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On the robustness of the integrable trajectories of the control systems with limited control resources

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The control system described by Urysohn type integral equation is considered where the system is nonlinear with respect to the phase vector and is affine with respect to the control vector. The control functions are chosen from the closed ball of the space $L_q(\Omega; \mathbb{R}^m)$, $q > 1$, with radius r and centered at the origin. The trajectory of the system is defined as p -integrable multivariable function from the space $L_p(\Omega; \mathbb{R}^n)$, $\frac{1}{q} + \frac{1}{p} = 1$, satisfying the system's equation almost everywhere. It is shown that the system's trajectories are robust with respect to the fast consumption of the remaining control resource. Applying this result it is proved that every trajectory can be approximated by the trajectory obtained by full consumption of the total control resource.

Key words: nonlinear control system, integral equation, integral constraint, integrable trajectory, robustness.

1. Introduction

The control systems described by integral equations is one of the important chapters of the control systems theory. The integral models undoubtedly have some advantages over differential ones, since the integral models allow to use continuous, and even integrable functions as the system's trajectory. It should be also underlined that the solution concepts for different type of initial and boundary value problems for differential equations can be reduced to solution notions for appropriate integral equations. Note that the theory of the linear integral equations

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is considered one of the origins of the contemporary functional analysis (see, e.g., [3, 4, 6, 10, 11]).

The integral constraint on the control functions is inevitable, if the control resource is exhausted by consumption such as energy, fuel, finance, etc. For example, the motion of the flying object with rapidly changing mass is described by a control system with integral constraint on the control functions (see, e.g., [2, 12]). The integral constraint on the control functions differs from geometric constraint, since the integral boundedness of a function does not guarantee its geometric boundedness. Therefore, the control systems with integral constraints on the control functions have special behaviour and investigation of these systems requires different approaches (see, e.g., [5, 8, 9, 12, 14, 16] and references therein).

In this paper the robustness of p -integrable trajectories of the control system with respect to the fast consumption of the remaining control resource is discussed. The admissible control functions are chosen from the closed ball of the space L_q with radius r and centered at the origin. The behaviour of the control system is described by Urysohn type integral equation and the trajectory of the system is defined as multivariable p -integrable function ($1/q + 1/p = 1$) which satisfies the system's equation almost everywhere. Note that this kind of trajectories often arise in different problems of contemporary physics (see, e.g., [1, 13, 15]). It is shown that aggressive consumption of the control resource is not advisable way to spend the control resource, and to achieve the essential change in the system's trajectory, the control resource should be consumed in economy mode, i.e. with small portions in the domains which have a sufficiently small Lebesgue measure.

The paper is organised as follows. In section 2 the basic conditions and preliminary propositions, which are used in following arguments, are given. In Section 3 it is proved that every trajectory is robust with respect to the fast consumption of the remaining control resource (Theorem 1). It is also proved the set of trajectories coincides with the closure of the set of trajectories obtained by full consumption of the total control resource (Theorem 2).

2. Preliminaries

Consider the control system described by the Urysohn type integral equation

$$x(\xi) = f(\xi, x(\xi)) + \int_{\Omega} [K_1(\xi, s, x(s)) + K_2(\xi, s, x(s)) u(s)] ds, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, $\xi \in \Omega$, $\Omega \subset \mathbb{R}^k$ is a compact set.

For given $q > 1$ and $r > 0$ we denote

$$U_{q,r} = \{u(\cdot) \in L_q(\Omega; \mathbb{R}^m) : \|u(\cdot)\|_q \leq r\},$$

where $L_q(\Omega; \mathbb{R}^m)$ is the space of Lebesgue measurable functions $u(\cdot) : \Omega \rightarrow \mathbb{R}^m$

such that $\|u(\cdot)\|_q < +\infty$, $\|u(\cdot)\|_q = \left(\int_{\Omega} \|u(s)\|^q ds \right)^{\frac{1}{q}}$, $\|\cdot\|$ denotes the Euclidean norm.

$U_{q,r}$ is called the set of admissible control functions and every $u(\cdot) \in U_{q,r}$ is said to be an admissible control function.

For given $q \in (1, +\infty)$ let $p \in (1, +\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. It is assumed that the following conditions are satisfied.

