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A formula for the lower Bohl exponent of discrete time-varying systems

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In this note, a formula for the lower Bohl exponent of a discrete system with variable coefficients and weak variation was proved. This formula expresses the Bohl exponent through the eigenvalues of the coefficient matrix. Based on these formulas a necessary and sufficient condition for an uniform exponential instability of such systems is also presented.

Key words: time-varying systems, asymptotic stability, Bohl exponent

1. Introduction

One of the most frequently used models in control theory are linear time-invariant systems, however nowadays to cope with growing requirements formulated for control systems in the process of the building model we use linear time-varying systems (see e.g. [8, 17, 20] and the references therein). The dynamic properties of linear systems with variable coefficients can be described by various types of numerical characteristics such as Lapunov, Bohl, Perron exponents or central exponents (see [9]). These exponents are defined by solution norms or transition matrix norms, which makes them difficult to calculate. The computational difficulties of determining these characteristics are additionally increased by the fact that most of the characteristics are not a continuous function of the coefficients (see [16]) and therefore small inaccuracies of the system coefficients may cause large changes in the values of the characteristics. On the other hand, for time-invariant systems, a comprehensive description of the

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dynamic properties can be obtained through the spectrum of the system matrix. This brings to mind an attempt to express the numerical characteristics of systems with variable coefficients through the eigenvalues of the matrix of coefficients. In the general case, it is unfortunately impossible, because there are examples of exponentially uniformly stable continuous systems, whose coefficient matrices have spectra lying in the right half-plane, as well as examples of unstable systems with coefficient matrices with only eigenvalues with a negative real part (see e.g. [14, p. 257]). There are also analogous examples for discrete systems. It turns out, however, that if the coefficients of the system change slowly enough, then from the location of the spectra of the coefficient matrix, certain properties concerning the asymptotic properties of solutions can be deduced. This is the basic idea behind the so-called 'freezing method' initiated by Desoer's work [12]. A summary of the results obtained using this technique can be found in Section 10.1 of [13].

In this work we deal with the relationship between the smallest in absolute value eigenvalues of the coefficient matrix and the lower Bohl exponent of a discrete time-varying system. This exponent characterizes an uniform exponential instability, it is the smallest element of the spectrum of the exponential dichotomy and it is the exact boundary of mobility down of the lower Bohl exponents of trajectories of the solutions of perturbed systems under arbitrarily small perturbations of the coefficients matrix (see Theorem 1 below).

The main result of the work states that when the system has the so-called weak variance, then the lower Bohl exponent is the lower limit of the logarithms of the smallest module of eigenvalues of the coefficient matrix. A similar result for continuous systems and the upper Bohl exponent was obtained in the works of V.M. Millionschikov in [18] and by J. Daleckii and M.G. Krein in the monograph [11], p. 200. On the basis of the main result of the work, we formulated the necessary and sufficient condition for the uniform exponential instability of the discrete time-varying system with weak-variation. The work also contains a numerical example illustrating the obtained result.

In the work, we will use the following notation conventions: a sum $\sum\limits_{j=a}^{b}$ is equal to zero if $b < a$, $\mathbb{R}^{d\times d}$ is the set of all real matrices of size $d \times d$, $GL_d(\mathbb{R})$ is the subset of $\mathbb{R}^{d\times d}$ consisting of invertible matrices. For $A \in \mathbb{R}^{d\times d}$, $\mu(A)$ is the smallest and $\lambda(A)$ the greatest absolute value of the eigenvalues of matrix A, I_d is the identity matrix of size $d\times d$, $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^d$ and $\|A\|$ is the operator norm induced by Euclidean norm of a matrix $A \in \mathbb{R}^{d\times d}$.

