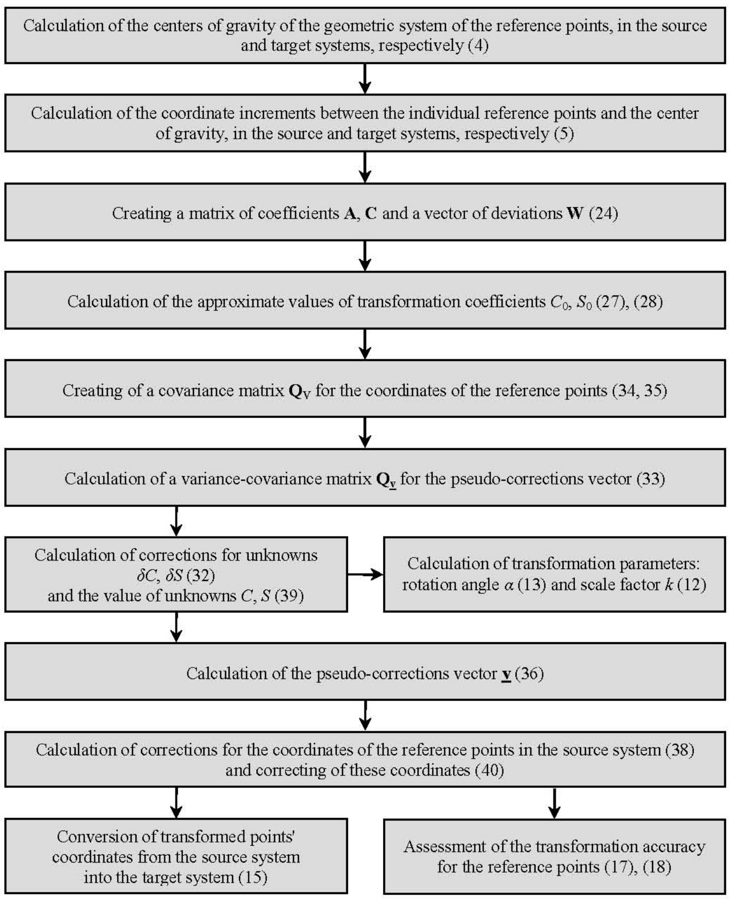

Fig. 1. Diagram of the proposed algorithm of Helmert transformation adjustment (with reference to the corresponding equations)

4. Numerical example. Results and discussion

The method for adjusting the Helmert transformation proposed here (Section 3) has been tested in practice on a dataset (Table 1).

Table 1. Input data for the Helmert transformation between coordinate systems

Point no.	Source system		Target system	
	x	y	X	Y
1*	1000.000	1000.000	5552693.250	6583648.165
2*	998.301	1074.615	5552689.790	6583573.590
3*	917.260	1117.813	5552767.584	6583524.860
101	1000.000	1024.949		
102	1000.968	1049.891		
103	988.870	1097.184		
104	965.361	1104.535		
105	941.150	1110.333		

* – reference points

The calculations were performed in four variants depending on reference point coordinate weighing. In the first transformation variant (I), the weights were equal to inverses of increment moduli a_i, b_i (see Eq. (35)). The second variant (II) involved weights inversely proportional to squares of increments a_i, b_i:

$$p_{x_i} = \frac{1}{a_i^2}, \quad p_{y_i} = \frac{1}{b_i^2}. \tag{41}$$

The weights in the third variant (III) followed the formula:

$$p_{x_i} = p_{y_i} = \frac{1}{a_i^2 + b_i^2}. \tag{42}$$

In the fourth variant (IV), the weights were the inverse of the distance from the i-th point to the centre of gravity x_0, y_0:

$$p_{x_i} = p_{y_i} = \frac{1}{\sqrt{a_i^2 + b_i^2}}. \tag{43}$$

The results of the Helmert transformation in the proposed method of adjusting conditioned observations with unknowns were summarized in Tables 2–5.

For comparison's sake, the coordinates were also transformed using the classical method (presented in Section 2), also including the Hausbrandt post-transformation correction (Tables 6, 7).

The results of the four variants of transformations were juxtaposed with the classical method results. The transformations done with adjustment of conditioned observations with unknowns and classically with the intermediate method employing the Hausbrandt corrections yielded similar results regarding transformation parameters and accuracy.

