BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 71(4), 2023, Article number: e146287
DOI: 10.24425/bpasts.2023.146287

CONTROL, INFORMATICS AND ROBOTICS

Results on the controllability of Caputo's fractional descriptor systems with constant delays

Beata Sikora*

Department of Applied Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland

Abstract. The paper investigates the controllability of fractional descriptor linear systems with constant delays in control. The Caputo fractional derivative is considered. Using the Drazin inverse and the Laplace transform, a formula for solving of the matrix state equation is obtained. New criteria of relative controllability for Caputo's fractional descriptor systems are formulated and proved. Both constrained and unconstrained controls are considered. To emphasize the importance of the theoretical studies, an application to electrical circuits is presented as a practical example.

Key words: descriptor systems; fractional-order systems; Caputo derivative; Drazin inverse; controllability.

1. INTRODUCTION

Differential calculus of fractional order is the field of mathematics that is developed most rapidly in the 21st century. With the rapid development of modern science and technology, it turns out that the theory of fractional differential calculus, as a new theoretical basis and mathematical tool, contributes to the development of many scientific fields [1–4]. One of the reasons for using fractional operators in the description of real processes is their nature of inheritance. It is particularly suitable for describing many physical processes, such as viscoelastic systems [5], heat flow models [6], ultracapacitor models [7], epidemiological models [8] and so on, with memory characteristics and some historical dependencies. Many the systems are descriptor (singular) systems, see also [9–13].

The study of controllability is a fundamental issue in control theory. Controllability of fractional-order linear control systems has been studied in many monographs and papers [14–19]. However, for many processes, the final state depends not only on the input data of the system, but also on past states and controls. This nonlocal property means that the equations describing the processes contain delays in the state or control. Because of the multitude of mathematical models describing systems with delays in control, the study of the controllability of such control systems seems particularly important. The controllability issues for continuous fractional-order systems with delayed control were investigated in [20–28], among others.

The Drazin matrix has rarely been applied to study the controllability of continuous fractional control systems. Controllability and observability for linear fractional systems without delays were considered in [29], the existence of solution for fractional-order descriptor systems was derived in [30], the minimum energy control of descriptor fractional-order systems was studied [31], sliding-mode control for nonlinear fractional-order systems was studied in [32]. Thus, the goal of the paper is to investigate controllability issues for linear continuous-time Caputo's descriptor systems with constant delays in the control, to find the solution of the state equation as well as to give criteria for the controllability of the systems with constrained and unconstrained controls.

The structure of the paper is as follows. Section 2 contains basic formulas and definitions needed in this study. It also explains the notation used. Section 3 presents the mathematical model of the studied Caputo's descriptor systems with delays in control. The solution of the discussed fractional matrix differential equation is derived. The main results of the study – new controllability criteria – are given in Section 4. Section 5 provides a practical example to illustrate the theoretical results. Finally, concluding remarks are made in Section 6.

2. PRELIMINARIES

This section introduces some definitions of basic terms that will be used in later sections, and indicates some notations used throughout the paper.

For the system description we use the Caputo fractional derivative. Using the Caputo derivative we ensure that the Cauchy conditions for differential equations of fractional order are similar to those for the case of integer order, which makes them interpretable in the same way [1].

Definition 1. If $f\colon \mathbb{R}_+ \to \mathbb{R}$, $\alpha \in \langle n-1, n)$, $n \in \mathbb{N}$, and Γ stands for the gamma function, then

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\alpha-n+1}}\,\mathrm{d}\eta, \qquad (1) $$

is called the Caputo fractional derivative of order α, provided it exists.

*e-mail: beata.sikora@polsl.pl

Manuscript submitted 2023-01-16, revised 2023-05-14, initially accepted for publication 2023-06-09, published in August 2023.

In the search for solutions for differential equations of fractional order, the one- and two-parameter Mittag-Leffler functions E_α, $E_{\alpha,\beta}$ are of essential importance. They are defined as

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z \in \mathbb{C}, \ \ \alpha > 0, \tag{2}$$

and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C}, \ \ \alpha > 0, \ \ \beta > 0. \tag{3}$$

On the basis of the Mittag-Leffler functions the following matrices can be defined [4, 15].

Definition 2. Let A be a square system matrix of ${}^CD^\alpha(t) = Ax(t)$, $\alpha > 0$. The matrix

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{+\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}$$

is called the *pseudo-transition matrix* of the system. Moreover,

$$\begin{aligned}\Phi(t) &= t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) \\ &= t^{\alpha-1} \sum_{k=0}^{+\infty} \frac{A^k t^{k\alpha}}{\Gamma((k+1)\alpha)}.\end{aligned}$$

It follows from Definition 2 that for $\alpha \in (0,1)$,

$$\Phi_0(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \Phi(t).$$

The following two definitions will also be helpful in further considerations.

Definition 3. [33] Let A be a n-th order square matrix of complex variable. The index of A ($\operatorname{Ind}(A)$) is the smallest number $q \in \mathbb{N} \cup \{0\}$ such that

$$\operatorname{rank}(A^q) = \operatorname{rank}(A^{q+1}).$$

Definition 4. [34] The unique solution of all the following equations

1. $AX = XA$
2. $XAX = X$
3. $XA^{q+1} = A^q$, where $q = \operatorname{Ind}(A)$.

is called the Drazin inverse of A. It is denoted as A^D.

Remark 1. Given a square matrix A, its Drazin inverse A^D exists uniquely, and assuming $\det A \neq 0$, we have $A^D = A^{-1}$.