We will consider systems of the following form

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{N}, \tag{1}$$

where $A = (A(n))_{n\in\mathbb{N}}$ is a sequence of invertible d by d matrices such that

$$a := \sup_{n\in\mathbb{N}} \max\left\{\|A(n)\|, \left\|A^{-1}(n)\right\|\right\} < \infty.$$

Let us denote by $(\Phi_A(n,m))_{n,m\in\mathbb{N}}$ the transition matrix of system (1) defined as follows

$$\Phi_A(n,m) = \begin{cases} A(n-1)\dots A(m) & \text{for } n > m, \\ I_d & \text{for } n = m, \\ \Phi_A^{-1}(m,n) & \text{for } n < m. \end{cases}$$

Definition 1 *The lower Bohl exponent $\omega(A)$ of system (1) is defined as follows*

$$\omega(A) = \liminf_{n-m\to\infty} \frac{1}{n-m} \ln \left\|\Phi_A^{-1}(n,m)\right\|^{-1}.$$

In the literature, together with the lower Bohl exponent, the upper Bohl exponent $\Omega(A)$ (see e.g. [14]) is also considered, defined as follows

$$\Omega(A) = \limsup_{n-m\to\infty} \frac{1}{n-m} \ln \|\Phi_A(n,m)\|.$$

It should also be noted that quantities $\Omega(A)$ and $\omega(A)$ are sometimes referred to in the literature as singular exponents ([19]) and general exponents [6]. The notion of Bohl exponent for continuous-time systems is due to P. Bohl [7].

The significance of the lower Bohl exponent for the theory of linear systems with variable coefficients results from the following theorem.

Theorem 1 *The lower Bohl exponent has the following properties:*

1) *system (1) uniformly exponentially unstable if and only if $\omega(A) > 0$.*

2) *$\omega(A)$ it is the smallest element of the spectrum of the monotonous exponential dichotomy of a system (1).*

3) *$\omega(A)$ is the exact boundary of mobility down of the lower Bohl exponents of trajectories of the solutions of perturbed systems under arbitrarily small perturbations of the coefficients matrix of the system (1).*

Proof. Point 1 results from point 2 and the definition of a spectrum for a uniform exponential dichotomy (see [3]). Point 2 was proved in [2], Lemma 4.1. Point 3 was proved in [5], Theorem 3.1. □

In our further consideration we will use the following results from the literature.

Theorem 2 (Gronwall's inequality) *(see [1], Theorem 4.1.9) Suppose that for all $n,\ m \in \mathbb{N}$, $m < n$ a sequence of non-negative numbers $(u(n))_{n\in\mathbb{N}}$ satisfies*

$$u(m) \leqslant pu(n) + q\sum_{i=m}^{n-1} u(i)$$

for certain $p,\ q > 0$, then

$$u(m) \leqslant pu(n)(1+q)^{m-n} \tag{2}$$

for all $n, m \in \mathbb{N}$, $m < n$.

Consider together with system (1) a nonhomogeneous system

$$y(n+1) = A(n)y(n) + f(n), \quad n \in \mathbb{N}, \tag{3}$$

where $f = (f(n))_{n\in\mathbb{N}}$ is a sequence of vectors from $\mathbb{R}^d$ and denote by $y = y(n, k_0, y_0, f)_{n\in\mathbb{N}}$, $n \geqslant k_0$ its solution with an initial condition $y(k_0, k_0, y_0, f) = y_0$. If $k_0 = 0$ we will write $y(n, y_0, f)_{n\in\mathbb{N}}$ instead of $y(n, 0, y_0, f)_{n\in\mathbb{N}}$.

Theorem 3 *(see section 2.5 in [1]) For any solution $y = y(n, k_0, y_0, f)_{n\in\mathbb{N}}$, $n \geqslant k_0$ of (3) we have*

$$y(n, k_0, y_0, f) = \Phi_A(n, k_0)\, y_0 + \sum_{j=k_0}^{n-1} \Phi_A(n, j+1) f(j), \quad n \in \mathbb{N}, \quad n \geqslant k_0\,. \tag{4}$$

Theorem 4 *(see [10]) The following equality holds*

$$\omega(A) = \lim_{N\to\infty}\left(\liminf_{k\to\infty} \ln \|\Phi_A(k, k+N)\|^{-1}\right). \tag{5}$$

2. Main result

Before we formulate and prove the main result, we will provide a lemma that will be used in the proof.