Table 2. Results of the Helmert transformation according to the proposed algorithm – Variant I

Point no.	Source coordinates (adjusted)		Target coordinates		Corrections of source coordinates	
	x_W	y_W	X	Y	V_x	V_y
1*	1000.019	999.991	5552693.250	6583648.165	0.019	–0.009
2*	998.272	1074.625	5552689.790	6583573.590	–0.029	0.010
3*	917.270	1117.812	5552767.584	6583524.860	0.010	–0.001
101	1000.000	1024.949	5552691.529	6583623.266	$M_x = 0.0211$	
102	1000.968	1049.891	5552688.824	6583598.452	$M_y = 0.0078$	
103	988.870	1097.184	5552697.596	6583550.430	$M_t = 0.0225$	
104	965.361	1104.535	5552720.536	6583541.458	$k = 1.000011$	
105	941.150	1110.333	5552744.284	6583533.986	$\alpha = 204.4418^g$	

* – reference points

Table 3. Results of the Helmert transformation according to the proposed algorithm – Variant II

Point no.	Source coordinates (adjusted)		Target coordinates		Corrections of source coordinates	
	x_W	y_W	X	Y	V_x	V_y
1*	1000.023	999.993	5552693.250	6583648.165	0.023	–0.007
2*	998.271	1074.626	5552689.790	6583573.590	–0.030	0.011
3*	917.268	1117.809	5552767.584	6583524.860	0.008	–0.004
101	1000.000	1024.949	5552691.531	6583623.268	$M_x = 0.0222$	
102	1000.968	1049.891	5552688.825	6583598.454	$M_y = 0.0081$	
103	988.870	1097.184	5552697.594	6583550.431	$M_t = 0.0236$	
104	965.361	1104.535	5552720.533	6583541.457	$k = 1.000015$	
105	941.150	1110.333	5552744.281	6583533.984	$\alpha = 204.4456^g$	

* – reference points

Table 4. Results of the Helmert transformation according to the proposed algorithm – Variant III

Point no.	Source coordinates (adjusted)		Target coordinates		Corrections of source coordinates	
	x_W	y_W	X	Y	V_x	V_y
1*	1000.016	999.991	5552693.250	6583648.165	0.016	–0.009
2*	998.271	1074.624	5552689.790	6583573.590	–0.030	0.009
3*	917.274	1117.813	5552767.584	6583524.860	0.014	0.000
101	1000.000	1024.949	5552691.527	6583623.266	$M_x = 0.0210$	
102	1000.968	1049.891	5552688.823	6583598.451	$M_y = 0.0070$	
103	988.870	1097.184	5552697.597	6583550.429	$M_t = 0.0222$	
104	965.361	1104.535	5552720.537	6583541.457	$k = 1.000034$	
105	941.150	1110.333	5552744.286	6583533.986	$\alpha = 204.4396^g$	

* – reference points

Table 5. Results of the Helmert transformation according to the proposed algorithm – Variant IV

Point no.	Source coordinates (adjusted)		Target coordinates		Corrections of source coordinates	
	x_W	y_W	X	Y	V_x	V_y
1*	1000.015	999.990	5552693.250	6583648.165	0.015	–0.010
2*	998.272	1074.623	5552689.790	6583573.590	–0.029	0.008
3*	917.274	1117.814	5552767.584	6583524.860	0.014	0.001
101	1000.000	1024.949	5552691.526	6583623.265	$M_x = 0.0207$	
102	1000.968	1049.891	5552688.823	6583598.451	$M_y = 0.0074$	
103	988.870	1097.184	5552697.597	6583550.428	$M_t = 0.0220$	
104	965.361	1104.535	5552720.538	6583541.457	$k = 1.000027$	
105	941.150	1110.333	5552744.287	6583533.987	$\alpha = 204.4385^g$	