Other notations used in the paper are: $\mathbb{R}^m$ – the real vector space of dimension m, $L^2_{\text{loc}}(\langle 0,+\infty),\mathbb{R}^m)$ – the space of locally square integrable functions with values in $\mathbb{R}^m$, $\mathscr{M}_{n\times n}(\mathbb{R})$ – the set of n-th order matrices with real entries.

3. SYSTEM DESCRIPTION

Consider linear fractional-order descriptor systems with constant delays in control functions. The systems are described by the matrix differential state equation with the Caputo derivative as follows

$$E\,{}^CD^\alpha x(t) = Ax(t) + \sum_{i=0}^{M} B_i u(t-h_i), \quad t \geq 0, \tag{4}$$

where: $\alpha \in (0,1)$, $x(t) \in \mathbb{R}^n$ is the pseudo-state vector, $u(t) \in \mathbb{R}^m$ is the input vector (control), $u \in L^2_{\text{loc}}(\langle 0,+\infty),\mathbb{R}^m)$, $E \in \mathscr{M}_{n\times n}(\mathbb{R})$ is singular ($\det E = 0$) with rank $n_e < n$, $A \in \mathscr{M}_{n\times n}(\mathbb{R})$, for each $i = 0,1,\ldots,M$, $B_i \in \mathscr{M}_{n\times m}(\mathbb{R})$, and $h_i\colon \langle 0,t_1\rangle \to \mathbb{R}$, $i = 0,1\ldots,M$, are such constant delays, that $\forall_{i=0,1\ldots,M-1}\ h_i < h_{i+1}$.

For the system (4) the initial complete state $z(0) = \{x(0), u_0\}$, where $u_t(s) = u(s)$ for $s \in \langle t-h_M, t)$, is considered.

In the case of $\alpha \in (0,1)$, $n = 1$, thus

$${}^CD^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(\eta)}{(t-\eta)^\alpha}\,\mathrm{d}\eta,$$

where $x'(t) = \dfrac{\mathrm{d}x}{\mathrm{d}t}$.

We assume that for the singular matrix E the pencil (E,A) of the fractional control system (4) is regular, i.e. there exists $s \in \mathbb{C}$ such that

$$\det[Es^\alpha - A] \neq 0.$$

Thus, let us select a number $s_e \in \mathbb{C}$ for which $\det[Es_e^\alpha - A] \neq 0$. Then $[Es_e^\alpha - A]$ has the inverse $[Es_e^\alpha - A]^{-1}$. Premultiplying the equation (4) by the inverse, we obtain

$$\begin{aligned}{[Es_e^\alpha - A]}^{-1} E\,{}^CD^\alpha x(t) &= [Es_e^\alpha - A]^{-1} Ax(t) \\ &\quad + \sum_{i=0}^{M} [Es_e^\alpha - A]^{-1} B_i u(t-h_i).\end{aligned}$$

Introducing the following denotation

$$\begin{aligned}&\widetilde{E} = [Es_e^\alpha - A]^{-1} E, \quad \widetilde{A} = [Es_e^\alpha - A]^{-1} A, \\ &\widetilde{B}_i = [Es_e^\alpha - A]^{-1} B_i\end{aligned} \tag{5}$$

for $i = 0,1\ldots,M$, we have

$$\widetilde{E}\,{}^CD^\alpha x(t) = \widetilde{A}x(t) + \sum_{i=0}^{M} \widetilde{B}_i u(t-h_i). \tag{6}$$

Remark 2. The fractional differential equations (4) and (6) are equivalent, which means that the solution $x = x(t)$ for (4) is also the solution for (6).

There exist (and are unique) the Drazin inverses $\widetilde{A}^D$ and $\widetilde{E}^D$ of the matrices $\widetilde{A}$ and $\widetilde{E}$, respectively.

Lemma 1. [30, 34] For $\widetilde{A}$ and $\widetilde{E}$ given by the formulas (5), all the following conditions hold

1. $\widetilde{A}\widetilde{E} = \widetilde{E}\widetilde{A}$, $\widetilde{A}^D\widetilde{E} = \widetilde{E}\widetilde{A}^D$, $\widetilde{A}\widetilde{E}^D = \widetilde{E}^D\widetilde{A}$, $\widetilde{A}^D\widetilde{E}^D = \widetilde{E}^D\widetilde{A}^D$,
2. $\ker\widetilde{A} \cap \ker\widetilde{E} = \{0\}$,

3. $\widetilde{E} = C \begin{bmatrix} F & 0 \\ 0 & N \end{bmatrix} C^{-1}$, $\widetilde{E}^D = C \begin{bmatrix} F^{-1} & 0 \\ 0 & 0 \end{bmatrix} C^{-1}$
   where $\det C \neq 0$, $F \in \mathscr{M}_{n_1 \times n_1}(\mathbb{R})$ and $\det F \neq 0$, $N \in \mathscr{M}_{n_2 \times n_2}(\mathbb{R})$ is nilpotent, $n_1 + n_2 = n$,
4. $(I_n - \widetilde{E}\widetilde{E}^D)\widetilde{A}\widetilde{A}^D = I_n - \widetilde{E}\widetilde{E}^D$, $(I_n - \widetilde{E}\widetilde{E}^D)(\widetilde{E}\widetilde{A}^D)^q = 0$,
   where $I_n \in \mathscr{M}_{n\times n}(\mathbb{R})$ – the identity matrix, $q = \operatorname{Ind}(\widetilde{E})$.

The next theorem gives the solution for the fractional differential equation (4).

