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Necessary optimality conditions for quasi-singular controls for systems with Caputo fractional derivatives

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In this paper, we consider an optimal control problem in which a dynamical system is controlled by a nonlinear Caputo fractional state equation. First we get the linearized maximum principle. Further, the concept of a quasi-singular control is introduced and, on this basis, an analogue of the Legendre-Clebsch conditions is obtained. When the analogue of Legendre-Clebsch condition degenerates, a necessary high-order optimality condition is derived. An illustrative example is considered.

Key words: fractional derivative, fractional optimal control, necessary optimality condition.

1. Introduction

Fractional differential equations have been of great interest for the past three decades [2, 4, 5, 9, 11, 34, 37, 47]. This is due to the intensive development of the theory of fractional calculus itself as well as its applications. Note that optimal control problems for systems with fractional Riemann-Liouville and Caputo derivatives are currently being actively studied. Such problems arise in a wide range of applications, including, e.g., biology, chemistry, economics, electrical engineering, and medicine [3, 9, 28].

It is known that fractional optimal control problems described by ordinary fractional differential equations can be regarded as a generalization of classic optimal control problems. The Pontryagin maximum principle is a fundamental result of the theory of necessary optimality conditions of the first order, which initially was proved in [39] for optimal control problems described by ordinary

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differential equations. Further, various necessary conditions for optimality of the first and higher order are obtained for various systems [15, 24, 30–33, 35, 44–46].

For an optimal control problem with fractional derivatives, various formulations are considered, including linear-quadratic problems, problems of minimizing the integral quality index, problems of transferring the system to a given state in the shortest time or with a minimum control norm [1, 19, 29, 36].

The article [7] describes fractal dynamic games as an analytical tool for modeling and predicting human dynamics. Based on the description of the statistical physics of interactions between agents and the observed statistical properties of economic indicators, basic equations are constructed that characterize the dynamics of cost functionals as stochastic variables, which are influenced by additive and multiplicative noise forces. Using the concepts of optimal control theory, a continuum formulation for optimizing the dynamics of car traffic is derived, which leads to a nonlinear fractional partial differential equation. In [26], a fractional order system is presented as a model in the adaptive internal model control (IMC) structure to obtain a fractional adaptive IMC scheme. It is shown that this adaptive control scheme always provides theoretical guarantees of stability when a stable fractional transfer function is used as the IMC parameter.

The paper [42] presents some results for existence of global solutions and attractivity for mulidimensional fractional differential equations involving Riemann-Liouville derivative. First, by using a Bielecki type norm and Banach fixed point theorem, it is proved a Picard type theorem on the existence and uniqueness of solutions. Then, applying the properties of Mittag-Leffler functions, are described the attractivity of solutions to some classes of Riemann-Liouville linear fractional differential systems. In [20], the conditions for the positivity and stability of a class of fractional-nonlinear systems with continuous time are established. It is assumed that the nonlinear vector function is continuous, satisfies the Lipschitz condition, and the linear part is described by the Metzler matrix. The stability conditions are established by generalizing the Lyapunov method to positive fractional nonlinear systems. The article [38] considers a simple identification problem for a fractional differential equation of Caputo type. This is the problem of estimating the parameters for which the quadratic criterion is minimized. To solve this problem, a nonlinear programming method based on the Marquardt algorithm was used. In the article [13], the study of the existence of a homogeneous Lyapunov function for a class of homogeneous fractional systems was started, then it was proved that the local and global behavior are the same. The uniform Mittag-Leffler stability of homogeneous fractional time-varying systems is studied. The article [16] considers the problem of optimal control in a dynamical system described by a linear differential equation with the Caputo fractional derivative. The aim of the control is to minimize the Bolza-type cost functional. To solve this problem, it is proposed to reduce it to some auxiliary optimal control problem, whose reduction is based on the formula for representing solutions of linear-fractional differential equations. The work [10] presents the formulation of the optimal control and the numerical algorithm for a singular system of fractional order with discrete time for a fixed final state and a fixed final state of finite time. The productivity index has a quadratic form, and the dynamics of the system – in the sense of a fractional Riemann-Liouville derivative. To obtain the necessary conditions, the Hamilton method is used. In [40] the stability analysis of linear-fractional systems with discrete time and delays is presented. A state-space model with a time difference shift is considered. Necessary and sufficient conditions for practical stability and asymptotic stability are established. In addition, parametric descriptions of the boundaries of the regions of practical stability and asymptotic stability are presented. The work [27] presents a general approach to the explicit selection of domains of fractional powers of matrix-valued operators. The advantage of the present approach, which becomes apparent in this illustrative case, is that it is more conceptual and less computationally demanding than the previous approach. The article [23] considers an unconstrained local controlability problem of finite-dimensional fractional-discrete semi-linear systems with multiple control delays and constant coefficients. Using general formula of solution of difference state equation algebraic sufficient condition for local unconstrained controllablity in a given number of steps is formulated and proved.

The article [17] considers partial stability in a finite time and uniform partial stability in a finite time for nonlinear dynamical systems. In particular, Lyapunov conditions are provided, including a Lyapunov function that is positive definite and decreasing with respect to a part of the state of the system, and satisfies a differential inequality involving fractional powers to ensure partial stability over a finite time. The book [25] presents a wide and exhaustive range of questions and problems related to fractional order dynamical systems. It is intended to provide a full, comprehensive presentation of the many aspects associated with widely accepted fractional-order dynamical systems, which are an extension of traditional integer-order type descriptions.The article [12] solves the control problem for a linear stochastic system controlled by a noise process, which is an arbitrary zero-mean integrable stochastic process with continuous discrete paths and a cost functional quadratic in the system state and control. The article [43] establishes a sufficient condition to obtain the optimal control of discounted linear quadratic regulator optimization problem subject to disturbanced singular system where the disturbance is time varying. Combining control theory and modeling, the textbook [8] introduces and develops methods for modeling and solving specific problems in various applied sciences.

The authors of the paper study in [34] local and nonlocal boundary value problems for general hyperbolic equations with variable coefficients and a fractional Caputo derivative. To study the stated problem, a certain fractional-order functional space is introduced. The problem posed is reduced to an integral equation, and the existence of its solution is proved using an a priori estimate.

Pontryagin's maximum principle for fractional optimal control problems proved in [6, 14, 21, 48, 49]. In the paper [6] the Pontryagin maximum principle is proved for general Caputo fractional optimal problems with Bolza costs and terminal constraints. A linearized maximum principle is also obtained. Note that in this paper the adjoint function may have a singularity. The paper [14] proves Pontryagin's type necessary optimality conditions for a class of fuzzy fractional optimal control problems with a fuzzy fractional derivative described in the Caputo sense.

An admissible control is called a Pontryagin extremal if it satisfies the Pontryagin maximum condition. It follows from the maximum principle that any optimal control is a Pontryagin extremal. Thus, the optimal control problem is reduced to choosing the best control among the Pontryagin extremals. If an optimal control exists in the optimization problem and Pontryagin's extremal is unique, then this is the optimal control. Note that even in the simplest situations, the Pontryagin maximum principle is satisfied by the nonoptimal controls. In such cases, the maximum principle can no longer weed out all nonoptimal controls.

Choosing among the Pontryagin extremals a narrower subset that claims to be optimal is an important and difficult problem. Its complexity is explained by the fact that the maximum principle itself is a very strong necessary condition for optimality.

Therefore, the study of situations in which the Pontryagin maximum principle does not allow one to uniquely determine the optimal control is today one of the main directions in the development of the theory of optimal systems. Thus, the problem arises of constructing new necessary optimality conditions that sparse the set of controls that have passed the Pontryagin maximum principle. Therefore, by high-order necessary optimality conditions we mean conditions imposed on controls that satisfy the Pontryagin maximum principle and allow, generally speaking, to distinguish among such controls nonoptimal.

In the paper [48], a necessary first-order optimality condition is obtained in the form of the Pontryagin maximum principle, and in the case of degeneracy of the maximum principle, a necessary optimality condition for singular controls in the sense of the Pontryagin maximum principle. Note that if the Pontryagin maximum principle holds along the control but does not degenerate, then the results of [6,48] leave this control among the contenders for optimality. Therefore, it is natural to have new necessary optimality conditions that would sift out this control from among the candidates for optimality.

In the present paper we consider an optimal control problem for a dynamical system whose motion is described by a nonlinear differential equation with the Caputo fractional derivative of order $\alpha \in (0,\ 1)$. The time interval of the control process is fixed and finite. The goal of control is to minimize a given Bolza-type cost functional, which consists of two terms. One of them evaluates the state vector of the system realized at a fixed terminal time T, and the other is an integral evaluation of a control on the whole time interval $[0, T]$. The set of values of control functions is nonemty and convex.

The posed problem of optimal control is investigated using a new version of the increment method, in which the concept of a conjugate equation of an integral form is essentially used. First we get the linearized maximum principle. Further, the concept of a quasi-singular control is introduced and, on its basis, an analogue of the Legendre-Clebsch conditions is obtained. When the analogue of Legendre-Clebsch condition degenerates, a necessary high-order optimality condition is obtained.

Note that the singular control in the sense of the Pontryagin maximum principle is also quasi-singular. The converse is generally not true. In other words, quasi-singular control may not be singular in the sense of the Pontryagin maximum principle. The obtained results in a number of cases make it possible to establish the nonoptimality of those controls that satisfy the Pontryagin maximum condition and are not singular in the sense of the maximum principle. An illustrative example is considered.

Thus, the main novelty of this paper is one of the possible approaches to proving high-order optimality criteria in the general case, using a new version of the increment method.

The rest of the paper is organized as follows.

In Section 2, the definitions and basic properties of fractional order integrals and derivatives are recalled, and also some preliminary results are proved.

In Section 3 is given the formulation of the optimal control problem described by the equation with the Caputo fractional derivative.

In Section 4, to obtain the necessary condition for the optimality of the control function, the increment of the functional is calculated.

Section 5 is devoted to proving the existence and uniqueness of a continuous solution to the adjoint problem.

In Section 6, first of all, for the optimality of the control function, a linearized maximum principle is derived, and then, in the degenerate case of the linearized maximum principle, a necessary condition of the Legendre-Clebsch type is obtained. Finally, a necessary condition of a higher-order is derived.