Lemma 1 *For each $c,\ \varepsilon > 0$ there exists a constant $D_\varepsilon(c) > 0$ such that for each $A \in GL_d(\mathbb{R})$ with $\max\left\{\|A\|,\ \left\|A^{-1}\right\|\right\} \leqslant c$ we have*

$$\|A^{-n}\| \leqslant D_\varepsilon(c)\left(\mu^{-1}(A) + \varepsilon\right)^n \tag{6}$$

for all $n \in \mathbb{N}$.

Proof. Since $\mu^{-1}(A) = \lambda\left(A^{-1}\right)$, then the statement of the Lemma is equivalent to the following: For each $c, \varepsilon > 0$ there exists a constant $D_\varepsilon(c) > 0$ such that for each $B \in GL_d(\mathbb{R})$ with $\max\left\{\|B\|, \left\|B^{-1}\right\|\right\} \leqslant c$ we have

$$\|B^n\| \leqslant D_\varepsilon(c)\left(\lambda(B)+\varepsilon\right)^n . \tag{7}$$

To prove the last fact let us fix $c, \varepsilon > 0$, $B \in GL_d(\mathbb{R})$ with $\|B\| < c$ and consider the contour

$$\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = \varepsilon + \lambda(B)\} .$$

Then (see Section 4.1 in [13])

$$B^n = -\frac{1}{2\pi i}\int\limits_\Gamma \lambda^n \left(B - \lambda I_d\right)^{-1} \mathrm{d}\lambda$$

for all $n \in \mathbb{N}$. Consequently,

$$\|B^n\| \leqslant M_\varepsilon \left(\varepsilon + \lambda(B)\right)^n , \tag{8}$$

where

$$M_\varepsilon = (\varepsilon + \lambda(B)) \sup_{|\lambda|=\varepsilon+\lambda(B)} \left\|(B - \lambda I_d)^{-1}\right\| .$$

For any $X \in \mathbb{R}^{d\times d}$ we have (see Theorems 5.6.9 and 1.2.12 in [15])

$$\lambda(X) \leqslant \|X\| \quad \text{and} \quad |\det X| \geqslant \mu^d(X)$$

and if additionally $M \in GL_d(\mathbb{R})$ then

$$\left\|X^{-1}\right\| \leqslant \frac{\|X\|^{d-1}}{|\det X|}$$

(see (2) in [4]). Using the last three inequalities and the fact $\|B\| \leqslant c$, we may estimate the constant M_ε as follows

$$\begin{aligned}
M_\varepsilon &\leqslant (\varepsilon + c) \sup_{|\lambda|=\varepsilon+\lambda(B)} \frac{\|B - \lambda I_d\|^{d-1}}{|\det(B - \lambda I_d)|} \\
&\leqslant (\varepsilon + c)\frac{\sup\limits_{|\lambda|=\varepsilon+\lambda(B)} \|B - \lambda I_d\|^{d-1}}{\inf\limits_{|\lambda|=\varepsilon+\lambda(B)} |\det(B - \lambda I_d)|} \leqslant (\varepsilon + c)\frac{\sup\limits_{|\lambda|=\varepsilon+\lambda(B)} (|B| + |\lambda|)^{d-1}}{\varepsilon^d} \\
&\leqslant (\varepsilon + c)\frac{(\|B\| + \varepsilon + \lambda(B))^{d-1}}{\varepsilon^d} \leqslant (\varepsilon + c)\frac{(2\|B\| + \varepsilon)^{d-1}}{\varepsilon^d} \\
&\leqslant (\varepsilon + c)\frac{(2c + \varepsilon)^{d-1}}{\varepsilon^d} .
\end{aligned}$$

Defining

$$D_\varepsilon(c) = (\varepsilon + c)\frac{(2c+\varepsilon)^{d-1}}{\varepsilon^d}$$

we get from (8) the inequality (7). □

Let us introduce the following definition.