* – reference points

Table 6. Results of the Helmert transformation according to the classical method

Point no.	Source coordinates (adjusted)		Target coordinates		Corrections of source coordinates	
	x	y	X_W	Y_W	V_x	V_y
1*	1000.000	1000.000	5552693.263	6583648.152	0.013	–0.013
2*	998.301	1074.615	5552689.762	6583573.600	–0.028	0.010
3*	917.260	1117.813	5552767.599	6583524.864	0.015	0.004
101	1000.000	1024.949	5552691.526	6583623.263	$M_x = 0.0195$	
102	1000.968	1049.891	5552688.823	6583598.449	$M_y = 0.0098$	
103	988.870	1097.184	5552697.599	6583550.429	$M_t = 0.0218$	
104	965.361	1104.535	5552720.539	6583541.459	$k = 0.999997$	
105	941.150	1110.333	5552744.288	6583533.989	$\alpha = 204.4363^g$	

* – reference points

Table 7. The Hausbrandt post-transformation correction

Point no.	Source coordinates (adjusted)		Target coordinates		Corrections of source coordinates	
	x	y	X_W	Y_W	V_x	V_y
1*	1000.000	1000.000	5552693.250	6583648.165	–	–
2*	998.301	1074.615	5552689.790	6583573.590	–	–
3*	917.260	1117.813	5552767.584	6583524.860	–	–
101	1000.000	1024.949	5552691.521	6583623.272	0.0051	–0.0084
102	1000.968	1049.891	5552688.842	6583598.444	–0.0181	0.0050
103	988.870	1097.184	5552697.621	6583550.421	–0.0215	0.0078
104	965.361	1104.535	5552720.546	6583541.453	–0.0071	0.0053
105	941.150	1110.333	5552744.278	6583533.985	0.0096	0.0039

* – reference points

These characteristics stem from the least-squares method. Coordinates of the transformed points have been variation analysed. To this end, we determined differences among plane coordinates of the points following transformation.

$$\begin{cases} dX_j = X_j^H - X_j\,, \\ dY_j = Y_j^H - Y_j\,, \end{cases} \tag{44}$$

and their resultant value that is a linear measure of transformation translations:

$$d_j = \sqrt{dX_j^2 + dY_j^2}\,, \tag{45}$$

where: X^H, Y^H – the coordinates of transformed points (j) following the Hausbrandt correction.

Then, we extracted the maximum and minimum values (for standard deviations of differences) from the set of plane coordinates and their resultant and calculated their basic statistics: standard deviation (σ), mean deviation (D), and range (R).

Measures of dispersion of coordinate differences and their resultant around the mean demonstrated a significant consistency of the solutions achieved with the proposed method and the classical transformation procedure (Table 8). The maximum differences between the coordinates were around ±25 mm. The differences in transformation results clustered around the mean within ±15 mm and deviated from it not more than ±12 mm on average. The variability range of the features did not exceed 38 mm.

The measures of variability demonstrated a consistency of the results of the classical transformation using the Hausbrandt corrections with each of the four proposed variants of reference point coordinates weighing (Table 8). The differences in the coordinates varied within the mean error of each sample for all four weighing variants (Fig. 2).

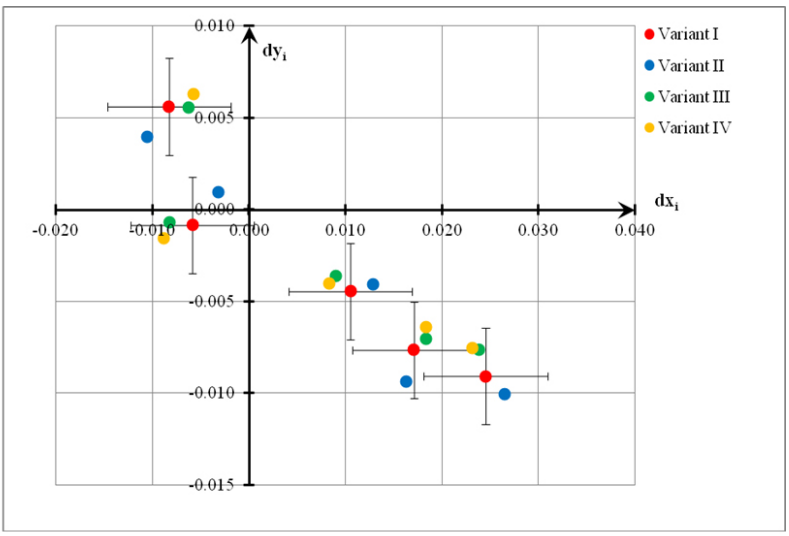

Fig. 2. Scatter plot of differences in coordinates of the transformed points for four weighing variants compared to the classical transformation (mean error bars for variant I)