2. Notations, definitions, and preliminary results

In this section, we give some definitions and basic concepts of fractional integrals and derivatives (for details, see [22, 41]).

Let $\mathbb{R}^n$and $\mathbb{R}^{n\times n}$ be the spaces of n-dimensional vectors and $(n \times n)$-matrices, and let $I \in \mathbb{R}^{n\times n}$ stand for identity matrix. By $\|\cdot\|$, we denote a norm in $\mathbb{R}^n$ and the corresponding norm in $\mathbb{R}^{n\times n}$. Let numbers $a,\ b \in \mathbb{R}$, $a < b$ be fixed, and let X be one of the spaces $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$.

Let $L^1([a,\, b],\, X)$ be the Lebesgue space of summable functions $\varphi(\cdot)$ defined on $[a,\, b]$ with values in X, endowd with its usual norm $\|\varphi(\cdot)\|_{L^1} = \int\limits_a^b \|\varphi(t)\| \mathrm{d}t$ and $L^\infty([a,\, b],\, X)$ is the Lebesgue space of essentially bounded functions $\varphi(\cdot)$ defined on $[a,\, b]$ with values in X, endowed with its usual norm $\|\varphi(\cdot)\|_{L^\infty} = \|\varphi(\cdot)\|_{[a,b]} = \operatorname*{ess\,sup}\limits_{t\in[a,b]} \|\varphi(t)\|$. $C([a,\, b], X)$ the space of continuous functions on $[a,\, b]$ with values in X. We denote by $C^a([a,\, b],\, X)$ the set of functions $\varphi(\cdot) \in C([a,\, b],\, X)$ such that $\varphi(a) = 0$.

The Euler gamma function $\Gamma(\cdot)$ is defined by the so called Euler integral of the second kind

$$\Gamma(\alpha) = \int\limits_0^\infty e^{-t} t^{\alpha-1} \mathrm{d}t, \quad \alpha > 0.$$

The beta function is defined by the Euler integral of the first kind:

$$B(\alpha,\, \beta) = \int\limits_0^1 t^{\alpha-1} (1-t)^{\beta-1} \mathrm{d}t, \quad \alpha,\, \beta > 0.$$

This function is connected with the gamma functions by the relation

$$B(\alpha,\, \beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,.$$

Definition 1 *Let $\alpha \in (0,\, 1)$. For a function $\varphi\colon [a,\, b] \longrightarrow X$, the left-sided and right-sided Riemann-Liouville fractional integrals of the order α are defined for $t \in [a,\, b]$ by*

$$\left(I^\alpha_{a+}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t (t-\tau)^{\alpha-1} \varphi(\tau) \mathrm{d}\tau, \ \textit{and} \ \left(I^\alpha_{b-}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^b (\tau-t)^{\alpha-1} \varphi(\tau) \mathrm{d}\tau,$$

respectively.

If $\varphi(\cdot) \in L^\infty([a,\, b],\, X)$, then the above functions are defined and finite everywhere on $[a,\, b]$.

Proposition 1 *If $\alpha > 0$ and $\varphi(\cdot) \in L^1([a,\, b],\, X)$, then $(I^\alpha_{a+}\varphi)(\cdot)$ and $(I^\alpha_{b-}\varphi)(\cdot) \in L^1([a,\, b],\, X)$.*

Proposition 2 *If* $\alpha_1 > 0$, $\alpha_2 > 0$ *and* $\varphi(\cdot) \in L^1([a,\, b],\, X)$, *then*

$$\left(I^{\alpha_1}_{a+}\left(I^{\alpha_2}_{a+}\varphi\right)\right)(t) = \left(I^{\alpha_1+\alpha_2}_{a+}\varphi\right)(t) = \left(I^{\alpha_2+\alpha_1}_{a+}\varphi\right)(t) = \left(I^{\alpha_2}_{a+}\left(I^{\alpha_1}_{a+}\varphi\right)\right)(t),$$
$$\left(I^{\alpha_1}_{b-}\left(I^{\alpha_2}_{b-}\varphi\right)\right)(t) = \left(I^{\alpha_1+\alpha_2}_{b-}\varphi\right)(t) = \left(I^{\alpha_2+\alpha_1}_{b-}\varphi\right)(t) = \left(I^{\alpha_2}_{b-}\left(I^{\alpha_1}_{b-}\varphi\right)\right)(t)$$

almost everywhere on $[a,\, b]$. *If moreover* $\varphi(\cdot) \in L^\infty([a,\, b],\, X)$ *and* $\alpha_1+\alpha_2 > 0$, *the above equality is satisfied everywhere on* $[a,\, b]$.

Proposition 3 *If* $\alpha > 0$ *and* $\varphi(\cdot) \in L^\infty([a,\, b],\, X)$, *then* $(I^\alpha_{a+}\varphi)(\cdot) \in C^a([a,\, b],\, X)$.

Definition 2 *Let* $\alpha \in (0,\, 1)$, $\varphi(\cdot) \in L^1([a,\, b],\, X)$. *For a function* $\varphi(\cdot)$ *the left-sided and right-sided Riemann-Liouville fractional derivatives of the order* α *are defined for* $t \in [a,\, b]$ *by*

$$\left(D^\alpha_{a+}\varphi\right)(t) = \frac{d}{dt}\left(I^{1-\alpha}_{a+}\varphi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_a^t (t-\tau)^{-\alpha}\varphi(\tau)\,\mathrm{d}\tau$$

and

$$\left(D^\alpha_{b-}\varphi\right)(t) = -\frac{d}{dt}\left(I^{1-\alpha}_{b-}\varphi\right)(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_t^b (\tau-t)^{-\alpha}\varphi(\tau)\,\mathrm{d}\tau,$$

if $\left(I^{1-\alpha}_{a+}\varphi\right)(\cdot)$ *and* $\left(I^{1-\alpha}_{b-}\varphi\right)(\cdot)$ *has an absolutely continuous representant on* $[a,\, b]$.

Definition 3 *Let* $\alpha \in (0,\, 1)$, *and* $\varphi(\cdot) \in C([a,\, b],\, X)$. *For a function* $\varphi(\cdot)$ *the left-sided and right-sided Caputo fractional derivatives of the order* α *are defined for* $t \in [a,\, b]$ *by*

$$\begin{aligned}\left({}^cD^\alpha_{a+}\varphi\right)(t) &= \left(D^\alpha_{a+}(\varphi(\cdot)-\varphi(a))\right)(t) = \frac{d}{dt}\left(I^{1-\alpha}_{a+}(\varphi(\cdot)-\varphi(a))\right)(t)\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_a^t (t-\tau)^{-\alpha}(\varphi(\tau)-\varphi(a))\,\mathrm{d}\tau,\end{aligned}$$

and

$$\begin{aligned}\left({}^cD^\alpha_{b-}\varphi\right)(t) &= \left(D^\alpha_{b-}(\varphi(\cdot)-\varphi(b))\right)(t) = -\frac{d}{dt}\left(I^{1-\alpha}_{b-}(\varphi(\cdot)-\varphi(b))\right)(t)\\ &= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_t^b (\tau-t)^{-\alpha}(\varphi(\tau)-\varphi(b))\,\mathrm{d}\tau,\end{aligned}$$

if $\left(I_{a+}^{1-\alpha}(\varphi(\cdot)-\varphi(a))\right)(\cdot)$ *and* $\left(I_{b-}^{1-\alpha}(\varphi(\cdot)-\varphi(b))\right)(\cdot)$ *has an absolutely continuous representant on* $[a,b]$.

By $AC_{\infty}^{\alpha}([a,b],X)$ *we denote the set of all functions* $\varphi:[a,b]\to X$, *such that*

$$\varphi(t)=\varphi(a)+\left(I_{a+}^{\alpha}\psi\right)(t),\quad t\in[a,b],$$

with $\psi(\cdot)\in L^{\infty}([a,b],X)$.

Proposition 4 *For any* $\varphi(\cdot)\in AC_{\infty}^{\alpha}([a,b],X)$, *the value* $\left({}^{c}D_{a+}^{\alpha}\varphi\right)(t)$ *is correctly defined for almost every* $t\in[a,b]$. *Moreover, the inclusion* $\left({}^{c}D_{a+}^{\alpha}\varphi\right)(\cdot)\in L^{\infty}([a,b],X)$ *holds (i.e., there exists* $\psi(\cdot)\in L^{\infty}([a,b],X)$ *such that* $\psi(t)=\left({}^{c}D_{a+}^{\alpha}\varphi\right)(t)$ *for almost every* $t\in[a,b]$*) and*

$$\left(I_{a+}^{\alpha}\left({}^{c}D_{a+}^{\alpha}\varphi\right)\right)(t)=\varphi(t)-\varphi(a),\quad t\in[a,b].$$

Proposition 5 *Let* $\alpha>0$ *and let* $\varphi(\cdot)\in L^{\infty}([a,b],X)$, *then* $\left({}^{c}D_{a+}^{\alpha}(I_{a+}^{\alpha}\varphi)\right)(t)=\varphi(t)$ *and* $\left({}^{c}D_{b-}^{\alpha}(I_{b-}^{\alpha}\varphi)\right)(t)=\varphi(t)$, *a.e.* $t\in[a,b]$.