Theorem 1. For any admissible control function $u \in L^2_{\text{loc}}(\langle 0, +\infty), \mathbb{R}^m)$ there exists a unique solution $x = x(t)$ of the state equation (4) given by

$$x(t) = \hat{\Phi}_0(t)\widetilde{E}\widetilde{E}^D v + \widetilde{E}^D \int_0^t \hat{\Phi}(t-\eta) \sum_{i=0}^M \widetilde{B}_i u(\eta - h_i)\,\mathrm{d}\eta$$
$$+ \left(\widetilde{E}\widetilde{E}^D - I_n\right) \sum_{k=0}^{q-1} \left(\widetilde{E}\widetilde{A}^D\right)^k \widetilde{A}^D \sum_{i=0}^M \widetilde{B}_i u^{(k\alpha)}(t - h_i), \quad t \geq 0, \tag{7}$$

where $v \in \mathbb{R}^n$ is an arbitrary vector,
$q = \operatorname{Ind}(\widetilde{E})$, $u^{(k\alpha)}(t) = {}^C\!D^{k\alpha} u(t)$, $\hat{\Phi}_0(t) = \sum_{k=0}^{+\infty} \frac{(\widetilde{E}^D\widetilde{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)}$, and
$\hat{\Phi}(t) = \sum_{k=0}^{+\infty} \frac{(\widetilde{E}^D\widetilde{A})^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}$.

Proof. To prove that $x = x(t)$ given by (7) is the solution of (4) we will apply Remark 2. We will prove that the function $x = x(t)$ is the solution of the matrix differential equation (6). We substitute (7) into equation (6), then we apply properties of the Caputo fractional derivative, Definition 2 for matrices $\widetilde{E}$ and $\widetilde{A}$, and Lemma 1. Hence, we have

$$\widetilde{E}\,{}^C\!D^\alpha x(t) = \widetilde{E}\,{}^C\!D^\alpha \Bigg[\hat{\Phi}_0(t)\widetilde{E}\widetilde{E}^D v + \widetilde{E}^D \int_0^t \hat{\Phi}(t-\eta) \sum_{i=0}^M \widetilde{B}_i u(\eta - h_i)\,\mathrm{d}\eta + (\widetilde{E}\widetilde{E}^D - I_n) \sum_{k=0}^{q-1} (\widetilde{E}\widetilde{A}^D)^k \widetilde{A}^D \sum_{i=0}^M \widetilde{B}_i u^{(k\alpha)}(t-h_i)\Bigg]$$
$$= \widetilde{E}\,{}^C\!D^\alpha [\widetilde{E}\widetilde{E}^D v] + \widetilde{E}\,{}^C\!D^\alpha \left[\sum_{k=0}^{+\infty} \frac{(\widetilde{E}^D\widetilde{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)} v\right]$$
$$+ \widetilde{E}\,{}^C\!D^\alpha \Bigg[\widetilde{E}^D \int_0^t \frac{(t-\eta)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^M \widetilde{B}_i u(\eta - h_i)\,\mathrm{d}\eta + \widetilde{E}^D \int_0^t \sum_{k=0}^{+\infty} \frac{(\widetilde{E}^D\widetilde{A})^{k+1}(t-\eta)^{(k+2)\alpha-1}}{\Gamma((k+2)\alpha)} \sum_{i=0}^M \widetilde{B}_i u(\eta-h_i)\,\mathrm{d}\eta\Bigg]$$
$$+ \widetilde{E}\,{}^C\!D^\alpha \left[(\widetilde{E}\widetilde{E}^D - I_n) \sum_{k=0}^{q-1} (\widetilde{E}\widetilde{A}^D)^k \widetilde{A}^D \sum_{i=0}^M \widetilde{B}_i u^{(k\alpha)}(t-h_i)\right]$$
$$= \sum_{k=0}^{+\infty} \frac{\widetilde{E}(\widetilde{E}^D\widetilde{A})^{k+1} t^{k\alpha}}{\Gamma(k\alpha+1)} v + \widetilde{E}^D \sum_{i=0}^M \widetilde{B}_i u(\eta - h_i) + (\widetilde{E}^D)^2 \widetilde{A} \int_0^t \hat{\Phi}(t-\eta) \sum_{i=0}^M \widetilde{B}_i u(\eta-h_i)\,\mathrm{d}\eta$$
$$+ (\widetilde{E}\widetilde{E}^D - I_n) \sum_{k=0}^{q-1} (\widetilde{E}\widetilde{A}^D)^k \widetilde{A}^D \sum_{i=0}^M \widetilde{B}_i u^{(k\alpha)}(t-h_i)$$
$$= A \Bigg[\hat{\Phi}_0(t)\widetilde{E}\widetilde{E}^D v + \widetilde{E}^D \int_0^t \hat{\Phi}(t-\eta) \sum_{i=0}^M \widetilde{B}_i u(\eta - h_i)\,\mathrm{d}\eta + (\widetilde{E}\widetilde{E}^D - I_n) \sum_{k=0}^{q-1} (\widetilde{E}\widetilde{A}^D)^k \widetilde{A}^D \sum_{i=0}^M \widetilde{B}_i u^{(k\alpha)}(t-h_i)\Bigg] + \sum_{i=0}^M B_i u(t-h_i),$$

since the Caputo derivative keeps the Mittag-Leffler function invariant, ${}^C\!D^\alpha[\widetilde{E}\widetilde{E}^D v] = 0$, $\widetilde{E}(\widetilde{E}^D\widetilde{A})^{k+1} = \widetilde{A}^{k+1}(\widetilde{E}^D)^k$, $\hat{\Phi}(t) = \sum_{k=0}^{+\infty} \frac{(\widetilde{E}^D\widetilde{A})^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \sum_{k=0}^{+\infty} \frac{(\widetilde{E}^D\widetilde{A})^{k+1}(t-\eta)^{(k+2)\alpha-1}}{\Gamma((k+2)\alpha)}$, and the fourth item of Lemma 1 holds.