2.A. The function $f(\cdot, x) : \Omega \rightarrow \mathbb{R}^n$ is Lebesgue measurable for every fixed $x \in \mathbb{R}^n$, $f(\cdot, 0) \in L_p(\Omega; \mathbb{R}^n)$ and there exists $\gamma_0(\cdot) \in L_{\infty}(\Omega; [0, +\infty))$ such that for almost all (a.a.) $\xi \in \Omega$ the inequality

$$\|f(\xi, x_1) - f(\xi, x_2)\| \leq \gamma_0(\xi) \|x_1 - x_2\|$$

is satisfied for every $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$, where $L_{\infty}(\Omega; \mathbb{R}^{n_0})$ is the space of Lebesgue measurable functions $g(\cdot) : \Omega \rightarrow \mathbb{R}^{n_0}$ such that $\|g(\cdot)\|_{\infty} < +\infty$, $\|g(\cdot)\|_{\infty} = \inf\{c > 0 : \|g(s)\| \leq c \text{ for a.a. } s \in \Omega\}$;

2.B. The function $K_1(\cdot, \cdot, x) : \Omega \times \Omega \rightarrow \mathbb{R}^n$ is Lebesgue measurable for every fixed $x \in \mathbb{R}^n$, $K_1(\cdot, \cdot, 0) \in L_p(\Omega \times \Omega; \mathbb{R}^n)$ and there exists $\gamma_1(\cdot, \cdot) : \Omega \times \Omega \rightarrow [0, +\infty)$ such that

$$\int_{\Omega} \left(\int_{\Omega} \gamma_1(\xi, s)^q ds \right)^{\frac{p}{q}} d\xi < +\infty$$

and for a.a. $(\xi, s) \in \Omega \times \Omega$ the inequality

$$\|K_1(\xi, s, x_1) - K_1(\xi, s, x_2)\| \leq \gamma_1(\xi, s) \|x_1 - x_2\|$$

is satisfied for every $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$;

2.C. The function $K_2(\cdot, \cdot, x) : \Omega \times \Omega \rightarrow \mathbb{R}^{n \times m}$ is Lebesgue measurable for every fixed $x \in \mathbb{R}^n$, $K_2(\cdot, \cdot, 0) \in L_p(\Omega \times \Omega; \mathbb{R}^{n \times m})$ and there exists $\gamma_2(\cdot, \cdot) \in L_{\infty}(\Omega \times \Omega; [0, +\infty))$ such that for a.a. $(\xi, s) \in \Omega \times \Omega$ the inequality

$$\|K_2(\xi, s, x_1) - K_2(\xi, s, x_2)\| \leq \gamma_2(\xi, s) \|x_1 - x_2\|$$

is satisfied for every $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$;

2.D. The inequality

$$6^{p-1} [\kappa_0^p + \kappa_1^p + r^p \kappa_2^p \mu(\Omega)] < 1$$

is satisfied, where $\mu(\Omega)$ denotes the Lebesgue measure of the set Ω ,

$$\kappa_0 = \|\gamma_0(\cdot)\|_\infty, \quad \kappa_2 = \|\gamma_2(\cdot, \cdot)\|_\infty, \quad (2)$$

$$\kappa_1 = \left(\int_\Omega \left(\int_\Omega \gamma_1(\xi, s)^q ds \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}. \quad (3)$$

Denote

$$\alpha_0 = \left(\int_\Omega \|f(\xi, 0)\|^p d\xi \right)^{\frac{1}{p}}, \quad \alpha_i = \left(\int_\Omega \int_\Omega \|K_i(\xi, s, 0)\|^p ds d\xi \right)^{\frac{1}{p}}, \quad i = 1, 2, \quad (4)$$

$$L_* = 6^{p-1} [\alpha_0^p + \alpha_1^p + r^p \alpha_2^p \mu(\Omega)]. \quad (5)$$

Condition 2.D implies that $L_* < 1$. Let us set

$$T_* = 6^{p-1} \left[\alpha_0^p + \alpha_1^p \mu(\Omega)^{\frac{p}{q}} + \alpha_2^p r^p \right],$$

$$\beta_* = \left[\frac{T_*}{1 - L_*} \right]^{\frac{1}{p}}, \quad (6)$$

where α_0 , α_1 and α_2 are defined by (4).