Lemma 1 *[18, 48] Suppose* $\alpha>0$, $a(\cdot)$ *is a nonnegative function locally integrable on* $[a,b)$ *and* $b(\cdot)$ *is a nonnegative, nondecreasing continuous function defined on* $t\in[a,b]$ *and suppose* $u(\cdot)$ *is nonnegative and locally integrable on* $[a,b)$ *with*

$$u(t)\leqslant a(t)+\frac{b(t)}{\Gamma(\alpha)}\int\limits_{a}^{t}(t-s)^{\alpha-1}u(s)\,\mathrm{d}s,\quad t\in[a,b).$$

Then

$$u(t)\leqslant a(t)+b(t)E_{\alpha,\alpha}(b(t)(t-a)^{\alpha})\int\limits_{a}^{t}(t-s)^{\alpha-1}a(s)\,\mathrm{d}s,\quad a\leqslant t<b,$$

where $E_{\alpha,\beta}(t)=\sum\limits_{k=0}^{\infty}\dfrac{t^{k}}{\Gamma(\alpha k+\beta)}$ *is Mittag-Leffler function with two parameters.*

Lemma 2 *[48] For an arbitrary function* $a(\cdot)\in L^{\infty}(0,T)$, *any of its Lebesgue points* $\theta\in(0,T)$, *and any numbers* $\alpha,\ \varepsilon$, $0<\alpha<1$, $0<\varepsilon<T-\theta$, *the equality*

$$\int\limits_{\theta}^{\theta+\varepsilon}(T-t)^{\alpha-1}(t-\theta)^{\alpha}a(t)\,\mathrm{d}t=\frac{(T-\theta)^{\alpha-1}a(\theta)}{\alpha+1}\varepsilon^{\alpha+1}+o(\varepsilon^{\alpha+1}),$$

holds, where $\lim\limits_{\varepsilon\to0}\dfrac{o(\varepsilon^{\alpha+1})}{\varepsilon^{\alpha+1}}=0$.

Lemma 3 *For an arbitrary function $a(\cdot) \in L^{\infty}(0, T)$ and any Lebesgue point $\theta \in (0, T)$ of this function, the equality*

$$\lim_{t \to \theta^+} \frac{1}{(t-\theta)^{\alpha}} \int_{\theta}^{t} (t-\tau)^{\alpha-1} |a(\tau) - a(\theta)| \,\mathrm{d}\tau = 0, \quad 0 < \alpha \leqslant 1, \tag{1}$$

holds.

Proof. Under the condition $p > \dfrac{1}{\alpha}$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, by Holder's inequality we have

$$\begin{aligned}
\int_{\theta}^{t} (t-\tau)^{\alpha-1} |a(\tau) - a(\theta)| \,\mathrm{d}\tau &\leqslant \left(\int_{\theta}^{t} (t-\tau)^{q(\alpha-1)} \,\mathrm{d}\tau \right)^{\frac{1}{q}} \left(\int_{\theta}^{t} |a(\tau) - a(\theta)|^p \,\mathrm{d}\tau \right)^{\frac{1}{p}} \\
&= \left(- \frac{(t-\tau)^{q(\alpha-1)+1}}{q(\alpha-1)+1} \Big|_{\theta}^{t} \right)^{\frac{1}{q}} \left(\int_{\theta}^{t} |a(\tau) - a(\theta)|^p \,\mathrm{d}\tau \right)^{\frac{1}{p}} \\
&= \frac{\left((t-\theta)^{q(\alpha-1)+1}\right)^{\frac{1}{q}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} \frac{(t-\theta)^{\frac{1}{p}}}{(t-\theta)^{\frac{1}{p}}} \left(\int_{\theta}^{t} |a(\tau) - a(\theta)|^p \,\mathrm{d}\tau \right)^{\frac{1}{p}} \\
&= \frac{(t-\theta)^{\alpha}}{(q(\alpha-1)+1)^{\frac{1}{q}}} \left(\frac{1}{t-\theta} \int_{\theta}^{t} |a(\tau) - a(\theta)|^p \,\mathrm{d}\tau \right)^{\frac{1}{p}} .
\end{aligned}$$

Denoting $d = (q(\alpha-1)+1)^{\frac{1}{q}}$, from this we have

$$\frac{1}{(t-\theta)^{\alpha}} \int_{\theta}^{t} (t-\tau)^{\alpha-1} |a(\tau) - a(\theta)| \,\mathrm{d}\tau \leqslant \frac{1}{d} \left(\frac{1}{t-\theta} \int_{\theta}^{t} |a(\tau) - a(\theta)|^p \,\mathrm{d}\tau \right)^{\frac{1}{p}} . \tag{2}$$

Note that, if there exists a measurable set $E \subset [\theta, t]$ such that

$$\lim_{t \to \theta^+} \frac{\operatorname{meas}\{[\theta, t] \cap E\}}{t-\theta} = 1 \quad \text{and} \quad \lim_{t \to \theta^+} a(t) = a(\theta), \ \ t \in E, \tag{3}$$

then θ is said to be the point of approximate continuity of $a(\cdot)$. Points of approximate continuity of a measurable functions comprise a set of full measure. Also recall that Lebesgue points of a measurable function are necessarily points of its approximate continuity. The converse is true if the function is essentially bounded.

It follows from (3) that for any $\varepsilon > 0$ there is $\delta_1 = \delta_1(\varepsilon) > 0$ such that $|a(\tau) - a(\theta)| < \dfrac{\varepsilon d}{2^{\frac{1}{p}}}$ for $|t-\theta| \leqslant \delta_1$, $\tau \in [\theta, t] \cap E$ and $\lim\limits_{t\to\theta} \dfrac{\text{meas}\big\{[\theta, t]|E\big\}}{t-\theta} = 0$. Denoting $M = \text{ess}\sup\{|a(\tau)| : \tau \in [\theta, t]\}$, we find from $\left(\dfrac{\varepsilon}{2^{1+\frac{1}{p}}M}\right)^p$ such $\delta_2 > 0$, such that for $|t-\theta| \leqslant \delta_2$ we have

$$\frac{\text{meas}\big\{[\theta, t] \cap E\big\}}{t-\theta} \leqslant \left(\frac{\varepsilon d}{2^{1+\frac{1}{p}}M}\right)^p.$$

Putting $\delta = \min\{\delta_1, \delta_2\}$ and taking into account the relations obtained, we arrive at $|t-\theta| \leqslant \delta$ to the inequalities

$$\begin{aligned}\frac{1}{t-\theta}\int_\theta^t |a(\tau)-a(\theta)|^p \,\mathrm{d}\tau &= \frac{1}{t-\theta}\int_{[\theta,t]\cap E} |a(\tau)-a(\theta)|^p \,\mathrm{d}\tau \\ &+ \frac{1}{t-\theta}\int_{[\theta,t]\setminus E} |a(\tau)-a(\theta)|^p \,d\tau \leqslant \frac{1}{t-\theta}\frac{\varepsilon^p d^p}{2}(t-\theta) \\ &+ \frac{1}{t-\theta}(2M)^p \frac{d^p\varepsilon^p(t-\theta)}{2\cdot(2M)^p} = (\varepsilon d)^p.\end{aligned}$$

Using this estimate, from (2) we see that

$$\frac{1}{(t-\theta)^\alpha}\int_\theta^t (t-\tau)^{\alpha-1}|a(\tau)-a(\theta)|\,\mathrm{d}\tau < \varepsilon.$$

Hence equality (1) follows. The proof is completed. □

Lemma 4 *For an arbitrary function $a(\cdot) \in L^\infty(0, T)$ and any Lebesgue point $\theta \in (0, T)$ of this function, the equality*

$$\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\int_\theta^{\theta+\varepsilon}(t-\tau)^{\alpha-1}|a(\tau)-a(\theta)|\,\mathrm{d}\tau = 0, \quad 0<\alpha\leqslant 1, \quad t\in(\theta+\varepsilon, T], \tag{4}$$

holds.

Proof. Since θ is the Lebesgue point for the function $a(\cdot)$, we have

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int\limits_{\theta}^{\theta+\varepsilon} |a(\tau) - a(\theta)| \mathrm{d}\tau = 0.$$

From here, by changing variables, we have

$$\begin{aligned}\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int\limits_{\theta}^{\theta+\varepsilon} |a(\tau) - a(\theta)| \mathrm{d}\tau &= \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int\limits_{0}^{1} |a(\theta+\varepsilon z) - a(\theta)| \varepsilon \mathrm{d}z \\ &= \lim_{\varepsilon\to 0+} \int\limits_{0}^{1} |a(\theta+\varepsilon z) - a(\theta)| \mathrm{d}z = 0.\end{aligned}$$

Now we change the variables on the left-hand side of equality (4) and estimate the resulting integral, we have

$$\begin{aligned}0 &\leqslant \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int\limits_{\theta}^{\theta+\varepsilon} (t-\tau)^{\alpha-1} |a(\tau) - a(\theta)| \mathrm{d}\tau \\ &= \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int\limits_{0}^{1} (t-\theta-\varepsilon z)^{\alpha-1} |a(\theta+\varepsilon z) - a(\theta)| \varepsilon \mathrm{d}z \\ &\leqslant \lim_{\varepsilon\to 0+} (t-\theta-\varepsilon)^{\alpha-1} \lim_{\varepsilon\to 0+} \int\limits_{0}^{1} |a(\theta+\varepsilon z) - a(\theta)| \mathrm{d}z = 0, \quad t \in (\theta+\varepsilon, T].\end{aligned}$$

Hence equality (4) follows. The proof is completed. $\square$

3. Problem statement

Consider a dynamical system whose motion is described by a differential equation with a fractional Caputo derivative of order $\alpha \in (0, 1)$:

$$\left({}^{c}D^{\alpha}_{0+}x\right)(t) = f(t, x(t), u(t)), \quad a.e.\ t \in [0, T], \tag{5}$$

with the initial condition

$$x(0) = x_0 . \tag{6}$$

Here $x(t)$ is n-dimensional vector of phase variables, $u(t)$ is r-dimensional measurable and bounded vector of controlling effects on the seqment $[0, T]$, $0 < T \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ are fixed, function $f(t, x, u)$ is continuous on totality of arguments on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^r$ together with partial derivatives with respect to x, u up to second order, inclusively.

As the set of admissible controls, we take the set of measurable bounded r-dimensional vector functions $u(\cdot)$, taking values from a given nonempty convex closed set $V \subset \mathbb{R}^r$:

$$u(t) \in V, \quad t \in [0, T]. \tag{7}$$

As a solution of the problem (5), (6) corresponding to the fixed control function $u(\cdot)$, we consider the function $x(\cdot) \in AC_\infty^\alpha([0, T], \mathbb{R}^n)$ satisfies differential equation (5) for almost every $t \in [0, T]$ and the initial condition (6).