This proves that the function $x = x(t)$ defined by (7) satisfies equation (6). Moreover, for $t = 0$ in (7) we have

$$x(0) = \widetilde{E}\widetilde{E}^D v + \left(\widetilde{E}\widetilde{E}^D - I_n\right) \sum_{k=0}^{q-1} \left(\widetilde{E}\widetilde{A}^D\right)^k \widetilde{A}^D \sum_{i=0}^M \widetilde{B}_i u^{(k\alpha)}(0-h_i) \tag{8}$$

for an arbitrary vector $v \in \mathbb{R}^n$, where $\sum_{i=0}^M u^{(k\alpha)}(-h_i) = u_0$. This means that initial conditions $z(0) = \{x(0), u_0\}$ should satisfy the equality (8). Especially, for $u_0 = 0$ we obtain $x(0) = \widetilde{E}\widetilde{E}^D v$, hence $x(0)$ is an element of the image of $\widetilde{E}\widetilde{E}^D$. □

Definition 5. A set

$$K(t) = \Bigg\{x(t) \in \mathbb{R}^n : x(t) = \hat{\Phi}_0(t)\widetilde{E}\widetilde{E}^D v + \widetilde{E}^D \int_0^t \hat{\Phi}(t-\eta) \sum_{i=0}^M \widetilde{B}_i u(\eta-h_i)\,\mathrm{d}\eta + (\widetilde{E}\widetilde{E}^D - I_n) \sum_{k=0}^{q-1} (\widetilde{E}\widetilde{A}^D)^k \widetilde{A}^D \sum_{i=0}^M \widetilde{B}_i u^{(k\alpha)}(t-h_i)\Bigg\} \tag{9}$$

for some $v \in \mathbb{R}^n$, is said to be the attainable set for the fractional system (4).

4. CONTROLLABILITY CRITERIA

In fractional differential systems, the so-called memory effect occurs. It refers to the fact that the output of such systems at a given time depends not only on the current input, but also on the past inputs and outputs of the system. In other words, the behavior of the system is affected by its history. This effect is a consequence of the nonlocal nature of fractional derivatives, which involve integrals of the input signal over a time interval, not just the current time.

In case of delays in control systems, the so-called relative controllability is taken into account. Thus, in this paper we consider relative controllability on an $\langle 0,t_1\rangle$. This means that the goal is to determine a control u that will take the considered system from the initial state $z(0)=(x(0),u_0)$ to a final state $x=x(t_1)$. Necessary definitions are presented below.

Definition 6. If for any complete initial state $z(0)=(x(0),u_0)$ and any vector $\hat{x}\in\mathbb{R}^n$ there exists a control $\hat{u}\in L^2_{\text{loc}}(\langle 0,+\infty),\mathbb{R}^m)$ for which

$$x(t_1)=x(t_1,z(0),\hat{u})=\hat{x},$$

then Caputo's fractional descriptor system (4) is called relatively controllable on $\langle 0,t_1\rangle$.

In particular, we say about null controllability as defined below.

Definition 7. If for any complete initial state $z(0)=(x(0),u_0)$ there exists a control $\hat{u}\in L^2_{\text{loc}}(\langle 0,+\infty),\mathbb{R}^m)$ for which the solution of (7) at $t=t_1$ is

$$x(t_1)=x(t_1,z(0),\hat{u})=0,$$

then Caputo's fractional descriptor system (4) is called relatively null controllable on $\langle 0,t_1\rangle$.

If control values are bounded, that is $u(t)\in U\subset\mathbb{R}^m$, the system is called relatively null U-controllable.

The memory effect of fractional systems can be understood in terms of the Mittag-Leffler function which appears in the solution of fractional differential equations. This function describes the decay of the system response over time, and its properties depend on the order of the fractional derivative and the initial conditions of the system. Linear fractional systems can exhibit long-tail effects depending on the nature of the system and the input signal. The long-tail effect can occur when the system has poles close to the imaginary axis or when it has a large number of poles with small damping coefficients. For this reason, for very general cases, the definitions of controllability and, in particular, null controllability are generalized (see [35]), taking into account the possibility of the occurrence of the long-tail effect.

Remark 3. The Kalman-type rank condition presented in this paper is based on the Mittag-Leffler function and the controls are sufficiently smooth to ensure controllability in a given interval in terms of Definition 6 and Definition 7. There is no need to generalize the definitions.

Using the substitution rule and the properties of definite integrals, we rewrite (7) as follows

$$x(t)=\hat{\Phi}_0(t)\widetilde{E}\widetilde{E}^Dv+\sum_{i=0}^{M}\int_{-h_i}^{t-h_i}\widetilde{E}^D\hat{\Phi}(t-\eta-h_i)\widetilde{B}_iu(\eta)\,\mathrm{d}\eta + (\widetilde{E}\widetilde{E}^D-I_n)\sum_{k=0}^{q-1}(\widetilde{E}\widetilde{A}^D)^k\widetilde{A}^D\sum_{i=0}^{M}\widetilde{B}_iu^{(k\alpha)}(t),\quad t\geq 0. \tag{10}$$

To formulate controllability criteria, we first define the following matrix

$$W_E(0,t)=\sum_{i=0}^{M}\int_{-h_i}^{t-h_i}\widetilde{E}^D\hat{\Phi}(t-\eta-h_i)\widetilde{B}_i\widetilde{B}_i^T\hat{\Phi}^T(t-\eta-h_i)(\widetilde{E}^D)^T\,\mathrm{d}\eta,$$

where the symbol ^{T} means the transpose.