Let $u(\cdot) \in U_{q,r}$ be a given admissible control function. A function $x(\cdot) \in L_p(\Omega; \mathbb{R}^n)$ satisfying the integral equation (1) for a.a. $\xi \in \Omega$ is said to be a trajectory of the system (1) generated by the admissible control function $u(\cdot) \in U_{q,r}$. The set of trajectories of the system (1) generated by all admissible control functions $u(\cdot) \in U_{q,r}$ is denoted by $\mathbf{X}_{p,r}$ and is called the set of trajectories of the system (1).

Now we will formulate some propositions the proofs of which are given in [7], and will be used in following arguments.

Proposition 1 [7] *Every admissible control function $u(\cdot) \in U_{q,r}$ generates unique trajectory of the system (1).*

Proposition 2 [7] *For each $x(\cdot) \in \mathbf{X}_{p,r}$ the inequality*

$$\|x(\cdot)\|_p \leq \beta_*$$

is satisfied where β_ is defined by (6).*

Proposition 3 [7] *The set of trajectories $\mathbf{X}_{p,r}$ is a compact and path-connected subset of the space $L_p(\Omega; \mathbb{R}^n)$.*

3. Robustness of the trajectories

The robustness of the system with respect to some parameter which takes values on a given set, usually means that a variation of the parameter within a given set generates an insignificant deviation of the phase state. The robustness of the trajectory of the control system with respect to the fast consumption of the remaining control resource means that no matter how much the remaining control resource is, applying special method for complete consumption of the remaining control resource, it is possible to obtain a small variation of the original trajectory.

Let

$$c_* = 2r \left[\frac{6^{p-1}}{1 - L_*} \right]^{\frac{1}{p}}, \quad (7)$$

where L_* is defined by (5).

Denote $g_*(\xi, s) = K_2(\xi, s, 0)$, $(\xi, s) \in \Omega \times \Omega$. According to the condition 2.C we have $g_*(\cdot, \cdot) \in L_p(\Omega \times \Omega; \mathbb{R}^{n \times m})$. It is known (see, [10, p. 318]) that for given $\varepsilon > 0$ there exists a continuous function $g_\varepsilon(\cdot, \cdot) : \Omega \times \Omega \rightarrow \mathbb{R}^{n \times m}$ such that

$$\int_{\Omega} \int_{\Omega} \|g_*(\xi, s) - g_\varepsilon(\xi, s)\|^p \, ds \, d\xi \leq \frac{\varepsilon^p}{3c_*^p}. \quad (8)$$

where c_* is defined by (7).

Now, let us set

$$M(\varepsilon) = \max \{ \|g_\varepsilon(\xi, s)\| : (\xi, s) \in \Omega \times \Omega \}. \quad (9)$$

From Proposition 3, i.e. from the compactness of the set of trajectories $\mathbf{X}_{p,r} \subset L_p(\Omega; \mathbb{R}^n)$ it follows the validity of the following proposition.

Proposition 4 For every $\varepsilon > 0$ there exists $\delta_*(\varepsilon) \in \left(0, \frac{\varepsilon^p}{3M(\varepsilon)^p c_*^p \mu(\Omega)}\right)$ such that for each Lebesgue measurable set $\Omega_* \subset \Omega$ such that

$$\mu(\Omega_*) \leq \delta_*(\varepsilon),$$

the inequality

$$\int_{\Omega_*} \|x(s)\|^p \, ds \leq \frac{\varepsilon^p}{3\kappa_2^p c_*^p \mu(\Omega)}$$

is satisfied for every $x(\cdot) \in \mathbf{X}_{p,r}$ where κ_2 is defined by (2), c_* is defined by (7).

Theorem 1 Let $\varepsilon > 0$ be a given number, $x(\cdot) \in \mathbf{X}_{p,r}$ be a trajectory of the system (1) generated by the admissible control function $u(\cdot) \in U_{q,r}$, $\|u(\cdot)\|_q = r_0 < r$, $\Omega_* \subset \Omega$ be Lebesgue measurable set,

$$v(\xi) = \begin{cases} u(\xi) & \text{if } \xi \in \Omega \setminus \Omega_*, \\ u_*(\xi) & \text{if } \xi \in \Omega_* \end{cases} \quad (10)$$

be such that $\|v(\cdot)\|_q = r$, $z(\cdot) \in \mathbf{X}_{p,r}$ be the trajectory of the system (1) generated by the admissible control function $v(\cdot) \in U_{q,r}$. If

$$\mu(\Omega_*) \leq \delta_*(\varepsilon) \quad (11)$$

then

$$\|x(\cdot) - z(\cdot)\|_p \leq \varepsilon$$

where $\delta_*(\varepsilon)$ is defined in Proposition 4.