The goal of the optimal control problem is the minimization of the functional

$$J(u) = \varphi(x(T)) + \frac{1}{\Gamma(\beta)} \int_0^T (T - t)^{\beta-1} f_0(t, x(t), u(t))\,\mathrm{d}t \tag{8}$$

determined in the solutions of problem (5), (6) for admissible control satisfying the condition (7). Here it is supposed that, $\varphi(\cdot)$-a given twice continuously differentiable scalar function defined in $\mathbb{R}^n$ and scalar function $f_0(t, x, u)$ is continuous on totality of arguments on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^r$ together with partial derivatives with respect to x, u up to second order, inclusively. Let $0 < \alpha < 1$ and $\beta \geqslant \alpha$ be fixed.

Let $x(\cdot)$ be a solution to problem (5), (6) corresponding to $u(\cdot)$, then the pair $(x(\cdot), u(\cdot))$ will be called an admissible process. We will assume that problem (5), (6) has a unique solution $x(\cdot)$ for each admissible control $u(\cdot)$. In this case, we call an admissible control $u(\cdot)$ that is the solution of problem (5)–(8) an optimal control, and its corresponding trajectory $x(\cdot)$, an optimal trajectory. Then the pair $(x(\cdot), u(\cdot))$ is said to be an optimal process.

4. The functional increment formula

Let $\{u(\cdot), x(\cdot)\}$ and $\{\widetilde{u}(\cdot) = u(\cdot) + \Delta u(\cdot), \widetilde{x}(\cdot) = x(\cdot) + \Delta x(\cdot)\}$ be two admissible processes. Then applying (5), (6), we obtain that the increment $\Delta x(\cdot)$ satisfies the problem

$$\begin{aligned} &({}^cD_{0+}^\alpha \Delta x)(t) = \Delta f(t, x(t), u(t)), \quad a.e.\ t \in [0, T], \\ &\Delta x(0) = 0, \end{aligned} \tag{9}$$

where $\Delta f(t, x, u) = f(t, \widetilde{x}, \widetilde{u}) - f(t, x, u)$ denotes the total increment of the function $f(t, x, u)$. Then we can represent the increment of the functional in the form

$$\Delta J(u) = J(\widetilde{u}) - J(u) = \Delta\varphi(x(T)) + \frac{1}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} \Delta f_0(t,\, x(t),\, u(t))\,\mathrm{d}t,$$

where $\Delta\varphi(x) = \varphi(\widetilde{x}) - \varphi(x)$, and $\Delta f_0(t,x,u) = f_0(t,\, \widetilde{x},\, \widetilde{u}) - f_0(t,x,u)$.

Let us introduce some nontrivial n-dimensional vector-function $\psi(t)$, $t \in [0,T]$. Then using the formula Teylora, increment of functional may be represented as

$$\begin{aligned}
\Delta J(u) &= \Delta\varphi(x(T)) + \frac{1}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} \Delta f_0(t,\, x(t),\, u(t))\,\mathrm{d}t \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \psi'(t) \left(({}^c D^{\alpha}_{0+}\Delta x)(t) - \Delta f(t,x(t),u(t))\right) \mathrm{d}t \\
&= \varphi'_x(x(T))\Delta x(T) + \frac{1}{2}\Delta x'(T)\varphi_{xx}(x(T))\Delta x(T) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \psi'(t) ({}^c D^{\alpha}_{0+}\Delta x)(t)\,\mathrm{d}t \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big[H'_x(t,x(t),u(t),\psi(t))\Delta x(t) \\
&\quad + H'_u(t,x(t),u(t),\psi(t))\Delta u(t) + \frac{1}{2}\Big(\Delta x'(t) H_{xx}(t,x(t),u(t),\psi(t))\Delta x(t) \\
&\quad + 2\Delta x'(t) H_{xu}(t,x(t),u(t),\psi(t))\Delta u(t) + \Delta u'(t) H_{uu}(t,x(t),u(t),\psi(t))\Delta u(t) \Big) \\
&\quad + o_H(\|\Delta x(t)\|^2 + \|\Delta x(t)\| \cdot \|\Delta u(t)\| + \|\Delta u(t)\|^2) \Big] \mathrm{d}t + o_\varphi(\|\Delta x(T)\|^2),
\end{aligned}$$

where $H(t,x,u,\psi) = \psi' f(t,x,u) - \dfrac{\Gamma(\alpha)}{\Gamma(\beta)} (T-t)^{\beta-\alpha} f_0(t,x,u)$.

Using relation

$$\Delta x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} ({}^c D^{\alpha}_{0+}\Delta x)(\tau)\,\mathrm{d}\tau,$$

we get

$$\Delta J(u) = \frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Bigg[\varphi_x'(x(T)) + \psi'(t)$$
$$- \frac{(T-t)^{1-\alpha}}{\Gamma(\alpha)} \int_t^T (T-\tau)^{\alpha-1} (\tau - t)^{\alpha-1} H_x'(\tau)\,\mathrm{d}\tau \Bigg] ({}^cD_{0+}^{\alpha} \Delta x)(t)\,\mathrm{d}t$$
$$- \frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Bigg[H_u'(t)\Delta u(t) + \frac{1}{2}\Delta x'(t) H_{xx}(t) \Delta x(t)$$
$$+ \Delta x'(t) H_{xu}(t) \Delta u(t) + \frac{1}{2} \Delta u'(t) H_{uu}(t) \Delta u(t)$$
$$+ o_H\left(\|\Delta x(t)\|^2 + \|\Delta x(t)\| \|\Delta u(t)\| + \|\Delta u(t)\|^2\right) \Bigg] \mathrm{d}t$$
$$+ \frac{1}{2} \Delta x'(T) \varphi_{xx}(x(T)) \Delta x(T) + o_{\varphi}(\|\Delta x(T)\|^2). \tag{10}$$

Further, we require that the vector function $\psi(\cdot)$ is a solution to the following integral equation

$$\psi(t) = -\varphi_x(x(T)) + \frac{(T-t)^{1-\alpha}}{\Gamma(\alpha)} \int_t^T (T-\tau)^{\alpha-1} (\tau - t)^{\alpha-1} H_x(\tau)\,\mathrm{d}\tau, \quad t \in [0, T]. \tag{11}$$

Then, increment formula (10) takes the form

$$\Delta J(u) = -\frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(H_u'(t)\Delta u(t) + \frac{1}{2} \Delta x'(t) H_{xx}(t) \Delta x(t)$$
$$+ \Delta x'(t) H_{xu}(t) \Delta u(t) + \frac{1}{2} \Delta u'(t) H_{uu}(t) \Delta u(t)$$
$$+ o_H\left(\|\Delta x(t)\|^2 + \|\Delta x(t)\| \cdot \|\Delta u(t)\| + \|\Delta u(t)\|^2\right) \Big) \mathrm{d}t$$
$$+ \frac{1}{2} \Delta x'(T) \varphi_{xx}(x(T)) \Delta x(T) + o_{\varphi}(\|\Delta x(T)\|^2). \tag{12}$$

5. Adjoint function

The problem (11) is said to be a conjugated problem. We will show that the function $\psi(\cdot)$ defined as a solution to the adjoint equation (11) is continuous. For this, we first prove the following auxiliary lemma.

Lemma 5 *For any $\Phi(\cdot) \in L^\infty([0, T], \mathbb{R}^n)$ the function*

$$\psi(t) = -\varphi_x(x(T)) + \frac{(T-t)^{1-\alpha}}{\Gamma(\alpha)} \int_t^T (T-\tau)^{\alpha-1}(\tau-t)^{\alpha-1}\Phi(\tau)\,\mathrm{d}\tau \tag{13}$$

is continuous on $[0, T]$.

Proof. ased on Lemma 3.2 from [41] equality (13) can be written in the form

$$\psi(t) = -\varphi_x(x(T)) + \frac{1}{\Gamma(\alpha)} \int_t^T (\tau-t)^{\alpha-1}\left(\Phi(\tau) + (R^\alpha_{T-}\Phi(\cdot))(\tau)\right)\mathrm{d}\tau,$$

where

$$(R^\alpha_{T-}\Phi(\cdot))(t) = \frac{(1-\alpha)\sin\alpha\pi}{\pi} \int_t^T K(T-t, T-\tau)\Phi(\tau)\,\mathrm{d}\tau,$$

$$K(\xi,\eta) = \eta^{\alpha-1} \int_0^1 \frac{z^\alpha}{(1-z)^\alpha(\eta + z(\xi-\eta))^\alpha}\,\mathrm{d}z, \quad \xi > \eta > 0.$$

The following estimate holds for $(R^\alpha_{T-}\Phi(\cdot))(t)$:

$$\begin{aligned}
\|(R^\alpha_{T-}\Phi(\cdot))(t)\| &= \Bigg\| \frac{(1-\alpha)\sin\alpha\pi}{\pi} \int_t^T (T-\tau)^{\alpha-1} \\
&\quad \times \int_0^1 \frac{z^\alpha}{(1-z)^\alpha(T-\tau+z(T-t-(T-\tau)))^\alpha}\,\mathrm{d}z\Phi(\tau)\,\mathrm{d}\tau \Bigg\| \\
&\leqslant \frac{(1-\alpha)\sin\alpha\pi}{\pi(T-t)^\alpha} \int_t^T (T-\tau)^{\alpha-1}\|\Phi(\tau)\|\,\mathrm{d}\tau \int_0^1 (1-z)^{-\alpha}\,\mathrm{d}z \\
&\leqslant \frac{\sin\alpha\pi}{\alpha\pi}\|\Phi(\cdot)\|_{[t,T]},
\end{aligned} \tag{14}$$

where $\|\Phi(\cdot)\|_{[t,T]} = \operatorname*{ess\,sup}_{\tau\in[t,T]} \|\Phi(\tau)\|$.

