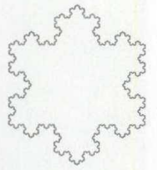
Fractals in holomorphic dynamics

Julia Sets

FELIKS PRZYTYCKI
Institute of Mathematics, Warsaw
Polish Academy of Sciences
feliksp@impan.gov.pl



In 1904, the Swedish mathematician Helge von Koch first described a geometrical extraordinary figure with a "self-similar" edge, which he dubbed a "snowflake." Nowadays such "self-similar" but "rough" sets are called fractals, well-known for their exceptional beauty, and are a subject of intense research

Enchanted with the receding infinity of von Koch's curve, the Italian mathematician Ernesto Cesàro wrote about it: "Had it been given life, it would not be possible to do away with it without destroying it altogether for it would rise again and again from the depths of its triangles, as life does in the Universe."

The prevalence of such figures in both mathematics and the natural world was noticed by Benoit Mandelbrot. He already had computers at his disposal, and he was the one who named them "fractals": as their geometrical dimension, which is greater than their topological dimension (the topological dimension of a curve is equal to 1, that of a surface 2, etc.), may be a *fraction* rather than a whole number.

Many fractals are characterized by self-similarity: within any given neighborhood, each of its parts bears a similarity to the whole fractal (or a large portion thereof). Because of this trait fractals also have applications in fields far remote from mathematics: generating computer graphics to realistically render landscapes, developing data compression techniques, studying the structure of the Universe as a whole, and even forecasting stock-exchange fluctuations.

The self-similarity exhibited by fractals often stems from the fact that there exists a locally stretching transformation f which maps the fractal onto itself, such that iterations of f transform small sets into large ones, deforming the shape only in a bounded way (called "limited distortion"). Of course the smaller the size (the smaller the set) the greater the time (the more iterations) it takes to reach a large scale. Thus micro-scale spatial properties turn out to be linked to the long-term behavior of the transformation trajectory.

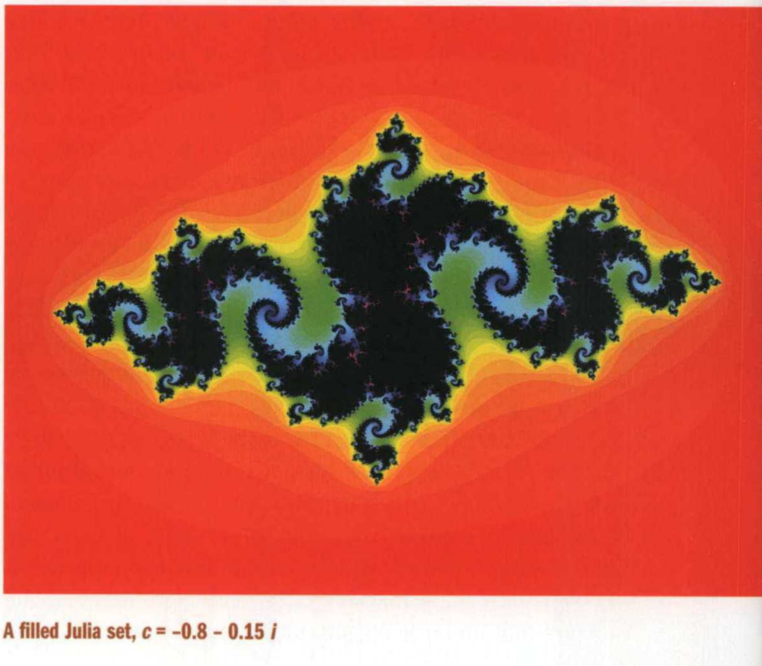
In search of stability

It is here that fractal analysis and geometry meet the theory of dynamic systems. Such systems describe the trajectory behavior of points as a certain transformation is iterated, or the trajectory followed by solutions to differential equation solutions describing physical processes over a long duration. This field of mathematics is concerned with the stability of systems and invariant sets, and its stretch back to celestial mechanics. The behavior of two-body systems was already understood by Kepler and Newton, but three-body systems (e.g. the Sun, Jupiter, and the Earth) are still not fully understood in mathematical terms. In such multiple-body situations, space generally breaks down into invariant sets of points whose movement is almost periodic, as well as others whose movement is "chaotic." The latter only came to be understood in recent decades: their motion turns out to be describable in terms of the simplest stochastic processes, hence the term "deterministic chaos." In situations where scale (and energy) does not have to be retained, there are also regions which are attracted towards periodic trajectories, attractors that are more complicated than individual trajectories, repellers, etc.