Proof. From (2), (10), Conditions 2.A, 2.B, 2.C, inclusion $u(\cdot) \in U_{q,r}$ and Hölder's inequality it follows that

$$\begin{aligned} \|x(\xi) - z(\xi)\| &\leq \kappa_0 \|x(\xi) - z(\xi)\| + \left(\int_{\Omega} \gamma_1(\xi, s)^q ds \right)^{\frac{1}{q}} \|x(\cdot) - z(\cdot)\|_p \\ &\quad + \kappa_2 r \|x(\cdot) - z(\cdot)\|_p + \int_{\Omega_*} \|K_2(\xi, s, z(s))\| \|u(s) - v(s)\| ds \end{aligned} \quad (12)$$

for a.a. $\xi \in \Omega$. (2) and Condition 2.C imply

$$\begin{aligned} \int_{\Omega_*} \|K_2(\xi, s, z(s))\| \|u(s) - v(s)\| ds &\leq \kappa_2 \int_{\Omega_*} \|z(s)\| \|u(s) - v(s)\| ds \\ &\quad + \int_{\Omega_*} \|K_2(\xi, s, 0)\| \|u(s) - v(s)\| ds \end{aligned} \quad (13)$$

for a.a. $\xi \in \Omega$.

Since $z(\cdot) \in \mathbf{X}_{p,r}$, then inclusions $u(\cdot) \in U_{q,r}$, $v(\cdot) \in U_{q,r}$, (11), Proposition 4 and Hölder's inequality yield

$$\begin{aligned} \int_{\Omega_*} \|z(s)\| \|u(s) - v(s)\| ds &\leq \left(\int_{\Omega_*} \|z(s)\|^p ds \right)^{\frac{1}{p}} \cdot \left(\int_{\Omega_*} \|u(s) - v(s)\|^q ds \right)^{\frac{1}{q}} \\ &\leq 2r \cdot \frac{\varepsilon}{c_* \kappa_2 [3\mu(\Omega)]^{\frac{1}{p}}}. \end{aligned} \quad (14)$$

From (9), inclusions $u(\cdot) \in U_{q,r}$, $v(\cdot) \in U_{q,r}$ and Hölder's inequality it follows that

$$\begin{aligned} \int_{\Omega_*} \|g_*(\xi, s)\| \cdot \|u(s) - v(s)\| \, ds &\leq \int_{\Omega_*} \|g_*(\xi, s) - g_\varepsilon(\xi, s)\| \cdot \|u(s) - v(s)\| \, ds \\ &+ \int_{\Omega_*} \|g_\varepsilon(\xi, s)\| \cdot \|u(s) - v(s)\| \, ds \\ &\leq 2r \left(\int_{\Omega} \|g_*(\xi, s) - g_\varepsilon(\xi, s)\|^p \, ds \right)^{\frac{1}{p}} + 2rM(\varepsilon) [\mu(\Omega_*)]^{\frac{1}{p}} \end{aligned} \quad (15)$$

for a.a. $\xi \in \Omega$ where $g_*(\cdot, \cdot) = K_2(\cdot, \cdot, 0)$, $g_\varepsilon(\cdot, \cdot)$ is defined in (8). Thus (13), (14) and (15) yield

$$\begin{aligned} \int_{\Omega_*} \|K_2(\xi, s, z(s))\| \|u(s) - v(s)\| \, ds &\leq 2r \cdot \frac{\varepsilon}{c_* [3\mu(\Omega)]^{\frac{1}{p}}} \\ &+ 2r \left(\int_{\Omega} \|g_*(\xi, s) - g_\varepsilon(\xi, s)\|^p \, ds \right)^{\frac{1}{p}} + 2rM(\varepsilon) [\mu(\Omega_*)]^{\frac{1}{p}} \end{aligned}$$