Let $t_1,\ t_2 \in [0,\ T]$ and $t_1 < t_2$. Then

$$\begin{aligned}
\left\|\psi(t_1)-\psi(t_2)\right\| &= \frac{1}{\Gamma(\alpha)}\left\| \int_{t_1}^{T} (\tau-t_1)^{\alpha-1}\big(\Phi(\tau)+(R^{\alpha}_{T-}\Phi(\cdot))(\tau)\big)\,\mathrm{d}\tau \right.\\
&\qquad \left. - \int_{t_2}^{T} (\tau-t_2)^{\alpha-1}\big(\Phi(\tau)+(R^{\alpha}_{T-}\Phi(\cdot))(\tau)\big)\,\mathrm{d}\tau \right\|\\
&\leqslant \frac{1}{\Gamma(\alpha)}\left\| \int_{t_2}^{T} \big((\tau-t_1)^{\alpha-1}-(\tau-t_2)^{\alpha-1}\big)\big(\Phi(\tau)+(R^{\alpha}_{T-}\Phi(\cdot))(\tau)\big)\,\mathrm{d}\tau \right\|\\
&\quad + \frac{1}{\Gamma(\alpha)}\left\| \int_{t_1}^{t_2} (\tau-t_1)^{\alpha-1}(\Phi(\tau)+(R^{\alpha}_{T-}\Phi(\cdot))(\tau)\big)\,\mathrm{d}\tau \right\|\\
&\leqslant \frac{1}{\Gamma(\alpha+1)}\Big(\big|(t_2-t_1)^{\alpha}-((T-t_1)^{\alpha}-(T-t_2)^{\alpha})\big|+(t_2-t_1)^{\alpha}\Big)\, M_{\alpha}\|\Phi(\cdot)\|_{[0,T]}\\
&\leqslant \frac{2}{\Gamma(\alpha+1)} M_{\alpha}\|\Phi(\cdot)\|_{[0,T]}(t_2-t_1)^{\alpha},
\end{aligned}$$

where $M_{\alpha} = 1 + \dfrac{\sin\alpha\pi}{\alpha\pi}$.

Hence the inclusion $\psi(\cdot) \in C([0,\ T],\ \mathbb{R}^n)$ hold.

This completes the proof. $\square$

Lemma 6 *The integral equation (11) has a unique solution in the space* $C_e([0,\ T],\ \mathbb{R}^n)$.

Proof. Denote by $C_e([0,\ T],\ \mathbb{R}^n)$ the space of continuous functions ψ, defined on the segment $[0,\ T]$ with the Bielecki norm

$$\|\psi(\cdot)\|_e = \max_{t\in[0,T]}\left(\|\psi(t)\|e^{-(T-t)k}\right).$$

Obvious the space $C_e([0,\ T],\ \mathbb{R}^n)$ is Banach. Consider the mapping $\Psi = A\psi$ defined by the formula

$$\Psi(t) = (A\psi)(t) \equiv -\varphi_x(x(T)) + \frac{(T-t)^{1-\alpha}}{\Gamma(\alpha)}\int_t^T (T-\tau)^{\alpha-1}(\tau-t)^{\alpha-1}H_x(\tau)\,\mathrm{d}\tau,$$

where $t \in [0,\ T]$.

We will prove that this mapping takes the complete space $C_e([0,T],\mathbb{R}^n)$ into itself and is a contraction in it. Let $\psi(\cdot)\in C_e([0,T],\mathbb{R}^n)$. Since $f_x(\cdot)\in L^\infty([0,T],\mathbb{R}^{n\times n})$, it follows that $f_x'(\cdot)\psi(\cdot)\in L^\infty([0,T],\mathbb{R}^n)$. Then it follows from the Lemma 5 that $\Psi(\cdot)\in C_e([0,T],\mathbb{R}^n)$. So this mapping takes the total space $C_e([0,T],\mathbb{R}^n)$ into itself. Now we will prove that this operator is a contraction. Let us denote $M_{f_x}=\|f_x(\cdot)\|_{[0,T]}$, and let $\psi_1(\cdot),\ \psi_2(\cdot)\in C_e([0,T],\mathbb{R}^n)$ are fixed. Then for any $t\in[0,T]$ we have

$$\|\Psi_1(t)-\Psi_2(t)\|\leqslant\frac{(T-t)^{1-\alpha}M_{f_x}}{\Gamma(\alpha)}\int\limits_t^T(T-\tau)^{\alpha-1}(\tau-t)^{\alpha-1}\|\psi_1(\tau)-\psi_2(\tau)\|\,\mathrm{d}\tau$$

$$=\frac{M_{f_x}}{\Gamma(\alpha)}\int\limits_t^T(\tau-t)^{\alpha-1}\left(\|\psi_1(\tau)-\psi_2(\tau)\|+(R^\alpha_{T-}\|\psi_1(\cdot)-\psi_2(\cdot)\|)(\tau)\right)\mathrm{d}\tau.$$

Using estimates (14) we have

$$\begin{aligned}\|\Psi_1(t)-\Psi_2(t)\|&\leqslant\frac{M_\alpha M_{f_x}}{\Gamma(\alpha)}\int\limits_t^T(\tau-t)^{\alpha-1}\|\psi_1(\cdot)-\psi_2(\cdot)\|_{[\tau,T]}d\tau\\&\leqslant\frac{M_\alpha M_{f_x}}{\Gamma(\alpha)}\|\psi_1(\cdot)-\psi_2(\cdot)\|_e\int\limits_t^T(\tau-t)^{\alpha-1}e^{(T-\tau)k}\,\mathrm{d}\tau\\&=\frac{M_\alpha M_{f_x}}{\Gamma(\alpha)}\|\psi_1(\cdot)-\psi_2(\cdot)\|_e\int\limits_0^{T-t}z^{\alpha-1}e^{(T-t-z)k}\,\mathrm{d}z\\&\leqslant\frac{M_\alpha M_{f_x}}{\Gamma(\alpha)}\|\psi_1(\cdot)-\psi_2(\cdot)\|_e e^{(T-t)k}\int\limits_0^\infty e^{-zk}z^{\alpha-1}\,\mathrm{d}z\\&=\frac{M_\alpha M_{f_x}}{\Gamma(\alpha)}\|\psi_1(\cdot)-\psi_2(\cdot)\|_e\frac{e^{(T-t)k}}{k^\alpha}\int\limits_0^\infty e^{-u}u^{\alpha-1}\,\mathrm{d}u\\&=\frac{M_\alpha M_{f_x}}{k^\alpha}e^{(T-t)k}\|\psi_1(\cdot)-\psi_2(\cdot)\|_e,\end{aligned}$$

where $M_\alpha=1+\dfrac{\sin\alpha\pi}{\alpha\pi}$.

Hence we have

$$\|\Psi_1(\cdot)-\Psi_2(\cdot)\|_e\leqslant\frac{M_\alpha M_{f_x}}{k^\alpha}\|\psi_1(\cdot)-\psi_2(\cdot)\|_e.$$

Thus, due to the choice of the number k, the operator A is a contraction. From this it follows that, according to the Banach principle, the equation $\psi = A\psi$ (i.e. equation (11)) has one and only one continuous solution in the space $C_e([0, T], \mathbb{R}^n)$.

This completes the proof. □

6. Neccesary optimality conditions

In this section we prove the main result of the paper. From the smoothness conditions imposed on the right hand side of system (5) it follows the solution of (9) satisfies also the following problem

$$\begin{aligned}({}^cD^{\alpha}_{0+}\Delta x)(t) &= f_x(t,x(t),u(t))\Delta x(t) + f_u(t,x(t),u(t))\Delta u(t)\\ &\quad + o\left(\|\Delta x(t)\| + \|\Delta u(t)\|\right), \quad a.e.\ t \in [0,T], \\ \Delta x(0) &= 0.\end{aligned} \tag{15}$$

Since the set V is convex, the special control increment $u(\cdot)$ can be determined by the formula

$$\Delta u(t,\ \widetilde{\varepsilon}) = \widetilde{\varepsilon}(\upsilon(t) - u(t)), \quad t \in 0,\ T], \tag{16}$$

where $0 < \widetilde{\varepsilon} \leqslant 1$, and $\upsilon(\cdot) \in L^{\infty}([0,\ T],\ V)$ is an arbitrary vector function.

Let $\Delta x(\cdot,\ \widetilde{\varepsilon})$ denote the special trajectory increment corresponding to the increment (16) of the control. Then $\Delta x(\cdot,\ \widetilde{\varepsilon})$ is determined from the system

$$\begin{aligned}({}^cD^{\alpha}_{0+}\Delta x)(t,\widetilde{\varepsilon}) &= f(t,x(t,\widetilde{\varepsilon}),u(t) + \Delta u(t,\widetilde{\varepsilon}))\\ &\quad - f(t,x(t),u(t)), \quad a.e.\ t \in (0,T], \quad \Delta x(0) = 0.\end{aligned} \tag{17}$$

Therefore, in integral form, problem (17) has the form

$$\begin{aligned}\Delta x(t,\widetilde{\varepsilon}) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Big[f\left(\tau,x(\tau) + \Delta x(\tau,\widetilde{\varepsilon}),u(\tau)\right.\\ \left. + \Delta u(\tau,\widetilde{\varepsilon})\right) - f(\tau,x(\tau),u(\tau))\Big]\mathrm{d}\tau, \quad t \in [0,T].\end{aligned} \tag{18}$$

Denoting by $L = L(\Delta)$ the Lipschitz constant of the function $f(t,x,u)$ in some Δ neighborhood of the trajectory $x(\cdot)$,

$$\begin{aligned}&\|f(t,x(t) + \Delta x(t,\widetilde{\varepsilon}),u(t) + \Delta u(t,\widetilde{\varepsilon})) - f(t,x(t),u(t))\|\\ &\qquad \leqslant L(\|\Delta x(t,\widetilde{\varepsilon})\| + \|\Delta u(t,\widetilde{\varepsilon})\|), \quad t \in [0,T]\end{aligned}$$

from equation (18) we obtain

$$\begin{aligned}\|\Delta x(t,\widetilde{\varepsilon})\| \leqslant \frac{L}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\Delta x(\tau,\widetilde{\varepsilon})\|\,\mathrm{d}\tau \\ + \frac{L\widetilde{\varepsilon}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\upsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau, \quad t\in[0,T].\end{aligned}$$