Theorem 2. The descriptor fractional system (4) is relatively controllable on $\langle 0,t_1\rangle$ if and only if

$$\operatorname{rank}W_E(0,t_1)=n. \tag{11}$$

Proof. $(\Rightarrow)$ Let the descriptor system (4) be relatively controllable on $\langle 0,t_1\rangle$. Assume that $\operatorname{rank}W_E(0,t_1)<n$. It follows that $W_E(0,t_1)$ is singular. It follows that there is a nonzero vector $\widetilde{x}$, such that

$$\widetilde{x}^TW_E(0,t_1)\widetilde{x}=0,$$

that is

$$\sum_{i=0}^{M}\int_{-h_i}^{t-h_i}\widetilde{x}^T\widetilde{E}^D\hat{\Phi}(t-\eta-h_i)\widetilde{B}_i\widetilde{B}_i^T\hat{\Phi}^T(t-\eta-h_i)(\widetilde{E}^D)^T\widetilde{x}\,\mathrm{d}\eta=0.$$

Thus, for $t\in\langle 0,t_1\rangle$, we have

$$\widetilde{x}^T\widetilde{E}^D\hat{\Phi}(t_1-t-h_i)\widetilde{B}_i=0. \tag{12}$$

Since the system (4) is controllable, it can be steered from z_0 to any final state $x(t_1)\in\mathbb{R}^n$. Therefore, there exists a control u_0 that takes the system from the initial state z_0 to zero, namely

$$0=\widetilde{E}^Dx(t_1)=\widetilde{E}^D\hat{\Phi}_0(t)\widetilde{E}\widetilde{E}^Dv + \widetilde{E}^D\sum_{i=0}^{M}\int_{-h_i}^{t-h_i}\widetilde{E}^D\hat{\Phi}(t-\eta-h_i)\widetilde{B}_iu_0(\eta)\,\mathrm{d}\eta. \tag{13}$$

Furthermore, there exists a control $\widetilde{u}$ that takes the system (4) from the initial state z_0 to the state $\widetilde{x}$, hence

$$\widetilde{E}^D\widetilde{x}=\widetilde{E}^D\hat{\Phi}_0(t_1)\widetilde{E}\widetilde{E}^Dv +\widetilde{E}^D\sum_{i=0}^{M}\int_{-h_i}^{t-h_i}\widetilde{E}^D\hat{\Phi}(t-\eta-h_i)\widetilde{B}_i\widetilde{u}(\eta)\,\mathrm{d}\eta. \tag{14}$$

Comparing the equalities (13)–(14), it follows that

$$\widetilde{E}^D\widetilde{x}-\widetilde{E}^D\sum_{i=0}^{M}\int_{-h_i}^{t-h_i}\widetilde{E}^D\hat{\Phi}(t-\eta-h_i)\widetilde{B}_i\left[\widetilde{u}(\eta)-u_0\right]\mathrm{d}\eta=0.$$

Next we multiply the above equality by $\widetilde{x}^T$ and apply (12). We find that $\widetilde{E}^D\widetilde{x}^T\widetilde{x}=0$ and thus $\widetilde{x}=0$. It contradicts the assumption. We proved that the $W_E(0,t_1)$ is nonsingular, and hence $\operatorname{rank}W_E(0,t_1)=n$.

($\Leftarrow$) To prove the sufficient condition, we take any complete initial state $z(0)=\{x(0),u_0\}$ of the fractional system (4) and any vector $x_1\in\mathbb{R}^n$. By the assumption $W_E(0,t_1)=n$, $W_E(0,t_1)$ is nonsingular. Thus, there exists $W_E^{-1}(0,t_1)$. Let us consider a control

$$\widetilde{u}(t)=\widetilde{E}^D\widetilde{B}_i^T\hat{\Phi}^T(t_1-t-h_i)\big(\widetilde{E}^D\big)^TW_E^{-1}(0,t_1)\\ \cdot\big(-E^D\hat{\Phi}_0(t)\widetilde{E}\widetilde{E}^D\nu\big)$$

for a given vector $\nu\in\mathbb{R}^n$. We will prove that the control defined above takes the system (4) to the final state $x(t_1)=x(t_1,z(0),\widetilde{u})$. From Theorem 1 and properties of the Drazin inverse it follows that

$$\widetilde{E}^Dx(t_1)=\widetilde{E}^D\hat{\Phi}_0(t_1)\widetilde{E}\widetilde{E}^D\nu\\ +\sum_{i=0}^{M}\int_{-h_i}^{t_1-h_i}\widetilde{E}^D\hat{\Phi}(t_1-\eta-h_i)\widetilde{B}_iu(\eta)\,\mathrm{d}\eta.$$

Therefore, for the control $\widetilde{u}(t)$ defined above we have

$$\widetilde{E}^Dx(t_1)=\widetilde{E}^D\hat{\Phi}_0(t_1)\widetilde{E}\widetilde{E}^D\nu\\ +W_E(0,t_1)W_E^{-1}(0,t_1)\left(-\widetilde{E}^D\hat{\Phi}_0(t)\widetilde{E}\widetilde{E}^D\nu\right)=0.$$

Premultiplying both sides by $\widetilde{A}^D$, we have

$$\widetilde{A}^D\widetilde{E}^Dx(t_1)=0.$$

From Lemma 1(1) implies, that $\widetilde{E}^Dx(t_1)\in\ker\widetilde{A}^D$ and $\widetilde{A}^Dx(t_1)\in\ker\widetilde{E}^D$. Since $\ker\widetilde{A}^D\cap\ker\widetilde{E}^D=\{0\}$, it follows that $x(t_1)=0$. Hence, Caputo's fractional system (4) is relatively controllable on $\langle 0,t_1\rangle$. □

Next, we consider constraints put on the control values, i.e., $u(t)\in U\subset\mathbb{R}^m$. To formulate a criterion for constrained controls, we introduce the asymptotic stability condition for linear descriptor systems of fractional order.