for a.a. $\xi \in \Omega$. Finally, from the last inequality, (11) and (12) we obtain

$$\begin{aligned} \|x(\xi) - z(\xi)\| &\leq \kappa_0 \|x(\xi) - z(\xi)\| + \left(\int_{\Omega} \gamma_1(\xi, s)^q \, ds \right)^{\frac{1}{q}} \|x(\cdot) - z(\cdot)\|_p \\ &+ \kappa_2 r \|x(\cdot) - z(\cdot)\|_p + 2r \cdot \frac{\varepsilon}{c_* [3\mu(\Omega)]^{\frac{1}{p}}} \\ &+ 2r \left(\int_{\Omega} \|g_*(\xi, s) - g_\varepsilon(\xi, s)\|^p \, ds \right)^{\frac{1}{p}} + 2rM(\varepsilon) [\delta_*(\varepsilon)]^{\frac{1}{p}} \end{aligned}$$

for a.a. $\xi \in \Omega$ and consequently

$$\begin{aligned} \|x(\xi) - z(\xi)\|^p &\leq 6^{p-1} \left[\kappa_0^p \|x(\xi) - z(\xi)\|^p + \left(\int_{\Omega} \gamma_1(\xi, s)^q \, ds \right)^{\frac{p}{q}} \|x(\cdot) - z(\cdot)\|_p^p \right. \\ &+ \kappa_2^p r^p \|x(\cdot) - z(\cdot)\|_p^p + 2^p r^p \cdot \frac{\varepsilon^p}{3\mu(\Omega)c_*^p} \\ &\left. + 2^p r^p \int_{\Omega} \|g_*(\xi, s) - g_\varepsilon(\xi, s)\|^p \, ds + 2^p r^p M(\varepsilon)^p \delta_*(\varepsilon) \right] \end{aligned}$$

for a.a. $\xi \in \Omega$. Integrating the last inequality on the set Ω and taking into consideration (3), (5), (8) and the inequality

$$\delta_*(\varepsilon) < \frac{\varepsilon^p}{3M(\varepsilon)^p c_*^p \mu(\Omega)}$$

we obtain

$$\begin{aligned} \|x(\cdot) - z(\cdot)\|_p^p &\leq 6^{p-1} \left[\kappa_0^p \|x(\cdot) - z(\cdot)\|_p^p \right. \\ &\quad + \int_{\Omega} \left(\int_{\Omega} \gamma_1(\xi, s)^q ds \right)^{\frac{p}{q}} d\xi \cdot \|x(\cdot) - z(\cdot)\|_p^p \\ &\quad + \kappa_2^p r^p \mu(\Omega) \|x(\cdot) - z(\cdot)\|_p^p + 2^p r^p \cdot \frac{\varepsilon^p}{3c_*^p} \\ &\quad \left. + 2^p r^p \int_{\Omega} \int_{\Omega} \|g_*(\xi, s) - g_\varepsilon(\xi, s)\|^p ds d\xi + 2^p r^p M(\varepsilon)^p \mu(\Omega) \delta_*(\varepsilon) \right] \\ &\leq 6^{p-1} \left[\kappa_0^p + \kappa_1^p + \kappa_2^p r^p \mu(\Omega) \right] \cdot \|x(\cdot) - z(\cdot)\|_p^p \\ &\quad + 6^{p-1} \left[2^p r^p \cdot \frac{\varepsilon^p}{3c_*^p} + 2^p r^p \cdot \frac{\varepsilon^p}{3c_*^p} + 2^p r^p M(\varepsilon)^p \mu(\Omega) \cdot \frac{\varepsilon^p}{3M(\varepsilon)^p c_*^p \mu(\Omega)} \right] \\ &= L_* \cdot \|x(\cdot) - y(\cdot)\|_p^p + 6^{p-1} \cdot [2r]^p \cdot \left[\frac{\varepsilon^p}{3c_*^p} + \frac{\varepsilon^p}{3c_*^p} + \frac{\varepsilon^p}{3c_*^p} \right] \\ &= L_* \cdot \|x(\cdot) - y(\cdot)\|_p^p + 6^{p-1} \cdot [2r]^p \cdot \frac{\varepsilon^p}{c_*^p}, \end{aligned}$$

and finally, by virtue of (7)

$$\|x(\cdot) - z(\cdot)\|_p \leq 2r \left[\frac{6^{p-1}}{1 - L_*} \right]^{\frac{1}{p}} \cdot \frac{\varepsilon}{c_*} = \varepsilon.$$

The proof is completed. □

Denote

$$U_{q,r}^* = \{u(\cdot) \in L_q(\Omega; \mathbb{R}^m) : \|u(\cdot)\|_q = r\},$$

and let $\mathbf{X}_{p,r}^*$ be the set of trajectories of the system (1) generated by all admissible control functions $u(\cdot) \in U_{q,r}^*$.