Hence, based on Lemma 1, and Beta function we have

$$\begin{aligned}\|\Delta x(t,\widetilde{\varepsilon})\| &\leqslant \frac{L\widetilde{\varepsilon}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\upsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau \\ &+ \frac{L^2\widetilde{\varepsilon}}{\Gamma(\alpha)}E_{\alpha,\alpha}(Lt^\alpha)\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\tau)^{\alpha-1}\|\upsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau\,\mathrm{d}s \\ &= \frac{L\widetilde{\varepsilon}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\upsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau \\ &+ \frac{L^2\widetilde{\varepsilon}}{\Gamma(\alpha)}E_{\alpha,\alpha}(Lt^\alpha)\int_0^t \|\upsilon(s)-u(s)\|\int_s^t (t-\tau)^{\alpha-1}(\tau-s)^{\alpha-1}\,\mathrm{d}\tau\,\mathrm{d}s \\ &= \frac{L\widetilde{\varepsilon}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\upsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau \\ &+ \frac{L^2\widetilde{\varepsilon}}{\Gamma(\alpha)}E_{\alpha,\alpha}(Lt^\alpha)\int_0^t (t-s)^{2\alpha-1}\|\upsilon(s)-u(s)\|\int_0^1 (1-z)^{\alpha-1}z^{\alpha-1}\,\mathrm{d}z\,\mathrm{d}s \\ &= \frac{L\widetilde{\varepsilon}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\upsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau \\ &+ \frac{L^2\widetilde{\varepsilon}\Gamma(\alpha)}{\Gamma(2\alpha)}t^\alpha E_{\alpha,\alpha}(Lt^\alpha)\int_0^t (t-\tau)^{\alpha-1}\|\upsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau, \quad t\in[0,T].\end{aligned}$$

Denoting

$$K = 2M\frac{T^\alpha}{\alpha}\left(\frac{L}{\Gamma(\alpha)} + \frac{L^2\Gamma(\alpha)}{\Gamma(2\alpha)}T^\alpha E_{\alpha,\alpha}(LT^\alpha)\right), \quad \|\upsilon(\cdot)\|_{L^\infty} \leqslant M, \quad \|u(\cdot)\|_{L^\infty} \leqslant M,$$

we have

$$\|\Delta x(t,\widetilde{\varepsilon})\| \leqslant K\widetilde{\varepsilon}, \quad t \in [0, T]. \tag{19}$$

Lemma 7 *For a special increment $\Delta x(\cdot,\varepsilon)$ of the trajectory $x(\cdot)$ of system (5), the following expansion*

$$\Delta x(t, \widetilde{\varepsilon}) = \widetilde{\varepsilon}\delta x(t) + o(\widetilde{\varepsilon}, t), \quad t \in [0, T] \tag{20}$$

takes place, where $\delta x(\cdot)$ is a solution to problem

$$\begin{aligned} ({}^c D^\alpha \delta x)(t) &= f_x(t, x(t), u(t))\delta x(t) \\ &\quad + f_u(t, x(t), u(t))(v(t) - u(t)), \quad a.e.\ t \in [0, T] \\ \delta x(0) &= 0. \end{aligned} \tag{21}$$

Proof. It follows from system (17) that $\Delta x(\cdot,\widetilde{\varepsilon})$ satisfies the following linearized system

$$\begin{aligned} ({}^c D^\alpha_{0+} \Delta x)(t,\widetilde{\varepsilon}) &= f_x(t)\Delta x(t,\widetilde{\varepsilon}) + f_u(t)\Delta u(t,\widetilde{\varepsilon}) \\ &\quad + o(\|\Delta x(t,\widetilde{\varepsilon})\| + \|\Delta u(t,\widetilde{\varepsilon})\|), \quad a.e.\ t \in [0, T], \\ \Delta x(0,\widetilde{\varepsilon}) &= 0. \end{aligned} \tag{22}$$

Interpreting equation (22) as a linear inhomogeneous fractional differential equation with respect to $\Delta x(\cdot, \widetilde{\varepsilon})$, taking into account the estimate (19) and the increment (16), on the basic of an analogue of the Caychy formula about the integral representation of solutions of such equations [48], we have

$$\begin{aligned} \Delta x(t,\widetilde{\varepsilon}) = \frac{\widetilde{\varepsilon}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(t,\tau) f_u(\tau)(v(\tau) - u(\tau))\,\mathrm{d}\tau \\ + o(t,\widetilde{\varepsilon}), \quad t \in [0, T], \end{aligned} \tag{23}$$

where the matrix-function $F(\cdot,\cdot)$ is a solution of the following integral equation [48]:

$$F(t,\tau) = I + \frac{(t-\tau)^{1-\alpha}}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\alpha-1} F(t,s) f_x(s)\,\mathrm{d}s, \quad 0 \leqslant \tau \leqslant t \leqslant T.$$

Denoting

$$\delta x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(t,\tau) f_u(\tau)(v(\tau) - u(\tau))\,\mathrm{d}\tau, \quad t \in [0, T], \tag{24}$$

of formula (23) we write in the form (20).

Now we will show that the function $\delta x(\cdot)$ defined by formula (24) is a solution to problem (21). For this, we write problem (21) in the form of an equivalent integral equation:

$$\delta x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [f_x(\tau)\delta x(\tau) + f_u(\tau)\delta u(\tau)]\,\mathrm{d}\tau, \quad t \in [0,T].$$

Here, taking into account formula (24), we have

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(t,\tau) f_u(\tau)\delta u(\tau) d\tau &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_x(\tau) \frac{1}{\Gamma(\alpha)} \\ &\times \int_0^\tau (\tau - s)^{\alpha-1} F(\tau,s) f_u(s)\delta u(s)\,\mathrm{d}s\,\mathrm{d}\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_u(\tau)\delta u(\tau)\,\mathrm{d}\tau, \quad t \in [0,T]. \end{aligned}$$

Using Dirichlet's formulas, we write this equality in the form:

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \Bigg[F(t,\tau) \\ &\qquad - \frac{(t-\tau)^{1-\alpha}}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\alpha-1} f_x(s) F(s,\tau)\,\mathrm{d}\tau \Bigg] f_u(\tau)\delta u(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_u(\tau)\delta u(\tau)\,\mathrm{d}\tau, \quad t \in [0,T]. \end{aligned} \tag{25}$$

Since the matrix function $F(\cdot,\cdot)$ with respect to the first argument satisfies the equation [48]:

$$F(t,\tau) = I + \frac{(t-\tau)^{1-\alpha}}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\alpha-1} f_x(s) F(s,\tau)\,\mathrm{d}s, \quad t \in [\tau,\, T],$$

then (25) implies the required result. With this the lemma is proved. □

Taking into account Lemma 7 and estimate (19), from (12) we obtain that

$$\Delta J(u) = J(u(t) + \Delta u(t, \widetilde{\varepsilon})) - J(u(t)) = \widetilde{\varepsilon}\delta J(u) + \frac{\widetilde{\varepsilon}^2}{2}\delta^2 J(u) + o(\widetilde{\varepsilon}^2), \tag{26}$$

where

$$\delta J(u) = -\frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} H_u'(t)(\upsilon(t) - u(t))\,\mathrm{d}t,$$

$$\begin{aligned}\delta^2 J(u) = -\frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big[&(\upsilon(t) - u(t))' H_{uu}(t)(\upsilon(t) - u(t)) \\ &+ 2(\upsilon(t) - u(t))' H_{ux}(t)\delta x(t) + \delta x'(t) H_{xx}(t)\delta x(t) \Big]\,\mathrm{d}t \\ &+ \delta x'(T)\varphi_{xx}(x(T))\delta x(T).\end{aligned}$$

In what follows we assume that $u(\cdot)$ is the optimal control. Then it follows from (26) that

$$\Delta J(u) \geqslant 0 \tag{27}$$

and

$$\int_0^T (T-t)^{\alpha-1} H_u'(t)(\upsilon(t) - u(t))\,\mathrm{d}t \leqslant 0. \tag{28}$$

Now we define $\upsilon(\cdot)$ by formula

$$\upsilon(t) = \begin{cases} \upsilon, & t \in [\theta,\ \theta+\varepsilon), \\ u(t), & t \in [0, T] \backslash [\theta,\ \theta+\varepsilon), \end{cases} \tag{29}$$

where $\upsilon \in V$, $\varepsilon > 0$ is a sufficiently small parameter, $\theta \in [0, T)$, $\theta + \varepsilon < T$, is the Lebesgue point. Then inequality (28) takes the form

$$\int_\theta^{\theta+\varepsilon} (T-t)^{\alpha-1} H_u'(t)(\upsilon - u(t))\,\mathrm{d}t \leqslant 0.$$

This implies the following theorem.

Theorem 1 *Let the admissible process $\{u(\cdot),\ x(\cdot)\}$ be optimal in problem (5)-(8) and let $\psi(\cdot)$ be a solution of conjugated problem (11) calculated on optimal process. Then for almost all $t \in [0,\ T]$ the following equality is fulfilled*

$$\max_{\upsilon \in V} H_u'(t, x(t), u(t), \psi(t))\upsilon = H_u'(t, x(t), u(t), \psi(t))u(t). \tag{30}$$

Condition (30) is a necessary first order optimality condition. Condition (30) is called the linearized maximum principle or differential maximum principle. Following [15], we introduce the concept of quasi-singular control.

Definition 4 *The control $u(\cdot)$ satisfying condition (30) is called quasi-singular if equality*

$$H_u'(t)(\upsilon - u(t)) = 0 \tag{31}$$

is satisfied for all $\upsilon \in V$ and a.e. $t \in [0, T]$.

It is obvious that for a quasi-singular control $u(\cdot)$ the differential maximum principle (30) becomes in effective. A control that is singular in the sense of the Pontryagin maximum principle, is also quasi-singular and the converse is generally not true. In other words, quasi-singular control may not be singular in the sense of the Pontryagin maximum principle. Therefore, the necessary conditions for the optimality of quasi-singular controls also make it possible, in many cases, to reveal the nonoptimality of those admissible controls for which the Pontryagin maximum principle holds without degeneration.

Now we proceed to the derivation of the necessary conditions for the optimality of quasi-singular controls in the considered problem (5)–(8).