Definition 2 *We will say that a sequence $A = (A(n))_{n\in\mathbb{N}}$ of d by d matrices or alternatively system (1) has a weak variation if*

$$\lim_{n\to\infty} \|A(n+1) - A(n)\| = 0.$$

The next theorem contains the main result of this note.

Theorem 5 *If system (1) has a weak variation, then*

$$\omega(A) = \liminf_{n\to\infty} \ln \mu\left(A(n)\right).$$

Proof. Let us fix $k \in \mathbb{N}$ and rewrite equation (1) in the following way

$$x(n+1) = A(k)x(n) + Q(n)x(n), \tag{9}$$

where

$$Q(n) = A(n) - A(k).$$

Moreover for $m,\ n \in \mathbb{N}$, $m < n$ denote

$$\delta_k(m,n) = \sup\left\{\|A(i) - A(k)\| : i \in \mathbb{N},\quad n-1 \geqslant i \geqslant m\right\}.$$

According to variation of constants formula (4)

$$\Phi_A(m,n) = A^{m-n}(k) - \sum_{j=m}^{n-1} A^{m-j-1}(k)Q(j)\Phi_A(j,n) \tag{10}$$

for all $m,\ n \in \mathbb{N}$, $m < n$. Let us fix $\varepsilon > 0$. Taking the norm on both sides of the last identity and using the inequality (6) we get

$$\begin{aligned}\|\Phi_A(m,n)\| \leqslant D_\varepsilon(a)\left(\mu^{-1}(A(k)) + \varepsilon\right)^{n-m} \\ + \delta_k(m,n)D_\varepsilon(a)\sum_{j=m}^{n-1}\left(\mu^{-1}(A(k)) + \varepsilon\right)^{j+1-m}\|\Phi_A(j,n)\|\end{aligned}$$

or equivalently

$$\left(\mu^{-1}(A(k))+\varepsilon\right)^{m}\|\Phi_A(m,n)\| \leqslant D_\varepsilon(a)\left(\mu^{-1}(A(k))+\varepsilon\right)^{n}$$
$$+\,\delta_k(m,n)D_\varepsilon(a)\left(\mu^{-1}(A(k))+\varepsilon\right)\sum_{j=m}^{n-1}\left(\mu^{-1}(A(k))+\varepsilon\right)^{j}\|\Phi_A(j,n)\|$$

for all $n, m \in \mathbb{N}$, $n > m$. Applying the Gronwall's inequality we obtain

$$\left(\mu^{-1}(A(k))+\varepsilon\right)^{m}\|\Phi_A(m,n)\|$$
$$\leqslant D_\varepsilon(a)\left(\mu^{-1}(A(k))+\varepsilon\right)^{n}\left(1+\delta_k(m,n)D_\varepsilon(a)\left(\mu^{-1}(A(k))+\varepsilon\right)\right)^{m-n},$$

and consequently

$$\frac{1}{n-m}\ln\|\Phi_A(m,n)\|^{-1} \geqslant \frac{1}{n-m}\ln D_\varepsilon^{-1}(a)$$
$$+\ln\left(\frac{1}{\mu^{-1}(A(k))+\varepsilon}+\delta_k(m,n)D_\varepsilon(a)\right),$$

for all $n, m \in \mathbb{N}$, $n > m$.

In particular for $m = k$ and $n = k + N$, $N \in \mathbb{N}$, $N > 0$ we have

$$\frac{1}{N}\ln\|\Phi_A(k,k+N)\|^{-1} \geqslant \frac{1}{N}\ln D_\varepsilon^{-1}(a)$$
$$+\ln\left(\frac{1}{\mu^{-1}(A(k))+\varepsilon}+\delta_k(k,k+N)D_\varepsilon(a)\right). \tag{11}$$