Theorem 2 *The equality*

$$\mathbf{X}_{p,r} = cl \left(\mathbf{X}_{p,r}^* \right)$$

is satisfied where cl denotes the closure of a set.

Proof. Let us choose an arbitrary $x(\cdot) \in \mathbf{X}_{p,r}$ generated by the control function $u(\cdot) \in U_{q,r}$ and let $\|u(\cdot)\|_q = r_0 < r$. For given $\varepsilon > 0$ choose Lebesgue measurable set $\Omega_* \subset \Omega$ where $\mu(\Omega_*) \leq \delta_*(\varepsilon)$ and $\delta_*(\varepsilon) > 0$ is defined as in Theorem 1. Assume that

$$\int_{\Omega \setminus \Omega_*} \|u(s)\|^q ds = r_1^q.$$

It is obvious that $r_1 \leq r_0$. Define control function

$$u_*(s) = \begin{cases} u(s) & \text{if } s \in \Omega \setminus \Omega_*, \\ \left[\frac{r^q - r_1^q}{\mu(\Omega_*)} \right]^{\frac{1}{q}} \cdot b_* & \text{if } s \in \Omega_*, \end{cases}$$

where $b_* \in \mathbb{R}^m$ is an arbitrary vector such that $\|b_*\| = 1$. One can show that $u_*(\cdot) \in U_{q,r}^*$. Let $x_*(\cdot): \Omega \rightarrow \mathbb{R}^m$ be the trajectory of the system (1) generated by the control function $u_*(\cdot) \in U_{q,r}^*$. Then $x_*(\cdot) \in \mathbf{X}_{p,r}^*$ and according to the theorem 1 we have

$$\|x(\cdot) - x_*(\cdot)\|_p \leq \varepsilon$$

which implies that

$$x(\cdot) \in \mathbf{X}_{p,r}^* + \varepsilon B_{L_p}(1), \quad (16)$$

where

$$B_{L_p}(1) = \{y(\cdot) \in L_p(\Omega; \mathbb{R}^m) : \|y(\cdot)\|_p \leq 1\}.$$

Since $\varepsilon > 0$ is an arbitrarily chosen number, then (16) yields that

$$x(\cdot) \in cl \left(\mathbf{X}_{p,r}^* \right)$$

and hence

$$\mathbf{X}_{p,r} \subset cl \left(\mathbf{X}_{p,r}^* \right). \quad (17)$$

Since $cl \left(\mathbf{X}_{p,r}^* \right) \subset \mathbf{X}_{p,r}$, then the inclusion (17) completes the proof of the theorem. \square

Corollary 1 *The equality*

$$h_{L_p} \left(\mathbf{X}_{p,r}, \mathbf{X}_{p,r}^* \right) = 0$$

is held where $h_{L_p}(\cdot, \cdot)$ stands for Hausdorff distance between the subsets of the space $L_p(\Omega; \mathbb{R}^n)$.

4. Conclusion

Theorem 1 implies that consuming the remaining control resource in the domain with a sufficiently small Lebesgue measure, will generate an insignificant deviation of the trajectory. This circumstance yields that if you have an undesired extra control resource and you need to get rid of this resource, then spending the remaining resource in the domain with a small measure, you will get insignificant deviation from the initial trajectory. This theorem allows to state that for a significant change in the trajectory of the system, it is advisable to avoid the aggressive spending of the control resource and to consume the control resource in economy mode, i.e. to consume the control resource in small portions. Possibility of the approximation of each trajectory with trajectory, generated by full consumption of the control resource permits in numerical construction methods of the set of trajectories to construct only the trajectories generated by full consumption of the control resource.

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