For quasi-singular optimal controls from (27) it follows that

$$\int_0^T (T-t)^{\alpha-1}\Big[(\upsilon(t)-u(t))'H_{uu}(t)(\upsilon(t)-u(t)) + 2(\upsilon(t)-u(t))'H_{ux}(t)\delta x(t) \\ + \delta x'(t)H_{xx}(t)\delta x(t)\Big]\mathrm{d}t - \Gamma(\alpha)\delta x'(T)\varphi_{xx}(x(T))\delta x(T) \leqslant 0. \tag{32}$$

To obtain effectively verifiable necessary optimality conditions for quasi-singular controls, we define $\upsilon(\cdot)$ in the form (29). First we find the expansion of $\delta x(t)$ in powers of ε. For $t \in [0, \theta]$ from (24) it follows $\delta x(t) \equiv 0$. For $t \in [\theta, \theta+\varepsilon]$ from formula (24) we obtain

$$\begin{aligned}
\delta x(t) &= \frac{1}{\Gamma(\alpha)}\int_\theta^t (t-\tau)^{\alpha-1}F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\,\mathrm{d}\tau \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\theta^t (t-\tau)^{\alpha-1}\big[F(t,\tau)f_u(\tau)(\upsilon-u(\tau)) - F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\big]\mathrm{d}\tau \\
&= \frac{(t-\theta)^\alpha}{\Gamma(\alpha+1)}F(t,\theta)f_u(\theta)(\upsilon-u(\theta)) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\theta^t (t-\tau)^{\alpha-1}\big[F(t,\tau)f_u(\tau)(\upsilon-u(\tau)) - F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\big]\mathrm{d}\tau.
\end{aligned}$$

Show that

$$\lim_{t\to\theta^+}\frac{1}{(t-\theta)^\alpha}\int\limits_\theta^t (t-\tau)^{\alpha-1}\big[F(t,\tau)f_u(\tau)(\upsilon-u(\tau))$$
$$-F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\big]\,\mathrm{d}\tau=0. \tag{33}$$

Denoting by $M_F=\max\limits_{(t,\tau)\in[0,T]\times[0,T]}\|F(t,\tau)\|$, $M_{f_u}=\operatorname*{ess\,sup}\limits_{0\leqslant t\leqslant T}\|f_u(t)\|$, then

$$\left\|\frac{1}{(t-\theta)^\alpha}\int\limits_\theta^t (t-\tau)^{\alpha-1}\big[F(t,\tau)f_u(\tau)(\upsilon-u(\tau))-F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\big]\,\mathrm{d}\tau\right\|$$
$$\leqslant\frac{M_F\cdot M_{f_u}}{(t-\theta)^\alpha}\int\limits_\theta^t (t-\tau)^{\alpha-1}\|u(\theta)-u(\tau)\|\,\mathrm{d}\tau$$
$$+\frac{M_F\|\upsilon-u(\theta)\|}{(t-\theta)^\alpha}\int\limits_\theta^t (t-\tau)^{\alpha-1}\|f_u(\tau)-f_u(\theta)\|\,\mathrm{d}\tau$$
$$+\frac{M_{f_u}\|\upsilon-u(\theta)\|}{(t-\theta)^\alpha}\int\limits_\theta^t (t-\tau)^{\alpha-1}\|F(t,\tau)-F(t,\theta)\|\,\mathrm{d}\tau.$$

By Lemma 3, each term on the right-hand side of this inequality tends to zero for $t\longrightarrow\theta^+$. Therefore, equality (33) is true.

Thus, for $t\in[\theta,\ \theta+\varepsilon]$ we got the equality:

$$\delta x(t)=\frac{(t-\theta)^\alpha}{\Gamma(\alpha+1)}F(t,\theta)f_u(\theta)(\upsilon-u(\theta))+o((t-\theta)^\alpha). \tag{34}$$

For $t\in(\theta+\varepsilon,\ T]$ from formula (24) we obtain:

$$\delta x(t)=\frac{1}{\Gamma(\alpha)}\int\limits_\theta^{\theta+\varepsilon}(t-\tau)^{\alpha-1}F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\,\mathrm{d}\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int\limits_\theta^{\theta+\varepsilon}(t-\tau)^{\alpha-1}\big[F(t,\tau)f_u(\tau)(\upsilon-u(\tau))-F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\big]\,\mathrm{d}\tau.$$

Show that

$$\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\int\limits_{\theta}^{\theta+\varepsilon}(t-\tau)^{\alpha-1}\big[F(t,\tau)f_u(\tau)(\upsilon-u(\tau))\\ -F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\big]d\tau=0,\qquad t\in(\theta+\varepsilon,T].\tag{35}$$

The following inequality is true

$$\left\|\frac{1}{\varepsilon}\int\limits_{\theta}^{\theta+\varepsilon}(t-\tau)^{\alpha-1}\big[F(t,\tau)f_u(\tau)(\upsilon-u(\tau))-F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\big]\,\mathrm{d}\tau\right\|$$

$$\leqslant\frac{M_F M_{f_u}}{\varepsilon}\int\limits_{\theta}^{\theta+\varepsilon}(t-\tau)^{\alpha-1}\|u(\theta)-u(\tau)\|\,\mathrm{d}\tau$$

$$+\frac{M_F\|\upsilon-u(\theta)\|}{\varepsilon}\int\limits_{\theta}^{\theta+\varepsilon}(t-\tau)^{\alpha-1}\|f_u(\tau)-f_u(\theta)\|\,\mathrm{d}\tau$$

$$+\frac{M_{f_u}\|\upsilon-u(\theta)\|}{\varepsilon}\int\limits_{\theta}^{\theta+\varepsilon}(t-\tau)^{\alpha-1}\|F(t,\tau)-F(t,\theta)\|\,\mathrm{d}\tau.$$

By Lemma 4, each term on the right-hand side of this inequality tends to zero for $\varepsilon\to 0^+$. Therefore, equality (35) is true.

Thus, for $t\in(\theta+\varepsilon,\,T]$ we got the equality:

$$\delta x(t)=\frac{1}{\Gamma(\alpha+1)}\big((t-\theta)^{\alpha}-(t-(\theta+\varepsilon))^{\alpha}\big)F(t,\theta)f_u(\theta)(\upsilon-u(\theta))+o(\varepsilon)$$

or

$$\delta x(t)=\frac{\varepsilon}{\Gamma(\alpha)}(t-\theta)^{\alpha-1}F(t,\theta)f_u(\theta)(\upsilon-u(\theta))+o(\varepsilon).\tag{36}$$

Taking into account this representation in (32), we obtain

$$\int\limits_{\theta}^{\theta+\varepsilon}(T-t)^{\alpha-1}(\upsilon-u(t))'H_{uu}(t)(\upsilon-u(t))\,\mathrm{d}t+\frac{2}{\Gamma(\alpha+1)}\int\limits_{\theta}^{\theta+\varepsilon}(T-t)^{\alpha-1}\\ \times(t-\theta)^{\alpha}(\upsilon-u(\theta))'H_{ux}(t)F(t,\theta)f_u(\theta)(\upsilon-u(\theta))\,\mathrm{d}t+\eta\leqslant 0,\tag{37}$$

where $\eta=\sum\limits_{i=1}^{4}\eta_i$,

$$\eta_1 = \frac{1}{\alpha^2\Gamma^2(\alpha)}(\upsilon - u(\theta))' f_u'(\theta) \int\limits_\theta^{\theta+\varepsilon} (T-t)^{\alpha-1}(t-\theta)^{2\alpha} F'(t,\theta) H_{xx}(t) F(t,\theta)\,\mathrm{d}t$$
$$\times f_u(\theta)(\upsilon - u(\theta)),$$
$$\eta_2 = \frac{\varepsilon^2}{\Gamma^2(\alpha)}(\upsilon - u(\theta))' f_u'(\theta) \int\limits_{\theta+\varepsilon}^{T} (T-t)^{\alpha-1}(t-\theta)^{2\alpha-2} F'(t,\theta) H_{xx}(t) F(t,\theta)\,\mathrm{d}t$$
$$\times f_u(\theta)(\upsilon - u(\theta)),$$
$$\eta_3 = \frac{\varepsilon^2}{\Gamma(\alpha)}(T-\theta)^{2\alpha-2}(\upsilon-u(\theta))' f_u'(\theta) F'(T,\theta)\varphi_{xx}(x(T)) F(T,\theta) f_u(\theta)(\upsilon-u(\theta)),$$
$$\eta_4 = o(\varepsilon^{1+\alpha}).$$

Denoted by

$$M_1(\theta,\upsilon) = \operatorname*{ess\,sup}_{t\in[0,T]} \left| \frac{1}{\Gamma^2(\alpha)}(\upsilon-u(\theta))' f_u'(\theta) F'(t,\theta) H_{xx}(t) F(t,\theta) f_u(\theta)(\upsilon-u(\theta)) \right|,$$
$$M_2(\theta,\upsilon) = \left| \frac{1}{\Gamma(\alpha)}(\upsilon - u(\theta))' f_u'(\theta) F'(T,\theta)\varphi_{xx}(x(T)) F(T,\theta) f_u(\theta)(\upsilon - u(\theta)) \right|,$$

and estimate the remainder terms η_i, $i = 1,\ 2,\ 3$.

$$|\eta_1| \leqslant \frac{M_1(\theta,\upsilon)}{\alpha^2} \int\limits_\theta^{\theta+\varepsilon} (T-t)^{\alpha-1}(t-\theta)^{2\alpha}\,\mathrm{d}t < \varepsilon^{2\alpha}\frac{M_1(\theta,\upsilon)}{\alpha^2} \int\limits_\theta^{\theta+\varepsilon} (T-t)^{\alpha-1}\,\mathrm{d}t$$
$$= \varepsilon^{1+2\alpha}\frac{M_1(\theta,\upsilon)}{\alpha^2}(T-\xi)^{\alpha-1}, \tag{38}$$
$$|\eta_3| \leqslant \varepsilon^2 M_2(\theta,\upsilon)(T-\theta)^{2\alpha-2}, \tag{39}$$
$$|\eta_2| \leqslant \varepsilon^2 M_1(\theta,\upsilon) \int\limits_{\theta+\varepsilon}^{T} (T-t)^{\alpha-1}(t-\theta)^{2\alpha-2}\,\mathrm{d}t$$
$$= \varepsilon^2 M_1(\theta,\upsilon) \int\limits_{\theta+\varepsilon}^{\xi} (T-t)^{\alpha-1}(t-\theta)^{2\alpha-2}\,\mathrm{d}t$$
$$+\ \varepsilon^2 M_1(\theta,\upsilon) \int\limits_{\xi}^{T} (T-t)^{\alpha-1}(t-\theta)^{2\alpha-2}\,\mathrm{d}t, \tag{40}$$

where $\eta \in (\theta+\varepsilon,\ T)$ is a fixed point and $\xi = \dfrac{T+\eta}{2}$.