It is known (see e.g. [15]), that for any matrix $X \in \mathbb{R}^{d\times d}$ we have

$$\lambda(X) \leqslant \|X\|$$

and if additionally X is non-singular, then

$$\mu^{-1}(X) = \lambda\left(X^{-1}\right).$$

This implies that

$$a \geqslant \mu(A(k)).$$

The last inequality yields to the following estimate

$$\begin{aligned}
&\frac{1}{\mu^{-1}(A(k))+\varepsilon}+\delta_k(k,k+N)D_\varepsilon(a)\\
&\qquad\qquad= \frac{\mu(A(k))}{1+\varepsilon\mu(A(k))}+\delta_k(k,k+N)D_\varepsilon(a)\\
&\qquad\qquad= \mu(A(k))\left(\frac{1}{1+\varepsilon\mu(A(k))}+\frac{\delta_k(k,k+N)D_\varepsilon(a)}{\mu(A(k))}\right)\\
&\qquad\qquad\geqslant \mu(A(k))\left(\frac{1}{1+\varepsilon a}+\frac{\delta_k(k,k+N)D_\varepsilon(a)}{a}\right).
\end{aligned}$$

Using this estimate in (11) we get

$$\begin{aligned}
&\frac{1}{N}\ln\|\Phi_A(k,k+N)\|^{-1}\\
&\quad\geqslant \frac{1}{N}\ln D_\varepsilon^{-1}(a)+\ln\left(\mu(A(k))\left(\frac{1}{1+\varepsilon a}+\frac{\delta_k(k,k+N)D_\varepsilon(a)}{a}\right)\right)\\
&\quad= \frac{1}{N}\ln D_\varepsilon^{-1}(a)+\ln\left(\mu(A(k))\right)+\ln\left(\frac{1}{1+\varepsilon a}+\frac{\delta_k(k,k+N)D_\varepsilon(a)}{a}\right).
\end{aligned}\tag{12}$$

Since the sequence A has a weak variation, then

$$\lim_{k\to\infty}\delta_k(k,k+N)=0 \tag{13}$$

for all $N\in\mathbb{N}$ and therefore passing to the lower limit when k tends to infinity in (12) we obtain

$$\begin{aligned}
\liminf_{k\to\infty}\frac{1}{N}\ln\|\Phi_A(k,k+N)\|^{-1}\geqslant{}& \frac{1}{N}\ln D_\varepsilon^{-1}(a)\\
&+\liminf_{k\to\infty}\ln\left(\mu(A(k))\right)+\ln\left(\frac{1}{1+\varepsilon a}\right).
\end{aligned}$$

Passing to the limit when N tends to infinity in the last inequality and using (5) we get

$$\omega(A)\geqslant\liminf_{k\to\infty}\ln\left(\mu(A(k))\right)+\ln\left(\frac{1}{1+\varepsilon a}\right).$$

Due to the arbitrariness of $\varepsilon>0$, we obtain the inequality

$$\omega(A)\geqslant\liminf_{k\to\infty}\ln\left(\mu(A(k))\right). \tag{14}$$

Now we will prove the opposite inequality. Let us fix $k \in \mathbb{N}$ and consider again equation (1) in the form (9). Using (10) we have

$$A^{m-n}(k) = \Phi_A(m,n) + \sum_{j=m}^{n-1} A^{m-j-1}(k) Q(j) \Phi_A(j,n)$$

$$\Phi_A(m,n) \left(I_d + \sum_{j=m}^{n-1} \Phi_A^{-1}(m,n) A^{m-j-1}(k) Q(j) \Phi_A(j,n) \right).$$

Therefore

$$\|A^{m-n}(k)\| \leqslant \|\Phi_A(m,n)\| \left(1 + a^{2(n-m)} \delta_k(m,n) \sum_{j=m}^{n-1} \left\| A^{m-j-1}(k) \right\| \right)$$

and

$$\|\Phi_A(m,n)\|^{-1} \leqslant \|A^{m-n}(k)\|^{-1} \left(1 + a^{2(n-m)} \delta_k(m,n) \sum_{j=m}^{n-1} \left\| A^{-1}(k) \right\|^{j+1-m} \right)$$

$$\leqslant \|A^{m-n}(k)\|^{-1} \left(1 + a^{2(n-m)} \delta_k(m,n) \frac{a^{n+1} - a^{m+1}}{a^{m+1} - a^m} \right).$$