Table 8. Measures of variability in differences among plane coordinates of points following the transformation and their resultant, in meters

Variant		|Max|	|Min|	Mean	σ	D	R
I	dX_j	0.0245	0.0058	0.0076	0.0143	0.0117	0.0328
	dY_j	0.0091	0.0008	–0.0033	0.0059	0.0045	0.0147
	d_j	0.0262	0.0059	0.0145	0.0080	0.0064	
II	dX_j	0.0265	0.0032	0.0084	0.0150	0.0122	0.0371
	dY_j	0.0101	0.0010	–0.0037	0.0062	0.0049	0.0140
	d_j	0.0283	0.0033	0.0150	0.0093	0.0068	
III	dX_j	0.0239	0.0062	0.0074	0.0143	0.0117	0.0321
	dY_j	0.0076	0.0007	–0.0027	0.0054	0.0041	0.0132
	d_j	0.0251	0.0082	0.0142	0.0077	0.0065	
IV	dX_j	0.0232	0.0057	0.0071	0.0142	0.0115	0.0320
	dY_j	0.0076	0.0015	–0.0026	0.0055	0.0040	0.0139
	d_j	0.0244	0.0085	0.0141	0.0073	0.0063	

5. Summary and conclusions

The proposed algorithm for solving the plane Helmert transformation for $n > 2$ reference points simplifies the computing stages compared to the classical approach. The assignment of adjustment corrections to coordinates of reference points in the source system helps avoid correcting coordinates of the reference points in the target system and additional corrections for transformed points. Such reduced computations do not affect the qualitative or quantitative characteristics of the outcome. The procedural advantage of the proposed algorithm is summarized in the table with similarities and differences between the two approaches (Table 9).

Table 9. Similarities and differences between the classical and proposed approaches to solving the Helmert transformation

SIMILARITIES BETWEEN THE CLASSICAL TRANSFORMATION PROCEDURE AND THE NEW ADJUSTMENT ALGORITHM	
The number of reference points ($n > 2$) for calculations using the least-squares method	
Adjustment results are unique parameters of the transformation between the source and target systems	
Accuracy assessment	
Values of coordinates in the target system	
DIFFERENCES	
Classical approach	**Proposed adjustment algorithm**
Corrections are assigned to coordinates of reference points in the target system as a result of the transformation	Corrections are assigned to coordinates of reference points in the source system as a result of the transformation
The transformation problem is solved using the intermediate observations method	The transformation problem is solved using the conditioned observations with unknowns method
Coordinates of transformed points in the target system require the Hausbrandt correction after the transformation	Coordinates of transformed points in the target system after the transformation are final

Author contributions

Conceptualization: T.G.; formal analysis: T.G.; investigation: T.G. and P.G.; methodology: T.G.; supervision: T.G. and P.G.; writing – original draft: T.G.; writing – review and editing: P.G.

Data availability statement

The datasets analyzed in the study are available from the corresponding author on reasonable request.

Acknowledgements

This paper was funded from a subsidy by the Ministry of Education and Science for the Hugo Kołłątaj University of Agriculture in Krakow for the year 2022.

References

Baran, L.W. (1999). *Teoretyczne podstawy opracowania wyników pomiarów geodezyjnych*. Warszawa: PWN.

Bektas, S. (2022). A new algorithm for 3D similarity transformation with dual quaternion. *Arab. J. Geosci.*, 15(14), 1–9. DOI: 10.1007/s12517-022-10457-z.

Beluch, J. (2009). *Ścisła ocena dokładności przetransformowanych współrzędnych po wprowadzeniu korekt Hausbrandta*. Archiwum Fotogrametrii i Teledetekcji, 19, 1-10.

Gargula, T. (2009). The conception of integrated survey networks composed of modular networks and GPS vectors. *Surv. Rev.*, 41(313), 301–313. DOI: 10.1179/003962609X451555.

Gargula, T. (2011). GPS Vector Network Adjustment in a Local System of Coordinates Based on Linear-Angular Spatial Pseudo-Observations. *J. Surv. Eng*, 137(2), 60–64. DOI: .

Hausbrandt, S. (1971). *Rachunek wyrównawczy i obliczenia geodezyjne*. T. II. Warszawa: PPWK.