Lemma 2. [36] Assume that $\operatorname{spec}(E,A)=\{\lambda\in\mathbb{C}:\det(E\lambda^\alpha-A)=0\}$ is a set of finite modes for the pair (E,A). The descriptor fractional systems (4) is asymptotically stable if

$$|\arg\left(\operatorname{spec}(E,A)\right)|>\alpha\frac{\pi}{2}. \tag{15}$$

Theorem 3. Assume that U is a convex and compact subset of $\mathbb{R}^m$ and $0\in\operatorname{int}U$. If

$$\operatorname{rank}W_E(0,t_1)=n \tag{16}$$

and the pair (E,A) satisfies the condition (15), then Caputo's descriptor system (4) is relatively null U-controllable on $\langle 0,t_1\rangle$.

Proof. Let $\operatorname{rank}W_E(0,t_1)=n$. Hence, based on Theorem 2, the descriptor system (4) without constraints is relatively controllable on $\langle 0,t_1\rangle$. Moreover, if the pair (E,A) satisfies condition (15), then the system (4) is asymptotically stable according to Lemma 2. We will show that it can be steered to $0\in\mathbb{R}^n$ in finite time. Since $u\in L^2_{\mathrm{loc}}(\langle 0,+\infty),\mathbb{R}^m$, U is a convex and compact subset of $\mathbb{R}^m$ and $0\in\operatorname{int}U$, the attainable set $K(t)$ defined by (9) also is convex and compact and $0\in intK(t)$. Due to the asymptotic stability of the system (4), $x=0$ is the solution of (4) for the control $u(t)=0$. By means of this null control, the system is steered into a neighborhood $N(0)$ of $0\in\mathbb{R}^n$, i.e. $x(t,z(0),0)\in N(0)$ and $\lim_{t_1\to+\infty}x(t_1,z(0),0)=0$. Thus, it is possible to control any state $x(t_1,z(0),0)$ of the fractional descriptor system (4) to $0\in\mathbb{R}^n$. This proves the relative null U-controllability on $\langle 0,t_1\rangle$. □

5. PRACTICAL EXAMPLE

In this section, real-life system modeled by Caputo's fractional singular equation (4) is presented as possible applications of the obtained theoretical results. The system is an electrical circuit which scheme is presented in Fig. 1.

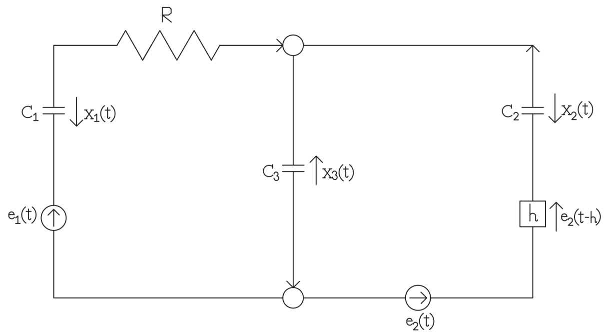

Fig. 1. Electrical circuit

The fractional electrical circuits are a generalization of the classical circuits. The fractional time components change the time constant and affect the transient response of the system. For example, it was proved in [37] that the obtained fractional differential system better describes the measurement of electrical impedance spectroscopy. The model of resistance, inductance, capacitance circuit using the fractional derivative is presented in [38]. For more details, see also [39–41].

In many electrical circuit applications, there are delays in state or control. A delay(s) of a few seconds or minutes may be required to ensure proper circuit operation, where without a specified delay the circuit could malfunction or even get damaged.

Example 1. Suppose that an electrical circuit presented in Fig. 1 is considered. A resistance R and capacitances C_1, C_2, C_3 are given constants. Also, we have constant delay h, and the current voltage sources (controls) e_1, e_2, for $0<\alpha<1$.

The equations describing the system can be derived from Kirchhoff's laws as follows

$$\begin{cases} e_1(t)=RC_1\,{}^CD^\alpha x_1(t)+x_1(t)+x_3(t),\\ e_2(t-h)=x_2(t)+x_3(t),\\ C_1\,{}^CD^\alpha x_1(t)+C_2\,{}^CD^\alpha x_2(t)-C_3\,{}^CD^\alpha x_3(t)=0.\end{cases}$$

Hence, the matrix state equation has the form

$$E\,{}^{C}D^{\alpha}x(t) = Ax(t) + B_0u(t) + B_1u(t-h), \tag{17}$$

for $t \geq 0$ and $u(t) \in \langle 0, +\infty)$, where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \qquad u(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix},$$

$$E = \begin{bmatrix} RC_1 & 0 & 0 \\ 0 & 0 & 0 \\ C_1 & C_2 & -C_3 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore, we get the descriptor fractional system (4) that has a regular pencil of the pair (E, A), because $\det E = 0$ and

$$\det[Es^{\alpha} - A] = \det \begin{bmatrix} RC_1s^{\alpha}+1 & 0 & 1 \\ 0 & 1 & 1 \\ C_1s^{\alpha} & C_2s^{\alpha} & -C_3s^{\alpha} \end{bmatrix}$$
$$= -(RC_1s^{\alpha}+1)(C_3s^{\alpha}+C_2s^{\alpha}) - C_1s^{\alpha}.$$