First, we estimate the second term on the right-hand side of relation (40):

$$\varepsilon^2 M_1(\theta,\upsilon)\int_{\xi}^{T}(T-t)^{\alpha-1}(t-\theta)^{2\alpha-2}\,\mathrm{d}t < \varepsilon^2 M_1(\theta,\upsilon)(\xi-\theta)^{2\alpha-2}\int_{\xi}^{T}(T-t)^{\alpha-1}\,\mathrm{d}t$$
$$= \varepsilon^2\frac{(T-\xi)^{\alpha}}{\alpha}(\xi-\theta)^{2\alpha-2}M_1(\theta,\upsilon). \tag{41}$$

Now let us estimate the first term on the right-hand side of relation (40). To do this, consider the following cases:

a) $0<\alpha<\dfrac{1}{2}$,

$$\varepsilon^2 M_1(\theta,\upsilon)\int_{\theta+\varepsilon}^{\xi}(T-t)^{\alpha-1}(t-\theta)^{2\alpha-2}\,\mathrm{d}t < \varepsilon^2(T-\xi)^{\alpha-1}M_1(\theta,\upsilon)\int_{\theta+\varepsilon}^{\xi}(t-\theta)^{2\alpha-2}\,\mathrm{d}t$$
$$= \varepsilon^2(T-\xi)^{\alpha-1}M_1(\theta,\upsilon)\left(\frac{(\xi-\theta)^{2\alpha-1}}{2\alpha-1}-\frac{\varepsilon^{2\alpha-1}}{2\alpha-1}\right)$$
$$< \frac{(T-\xi)^{\alpha-1}M_1(\theta,\upsilon)}{(1-2\alpha)}\varepsilon^{1+2\alpha}. \tag{42}$$

b) $\alpha=\dfrac{1}{2}$,

$$\varepsilon^2 M_1(\theta,\upsilon)\int_{\theta+\varepsilon}^{\xi}(T-t)^{-0{,}5}(t-\theta)^{-1}\,\mathrm{d}t < \frac{\varepsilon^2 M_1(\theta,\upsilon)}{\sqrt{T-\xi}}\ln\frac{\xi-\theta}{\varepsilon}. \tag{43}$$

c) $\dfrac{1}{2}<\alpha<1$,

$$\varepsilon^2 M_1(\theta,\upsilon)\int_{\theta+\varepsilon}^{\xi}(T-t)^{\alpha-1}(t-\theta)^{2\alpha-2}\,\mathrm{d}t < \frac{\varepsilon^2 M_1(\theta,\upsilon)}{(T-\xi)^{1-\alpha}}\int_{\theta+\varepsilon}^{\xi}(t-\theta)^{2\alpha-2}\,\mathrm{d}t$$
$$= \frac{\varepsilon^2 M_1(\theta,\upsilon)}{(T-\xi)^{1-\alpha}}\left(\frac{(\xi-\theta)^{2\alpha-1}}{2\alpha-1}-\frac{\varepsilon^{2\alpha-1}}{2\alpha-1}\right)$$
$$< \frac{\varepsilon^2(\xi-\theta)^{2\alpha-1}M_1(\theta,\upsilon)}{(2\alpha-1)(T-\xi)^{1-\alpha}}. \tag{44}$$

Using Lemma 2, we expand the second term in (37):

$$
\begin{aligned}
&\frac{2}{\Gamma(\alpha+1)}\int\limits_{\theta}^{\theta+\varepsilon}(t-\theta)^{\alpha}(T-t)^{\alpha-1}(\upsilon-u(\theta))'H_{ux}(t)F(t,\theta)\,\mathrm{d}t\,f_u(\theta)(\upsilon-u(\theta))\\
&=\frac{2\varepsilon^{\alpha+1}}{\Gamma(\alpha+2)}(T-\theta)^{\alpha-1}(\upsilon-u(\theta))'H_{ux}(\theta)f_u(\theta)(\upsilon-u(\theta))+o(\varepsilon^{1+\alpha}). \qquad (45)
\end{aligned}
$$

Taking into account estimates (38), (39), (41)–(44) and equality (45) in inequality (37), we have

$$
\begin{aligned}
&\int\limits_{\theta}^{\theta+\varepsilon}(T-t)^{\alpha-1}(\upsilon-u(t))'H_{uu}(t)(\upsilon-u(t))\,\mathrm{d}t+\frac{2\varepsilon^{\alpha+1}}{\Gamma(\alpha+2)}\\
&\quad\cdot(T-\theta)^{\alpha-1}(\upsilon-u(\theta))H_{ux}(\theta)f_u(\theta)(\upsilon-u(\theta))+o(\varepsilon^{1+\alpha})\leqslant 0. \qquad (46)
\end{aligned}
$$

Inequality (46) immediately implies the following theorem.

Theorem 2 *For the optimality of the quasi-singular control $u(\cdot)$ in problem (5)–(8), it is necessary that inequality*

$$(\upsilon-u(t))'H_{uu}(t)(\upsilon-u(t))\leqslant 0 \qquad (47)$$

be satisfied for all $\upsilon\in V$ and a.e. $t\in[0,T]$.

Note that if condition (30) degenerates, the necessary optimality condition (47) can reveal the nonoptimality of the quasi-singular control. However, examples can be constructed in which condition (47) is also expressed, i.e. performed trivially.

Definition 5 *An admissible control $u(\cdot)$ that is quasi-singular is called strongly quasi-singular if the equality*

$$(\upsilon-u(t))'H_{uu}(t)(\upsilon-u(t))=0 \qquad (48)$$

holds for all $\upsilon\in V$ and for a.e. $t\in[0,T]$.

Further, from inequality (46), we easily obtain the necessary optimality condition for strongly quasi-singular controls.

Theorem 3 *For the optimality of the strongly quasi-singular control $u(\cdot)$ in problem (5)–(8), it is necessary that inequality*

$$(\upsilon-u(t))'H_{ux}(t)f_u(t)(\upsilon-u(t))\leqslant 0 \qquad (49)$$

be satisfied for all $\upsilon\in V$ and a.e. $t\in[0,T]$.

To illustrate the effectiveness of the necessary optimality condition (49), consider the example.

Example. Consider the problem

$$
\begin{aligned}
&\left({}^cD^{\alpha}_{0+}x_1\right)(t) = u(t),\\
&\left({}^cD^{\alpha}_{0+}x_2\right)(t) = -u(t)x_1(t) + u^4(t), \quad x_i(0) = 0, \quad i = 1, 2, \quad t \in [0,\ 1],\\
&-1 \leqslant u \leqslant 0, \qquad J(u) = x_2(1) \longrightarrow \min .
\end{aligned}
\tag{50}
$$

We investigate the optimality of the control $u(t) = 0$, $t \in [0,\ 1]$. This control corresponds to the solution $x_i = 0$, $i = 1, 2$, of system (50). Along the process $\{0,\ 0\}$ we have:

$$
\psi_1(t) = 0, \qquad \psi_2(t) = -1, \quad t \in [0,\ 1].
$$

Along the control $u(t) \equiv 0$, $t \in [0,\ 1]$, the Pontyagin maximum principle $\Delta_{\upsilon}H = -\upsilon^4 \leqslant 0$, $\forall \upsilon \in [-1,\ 0]$ holds, and the result of work [6, 48] leaves this control among the candidates for optimality. On the other hand $H_u = 0$, $H_{uu} = 0$. Hence, the control $u(t) = 0$, $t \in [0, 1]$ is strongly quasi-singular. On this control the condition (49) takes the form

$$
H'_{ux}(t) f_u(t)\upsilon^2 = \upsilon^2 \leqslant 0, \quad t \in [0, 1],
$$

which is not satisfied for all $\upsilon \in [-1,\ 0)$. This shows that the control $u(t) = 0$, $t \in [0, 1]$ cannot be optimal.

Obviously, when controlling $u(t) = 0$, $t \in [0, 1]$, the quality criterion takes on the value $J(u) = x_2(1) = 0$. Let as see if there is another control function along which the values of the objective functional are less than zero. Calculate the value of J for admissible control $u(t) = -\frac{1}{2}$, $t \in [0, 1]$. For this function

$$
\begin{aligned}
x_1(t) &= -\frac{t^{\alpha}}{2\Gamma(\alpha+1)},\\
x_2(t) &= \frac{t^{\alpha}}{16\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{4\Gamma(2\alpha+1)}, \quad t \in [0, 1].
\end{aligned}
$$

Then we have

$$
J\left(-\frac{1}{2}\right) = \frac{\Gamma(2\alpha+1) - 4\Gamma(\alpha+1)}{16\Gamma(\alpha+1)\Gamma(2\alpha+1)} < 0 = J(0).
$$

This shows that the control $u(t) = 0$, $t \in [0,\ 1]$, is not optimal.

7. Conclusion

In this paper, we consider a fractional optimal control problem, when the state of the system is described by a nonlinear fractional order Caputo differential equation. The posed problem of optimal control is investigated using a new version of the increment method, in which the concept of a conjugate equation of an integral form is essentially used. Applying the Banach fixed point principle, the existence and uniqueness of a solution to the adjoint problem is proved for a fixed admissible control. The necessary optimality condition is obtained in the form of a linearized maximum principle. Further, the concept of a quasi-singular control is introduced and, on its basis, an analogue of the Legendre-Clebsch conditions is obtained. When expressing an analogue of the Legendre-Clebsch condition, one necessary high-order optimality condition is obtained. Note that the result obtained in some cases also excludes those controls that are not singular in the sense of the Pontryagin maximum principle. The approach presented here can be applied to the derivation of necessary optimality conditions in the form of a linearized maximum principle, the Legendre-Clebsch condition and a high order for an optimal control problem in which the system is controlled by a nonlinear fractional Caputo partial differential equation.

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