It is known (see e.g. [15]), that for any non-singular matrix $X \in \mathbb{R}^{d \times d}$ we have

$$\left\| X^{-1} \right\|^{-1} \leqslant \mu(X),$$

therefore the last inequality implies that

$$\|\Phi_A(m,n)\|^{-1} \leqslant \mu^{n-m}(A(k)) \left(1 + a^{2(n-m)} \delta_k(m,n) \frac{a^{n+1} - a^{m+1}}{a^{m+1} - a^m} \right)$$

for all $n, m \in \mathbb{N}$, $n \geqslant m$. Taking $m = k$ and $n = k + N$, $N \in \mathbb{N}$, $N > 0$ we obtain

$$\|\Phi_A(k, k+N)\|^{-1} \leqslant \mu^N(A(k)) \left(1 + a^{2N} \delta_k(k, k+N) \frac{a^{N+1} - a}{a - 1} \right)$$

and

$$\frac{1}{N} \ln \|\Phi_A(k, k+N)\|^{-1} \leqslant \ln\left(\mu(A(k))\right) + \ln \left(1 + a^{2N} \delta_k(k, k+N) \frac{a^{N+1} - a}{a - 1} \right)$$

Passing to the lower limit when k tends infinity and taking into account (13) we obtain

$$\liminf_{k \to \infty} \frac{1}{N} \ln \|\Phi_A(k, k+N)\|^{-1} \leqslant \liminf_{k \to \infty} \ln\left(\mu(A(k))\right)$$

Finally, passing to the limit when N tends to infinity in the last inequality and using (5) we get

$$\omega(A) \leqslant \liminf_{k\to\infty} \ln\left(\mu(A(k))\right). \tag{15}$$

From (14) and (15) we obtain the conclusion of the theorem. □

Considering the last theorem and Theorem 1 we express the following result.

Theorem 6 *System (1) with a weak variation is uniformly asymptotically unstable if and only if*

$$\liminf_{n\to\infty} \ln \mu\left(A(n)\right) > 0.$$

Now we will illustrate the obtained result on an example of the analysis of uniform exponential instability.

Example 1 Consider two dimensional system (1) with

$$A(n) = \begin{bmatrix} \frac{1}{3} & p_n \\ 1 & a \end{bmatrix},$$

where the bounded sequence $(p_n)_{n\in\mathbb{N}}$ of positive real numbers has a weak variation. Under this assumption about $(p_n)_{n\in\mathbb{N}}$ the sequence $(A(n))_{n\in\mathbb{N}}$ is bounded and has a weak variation. Suppose also that

$$\frac{a}{3} > \sup_{n\in\mathbb{N}} p_n. \tag{16}$$

Since $\det A(n) = \frac{1}{3}a - p_n$, then (16) implies that matrices $A(n)$, $n \in \mathbb{N}$ are invertible and the sequence $\left(A^{-1}(n)\right)_{n\in\mathbb{N}}$ is bounded. Moreover

$$\mu\left(A(n)\right) = \frac{1}{2}a - \sqrt{\left(\frac{a}{2} - \frac{1}{6}\right)^2 + p_n} + \frac{1}{6}$$

and by Theorem 5

$$\omega(A) = \liminf_{n\to\infty} \ln \lambda\left(A(n)\right) = \ln\left(\frac{1}{2}a - \sqrt{\left(\frac{a}{2} - \frac{1}{6}\right)^2 + \underline{p}} + \frac{1}{6}\right),$$

where

$$\underline{p} = \liminf_{n\to\infty} p_n.$$

From Theorem 6 it follows that the considered system is uniformly asymptotically unstable if and only if

$$a < -\frac{3}{2}\underline{p} + 1.$$

3. Conclusion

In this note, we showed that the lower Bohl exponent of discrete systems with a weak variation is equal to the lower limit of the logarithms of the minimal absolute value of the eigenvalues of the coefficient matrix. From this theorem it follows a necessary and sufficient condition for uniform exponential instability given in terms of eigenvalues of coefficient matrices. The obtained results are illustrated on a numerical example.

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