Taking specific values of constants, for example $R = 10, C_1 = 1, C_2 = 1, C_3 = 1$ with delay $h = 1$ and $\alpha = \frac{1}{2}$, on the interval $\langle 0, 2\rangle$ we have

$$W_E(0,2) = \sum_{i=0}^{1} \int_{-h_i}^{2-h_i} \widetilde{E}^D \hat{\Phi}(2-\eta-h_i)\widetilde{B}_i\widetilde{B}_i^T\hat{\Phi}^T(2-\eta-h_i)\left(\widetilde{E}^D\right)^T \mathrm{d}\eta,$$

where $h_0 = 0$ and $h_1 = h$. Let us find the matrices $\widetilde{E}$, $\widetilde{A}$ and $\widetilde{B}_i$. Taking $s_e = 1$ we have $\det\left[Es_e^{\frac{1}{2}} - A\right] = -23 \neq 0$. Therefore there exists the inverse

$$\left[Es_e^{\frac{1}{2}} - A\right]^{-1} = \begin{bmatrix} 11 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^{-1} = \frac{1}{23}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 12 & 11 \\ 1 & 11 & -11 \end{bmatrix}.$$

In consequence,

$$\widetilde{E} = \frac{1}{23}\begin{bmatrix} 21 & 1 & -1 \\ 1 & 11 & -11 \\ -1 & 11 & 11 \end{bmatrix}, \quad \widetilde{A} = \frac{1}{23}\begin{bmatrix} -2 & 1 & -1 \\ 1 & -12 & -11 \\ -1 & -11 & -12 \end{bmatrix}$$

and

$$\widetilde{B_0} = \frac{1}{23}\begin{bmatrix} 2 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \widetilde{B_1} = \frac{1}{23}\begin{bmatrix} 0 & -1 \\ 0 & 12 \\ 0 & 11 \end{bmatrix}.$$

Finally,

$$\widetilde{E}^D = \frac{1}{23}\begin{bmatrix} \frac{11}{230} & -\frac{1}{230} & 0 \\ 0 & \frac{1}{22} & \frac{1}{22} \\ \frac{1}{230} & -\frac{58}{1265} & \frac{1}{22} \end{bmatrix}$$

and, since $n = 3$,

$$\hat{\Phi}(t) = \sum_{k=0}^{2} \frac{\left(\widetilde{E}^D\widetilde{A}\right)^k t^{\frac{1}{2}(k-1)}}{\Gamma\left(\frac{1}{2}(k+1)\right)}.$$

With the help of the MATLAB environment we get

$$\operatorname{rank} W_E(0,2) = \operatorname{rank}\left[\int_0^2 \widetilde{E}^D\hat{\Phi}(2-\eta)\widetilde{B}_0\widetilde{B}_0^T\hat{\Phi}^T(2-\eta)\left(\widetilde{E}^D\right)^T \mathrm{d}\eta \right.$$
$$\left. + \int_{-1}^{1} \widetilde{E}^D\hat{\Phi}(1-\eta)\widetilde{B}_1\widetilde{B}_1^T\hat{\Phi}^T(1-\eta)\left(\widetilde{E}^D\right)^T \mathrm{d}\eta\right] = 3.$$

It follows that for given data the system is controllable on the time interval $\langle 0, 2\rangle$.

6. CONCLUDING REMARKS

Linear Caputo's fractional descriptor systems with constant delays in control and their application to electrical circuits were studied in the paper. It was proved that the solution of the discussed systems has the form (7). For this purpose the method of Laplace transformation and Drazin inverse were applied (Theorem 1). New relative controllability criteria for the discussed descriptor systems with delays were established and proved. Both constrained (Theorem 3) and unconstrained (Theorem 2) controls were considered. Practical application of the theoretical results was proposed. Further work could focus on extending the study for positive linear descriptor systems of fractional order.

ACKNOWLEDGEMENTS

The research is supported by the Silesian University of Technology, Poland under Grant No. BK-222/RMS0/2023.

REFERENCES

[1] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Mathematics in Science and Engineering, vol. 198, Academic Press, 1999.

[2] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, 2006.

[3] J. Sabatier, O.P. Agrawal, and J.A Tenreiro Machado, *Advances in Fractional Calculus*, Theoretical Developments and Applications in Physics and Engineering, Springer-Verlag, 2007.

[4] A. Monje, Y. Chen, B.M. Viagre, D. Xue, and V. Feliu, *Fractional-order Systems and Controls. Fundamentals and Applications*, Springer-Verlag, 2010.

[5] A. Lazopoulos, K. Karaoulanis, and D. Lazopoulos, "On Fractional Modelling of Viscoelastic Mechanical Systems," *Mech. Res. Commun.*, vol. 98, pp. 54–56, 2019.

[6] L. Chen, B. Basu, and D. McCabe, "Fractional order models for system identification of thermal dynamics of buildings," *Energy Build.*, vol. 133, pp. 381–388, 2016.

[7] A. Dzieliński, D. Sierociuk, and G. Sarwas, "Some applications of fractional order calculus," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 58, no. 4, pp. 583–592, 2010.

[8] A.T. Azar, A.G. Radwan, and S. Vaidyanathan, Fractional Order Systems. *Optimization, Control, Circuit Realizations and Applications*, Advances in Nonlinear Dynamics and Chaos, Academic Press, 2018.

[9] L. Dai, "Singular Control Systems," *Lectures Notes in Control and Information Sciences*, Springer-Verlag, Berlin, 1989.

[10] M. Dodig and M. Stosic, "Singular systems state feedbacks problems," *Linear Algebra Appl.*, vol. 431, pp. 1267–1292, 2009.

[11] D. Guang-Ren, *Analysis and Design of Descriptor Linear Systems*, Springer, New York, 2010.

[12] T. Kaczorek, "Singular fractional discrete-time linear systems," *Control Cybern.*, vol. 40, pp. 753–761, 2011.