Ioannidou, S., and Pantazis, G. (2020). Helmert Transformation Problem. From Euler Angles Method to Quaternion Algebra. *ISPRS Int. J. Geoinf*, 9(9), 494. DOI: 10.3390/ijgi9090494.

Kadaj, R. (2001). Wytyczne techniczne G-1.10. Formuły odwzorowawcze i parametry układów współrzędnych, wyd. 2 zmienione. Warszawa: GUGiK.

Kadaj, R. (2002a). Problematyka korekt układu "1965" i ich rozwiązanie w programie GEONET-unitrans. Retrieved August, 2022 from www.geonet.net.pl.

Kadaj R. (2002b). Polskie układy współrzędnych. Formuły transformacyjne, algorytmy i programy. Retrieved from www.geonet.net.pl.

Odziemczyk, W. (2020). Application of simulated annealing algorithm for 3D coordinate transformation problem solution. *Open Geosci.*, 12(1), 491–502.

Polish Journal of Laws (2020). Rozporządzenie Ministra Rozwoju z dnia 18 sierpnia 2020 r. w sprawie standardów technicznych wykonywania geodezyjnych pomiarów sytuacyjnych i wysokościowych oraz opracowywania i przekazywania wyników tych pomiarów do państwowego zasobu geodezyjnego i kartograficznego. Dz.U. 2020 poz. 1429.

Shen, Y-Z., Yi, Ch., and Zheng, D-H. (2006). A quaternion-based geodetic datum transformation algorithm. *J. Geod.*, 80.5, 233–239. DOI: 10.1007/s00190-006-0054-8.

Sjöberg, L.E. (2013). Closed-form and iterative weighted least squares solutions of Helmert transformation parameters. *J. Geod. Sci.*, 3(1), 7–11. DOI: 10.2478/jogs-2013-0002.

Swieton, T. (2010). *Analiza wybranych algorytmów obliczenia korekt LOK. PUW 1965*. Infrastruktura i ekologia terenów wiejskich, Komisja Technicznej Infrastruktury Wsi PAN, Oddział w Krakowie, Nr 12, s. 67–77.

Swieton, T. (2012). *Analiza zmian w topologii mapy cyfrowej po transformacji z zastosowaniem korekt Hausbrandta*. Zeszyty Naukowe Politechniki Rzeszowskiej, seria: Budownictwo i Inżynieria Środowiska, z. 59 (1/12/II), 347–354.

Vanicek, P., and Steeves, R.R. (1996). Transformation of coordinates between two horizontal geodetic datums. *J. Geod.*, 70(11), 740–745. DOI: 10.1007/BF00867152.

Wisniewski, Z. (2005). *Rachunek wyrównawczy w geodezji (z przykładami)*. Olsztyn: Wyd. UW-M.

Wysocki, P. (2011). *Transformacje osnów pomiarowych. Zastosowanie transformacji w procesie odnowienia osnowy pomiarowej podczas modernizacji zbiorów zintegrowanego systemu informacji o nieruchomościach*. Wydawnictwo Polskiego Internetowego Informatora Geodezyjnego. Olsztyn: I-NET.PL Sp. J.

Zeng, H., and Yi, Q. (2011). Quaternion-based iterative solution of three-dimensional coordinate transformation problem. *J. Comput.*, 6(7), 1361–1368. DOI: 10.4304/jcp.6.7.1361-1368.

Zeng, H. (2015). Analytical algorithm of weighted 3D datum transformation using the constraint of orthonormal matrix. *Earth, Planets and Space*, 67.1, 1–10. DOI: 10.1186/s40623-015-0263-6.

Zeng, H., Yi, Q., and Wu, Y. (2016). Iterative approach of 3D datum transformation with a non-isotropic weight. *Acta Geod. Geophys.*, 51, 557–570. DOI: 10.1007/s40328-015-0144-2.

Zeng, H., Fang, X., Chang, G. et al. (2018). A dual quaternion algorithm of the Helmert transformation problem. *Earth Planets Space*, 70, 26. DOI: 10.1186/s40623-018-0792-x.

Zeng, H., Chang, G., He, H. et al. (2019). Iterative solution of Helmert transformation based on a unit dual quaternion. *Acta Geod. et Geophys.*, 54(1), 123–141. DOI: 10.1007/s40328-018-0241-0.