Population dynamics

The simplest transformation that can be iterated to obtain interesting phenomena is the quadratic polynomial

Janet Chen



A filled Julia set, $c = -0.8 - 0.15\,i$

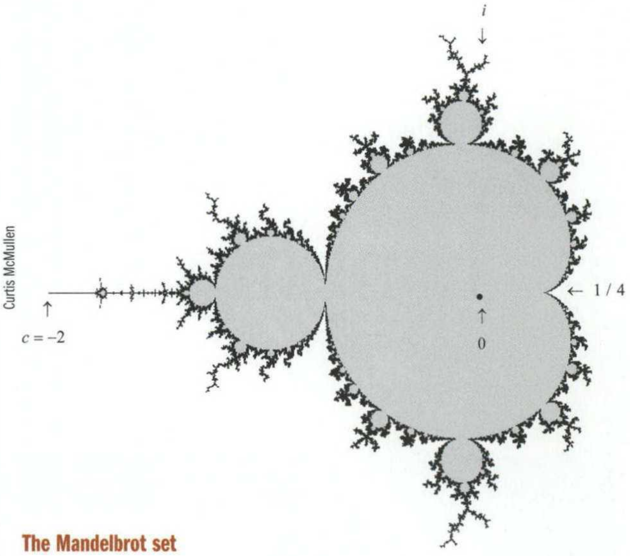
$x \mapsto ax\,(1 - x)$, with the positive coefficient a not greater than 4. The dynamics of this transformation yields the simplest model for the behavior of a population, e.g. a certain species of animal. If x is small it will increase in subsequent iterations (provided $a > 1$), meaning that the number of animals increases in subsequent generations. If x comes close to 1, however, the next generation will be small - an overpopulated habitat causes to a drop in numbers. Depending on the value of a, the future trajectory of a typical point (population size) may approach a periodic trajectory (orbit) or it may behave chaotically. Then, however, for nearly all parameters a, nearly every point of x will exhibit "the same" trajectory behavior - after a long duration, the entire span [0, 1] (or a certain finite number of its subsegments) will become "covered" by the trajectory, and the "density" of that coverage will not depend on x! This property is described as an "ergodic" equilibrium state.

Julia and Mandelbrot

Back in the 1980s, the present author researched iteration of holomorphic transformations of the plane. These are stretching transformations whose iterations (aside from certain special points called singularities) exhibit limited distortion, and which therefore imply self-similarity.

In order to understand the iterations of $ax\,(1 - x)$, we sometimes need to study the iterations $z \mapsto az\,(1 - z)$ or, after the appropriate change of coordinates, $z \mapsto z^2 - c$. Here z is a point on the plane, treated as a complex number, and the holomorphic transformation is simply a quadratic polynomial. Geometrically, complex numbers are added in the same way as one would add vectors running from the origin to these points, and they are multiplied by multiplying their lengths and orienting the resulting vector at an angle (running counterclockwise) with respect to the positive semiaxis (or argument) which is the sum of the arguments of those vectors. Given an arbitrary complex number c, if the point is sufficiently far from 0, during the iteration of the transformation f_c its trajectory approaches infinity. We then say that it lies in the basin of attraction of infinity. That basin has a boundary which is called a Julia set - after Gaston Julia, a French mathematician who defined such sets for f_c and for certain broader classes of functions in the early 20th century (independently of the sets' other discoverer, Pierre Fatou).

A Julia set is a "chaotic repeller." Please note that for f_0 the Julia set (f_0) is a circle with its center at the origin and radius of 1, which is in fact also a fractal. When c moves further away from the origin, moving outside the curve known as the cardioid, the resulting curve begins to self-attach and the Julia comes to look like the outline of a "dragon." When c moves outside a certain set known as the Mandelbrot set, in turn, the Julia set becomes totally disconnected.



The Mandelbrot set

For many values of c, the Julia set for f_c represents a sum of real line segments, "overgrown" with spines within the plane. Most of this "bush" (or "hedgehog") is self-similar, although not exactly so in view of the singularity 0 within the Julia set. The present author is now studying the statistical properties of such "non-uniform stretching" transformations of f_c, or more generally of rational functions (the polynomial quotient of a complex variable) and the more precise description of their local geometry.

We can also iterate the transformation $z \mapsto ae^z$, where for $z = (x, y)$ (a point on a plane with coordinates x, y) $e^z = e^x\,(\cos y + i \sin y)$ and i is the square root of -1. The Julia set of the transformation ae^z consists of an infinite number of "hairs" clung together into strands, called a "Cantor bouquet." In an outstanding doctorate thesis written under the present author's supervision, B. Karpińska has demonstrated that the Hausdorff geometric dimension of the ends of these hairs is equal to 2, while paradoxically the dimension of the entire bouquet without the ends (the dimension of the "stems" alone) is 1. This bouquet, therefore, is one of "very densily scatterd flowers."

Our Warsaw-based team, together with mathematicians from the United States, the United Kingdom, France, and Chile is involved in studying the Julia sets for general complex analytic functions of one or more variables, related to fractal dynamics. Some of this research is being pursued within the framework of the Marie Curie Research Training Network on "Conformal Structures and Dynamics" and the Marie Curie Transfer of Knowledge program on "Deterministic and Stochastic Dynamics, Fractals, Turbulence," hosted at the Institute of Mathematics, Polish Academy of Sciences. ■

Further reading:

Falconer K. (1990, 2003). *Fractal Geometry. Mathematical Foundations and Applications.* New York: J. Wiley.

Mandelbrot B. (1982). *The Fractal Geometry of Nature.* New York: W.H. Freeman and Company.

Kornfeld I.P., Fomin S.V., Sinai Ya.G. (1982). *Ergodic Theory.* New York: Springer-Verlag.

Peitgen H.-O., Jürgens H., Saupe D. (1992). *Chaos and Fractals: New Frontiers of Science.* New York: Springer-Verlag.

Przytycki F. (1986). Zoo na płaszczyźnie [Zoo in the plane]. *Delta, 2 (146)*, 1-6.