[13] A.A. Belov, O.G. Adrianova, and A.P. Kurdyukov, "Practical Application of Descriptor Systems," In: *Control of Discrete-Time Descriptor Systems. Studies in Systems, Decision and Control*, vol. 157, Springer, 2018.

[14] T. Kaczorek, "Positive linear systems with different fractional orders," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 58, no. 3, pp. 458–453, 2010.

[15] T. Kaczorek, "Selected Problems of Fractional Systems Theory," *Lecture Notes in Control and Information Science*, vol. 411, 2011.

[16] K. Balachandran and J. Kokila, "On the Controllability of Fractional Dynamical Systems," *Int. J. Appl. Math. Comput. Sci.*, vol. 22, no. 3, pp. 523–531, 2012.

[17] K. Balachandran and J. Kokila, "Controllability of fractional dynamical systems with prescribed controls," *IET Control Theory Appl.*, vol. 7, no. 9, pp. 1242–1248, 2013.

[18] Y.Q. Chen, H.S. Ahn, and D. Xue, "Robust controllability of interval fractional order linear time invariant systems," *Signal Process.*, vol. 86, pp. 2794–2802, 2006.

[19] A. Babiarz and M. Niezabitowski: "Controllability Problem of Fractional Neutral Systems: A Survey," *Math. Probl. Eng.*, vol. 2017, p. 4715861 (1–15), 2017.

[20] K. Balachandran, Y. Zhou and J. Kokila, "Relative controllability of fractional dynamical systems with delays in control," *Commun. Nonlinear. Sci. Numer. Simulat.*, vol. 17, pp. 3508–3520, 2012.

[21] K. Balachandran, J. Kokila, and J.J. Trujillo, "Relative controllability of fractional dynamical systems with multiple delays in control," *Comput. Math. Appl.*, vol. 64, pp. 3037–3045, 2012.

[22] J. Wei, "The controllability of fractional control systems with control delay," *Comput. Math. Appl.*, vol. 64, pp. 3153–3159, 2012.

[23] B. Sikora, "Controllability of time-delay fractional systems with and without constraints," *IET Control Theory Appl.*, vol. 10, pp. 1–8, 2016.

[24] B. Sikora, "Controllability criteria for time-delay fractional systems with a retarded state," *Int. J. Appl. Math. Comput. Sci.*, vol. 26, pp. 521–531, 2016.

[25] J. Klamka and B. Sikora, "New controllability Criteria for Fractional Systems with Varying Delays," *Lecture Notes in Electrical Engineering. Theory and Applications of Non-integer Order Systems*, vol. 407, pp. 333–344, 2017.

[26] B. Sikora and J. Klamka, "Constrained controllability of fractional linear systems with delays in control," *Syst. Control Lett.*, vol. 106, pp. 9–15, 2017.

[27] B. Sikora and J. Klamka, "Cone-type constrained relative controllability of semilinear fractional systems with delays," *Kybernetika*, vol. 53, pp. 370–381, 2017.

[28] B. Sikora, "On application of Rothe's fixed point theorem to study the controllability of fractional semilinear systems with delays," *Kybernetika*, vol. 55, pp. 675–689, 2019.

[29] A. Younus, I. Javaid, and A. Zehra, "On controllability and observability of fractional continuous-time linear systems with regular pencils," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 65, pp. 297–303, 2017.

[30] T. Kaczorek, "Drazin inverse marix method for fractional descriptor continuous-time linear systems," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 62, pp. 409–412, 2014.

[31] T. Kaczorek, "Minimum energy control of positive fractional descriptor continuous-time linear systems," *IET Control Theory Appl.*, vol. 362, pp. 1–7, 2013.

[32] T. Zhan, X. Liu, and S. Ma, "A new singular system approach to output feedback sliding mode control for fractional order nonlinear systems," *J. Franklin Inst.*, vol. 355, pp. 6746–6762, 2018.

[33] S.L. Campbell, C.D. Meyer, and N.J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," *SIAM J. Appl. Math.*, vol. 31, pp. 411–425, 1976.

[34] M.P. Drazin, "Pseudoinverses in associative rings and semigroups," *Amer. Math. Mountly*, vol. 65, pp. 506–514, 1958.

[35] Q. Lü and E. Zuazua, "On the lack of controllability of fractional in time ODE and PDE," *Math. Control Signals Syst.*, vol. 28, no. 2, pp. 1–21, 2016.

[36] S. Marir, M. Chadli, and D. Bouagada, "New Admissibility Conditions for Singular Linear Continuous-Time Fractional-Order Systems," *J. Franklin Inst.*, vol. 354, no. 2, pp. 752–766, 2017.

[37] M. Guía, F. Gómez, and J. Rosales, "Analysis on the Time and Frequency Domain for the RC Electric Circuit of Fractional Order," *Cent. Eur. J. Phys.*, vol. 11, no. 10, pp. 1366–1371, 2013.

[38] A. Atangana and B. Alkahtani, "Extension of the resistance, inductance, capacitance electrical circuit to fractional derivative without singular kernel," *Adv. Mech. Eng.*, vol. 7, pp. 1–6, 2015.

[39] A. Alsaedi, J. Nieto, and V. Venktesh, "Fractional electrical circuits," *Adv. Mech. Eng.*, vol. 7, pp. 1–7, 2015.

[40] T. Kaczorek and K. Rogowski, *Fractional linear systems and electrical circuits*, London, Springer, 2007.

[41] R. Banchuin, "Noise analysis of electrical circuits on fractal set," *Compel-Int. J. Comp. Math. Electr. Electron. Eng.*, vol. 41, pp. 1464–1490, 2022.