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Stability of convex linear combinations of continuous-time and discrete-time linear systems

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The asymptotic stability of the convex linear combination of continuous-time and discrete-time linear systems is considered. Using the Gershgorin theorem it is shown that the convex linear combination of the linear asymptotically stable continuous-time and discretetime linear systems is also asymptotically stable. It is shown that the above thesis is also valid (even simpler) for positive linear systems.

Key words: convex linear combination, linear system, continuous-time, discrete-time, positive, system, stability

1. Introduction

It is well-known that [1, 2, 5–14, 19] the dynamical properties of the linear systems essentially depend on the location of their poles and zeros in the complex plane. There exist many methods of the assignment of the poles and zeros in the continuous-time and discretetime linear systems [1, 2, 5–10, 14, 19].

A special class of dynamical systems are the positive systems. A dynamical system is called positive if its state variables and outputs take nonnegative values for any nonnegative inputs and nonnegative initial conditions [2, 6, 7, 9]. Models having positive behavior can be found in engineering, electrical circuits, economics, social sciences, biology medicine etc.

It is well-known [1,2,5,9,17,18] that if the pair $(A,\ B)$ of the linear systems is controllable then using the state feedback we may assign the poles of the closed-loop systems in the desired state positions. In single-input single-output linear systems by the use of the state feedbacks we may modify the positions only of

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its poles [2, 7, 9]. In multi-inputs multi-outputs linear systems by the use of the state feedbacks we may also modify the positions of the zeros of their transfer matrices. Practical stability, asymptotical stability and robust stability have been investigated in [15, 16]. Stabilization of descriptor fractional continuous-time and discrete-time systems have been analyzed in [17, 18] and global stability of nonlinear feedback systems with fractional positive linear parts in [4].

In this paper the stability of the convex linear combination of continuous-time and discrete-time linear system will be investigated.

The paper is organized as follows. In Section 2 the Geshgorin theorem is applied to analysis of the asymptotic stability of continuous-time and discrete-time linear systems and some its extension is proposed. The main results of the paper for continuous-time linear systems are given in Section 3 and for the discrete-time linear systems in Section 4. The results for the positive linear systems are extended in Section 5. Concluding remarks are given in Section 6.

The following notations will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n – the $n\times n$ identity matrix.

2. Gershgorin theorem and its application to stability of linear systems

Consider the autonomous continuous-time linear system

$$\dot{x}(t) = Ax(t), \tag{1a}$$

where $x(t) \in \Re^n$ is the state vector and

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \in \Re^{n\times n} \tag{1b}$$

is the state matrix.

The system (1) is called asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for all initial conditions $x(0) \in \Re^n$.

Theorem 1 *The system (1) is asymptotically stable if and only if* $\operatorname{Re} s_l < 0$ *for* $l = 1, \dots, n$ *where* s_l *are the eigenvalues of the matrix (1b).*

Note that

$$|s - a_{ii}| \leqslant R_i\,, \tag{2a}$$

where

$$R_i = \sum_{\substack{j=1\\ j\neq i}} |a_{ij}|, \qquad i = 1, \dots, n \tag{2b}$$

defines in the complex plane of s the circles C_i, $i = 1, \dots, n$ with the centers located in the points $s = a_{ii}$ and the radiuses R_i. In similar way we may define the circles C'_j, $j = 1, \dots, n$ with the centers in the points $s = a_{ii}$ and the radiuses

$$R'_j = \sum_{\substack{i=1 \\ i \neq j}} |a_{ij}|, \qquad j = 1, \dots, n \tag{2c}$$

Gershgorin Theorem. The eigenvalues of the matrix (1b) are located inside circles defined by (2).

Using Gershgorin theorem it is easy to prove the following theorem [3].

Theorem 2 *The linear continuous-time system (1) is asymptotically stable if all the circles C_i, $i = 1, \dots, n$ and C'_j, $j = 1, \dots, n$ are located in the left-hand part of the complex plane of s.*

Consider the two similar matrices $A \in \Re^{n \times n}$ and

$$\overline{A} = D\,A\,D^{-1}, \tag{3a}$$

where

$$D = \operatorname{diag}\left[d_1, \dots, d_n\right], \qquad d_i \neq 0, \quad i = 1, \dots, n. \tag{3b}$$

The matrices A and $\overline{A}$ have the same eigenvalues $\lambda_1, \dots, \lambda_n$ since

$$\begin{aligned} \det\left[I_n s - \overline{A}\right] &= \det\left[D\,D^{-1} s - D\,A\,D^{-1}\right] \\ &= \det D \det[I_n s - A] \det D^{-1} = \det[I_n s - A]. \end{aligned} \tag{4}$$

Note that

$$\begin{aligned} \overline{A} &= \operatorname{diag}\left[d_1, \dots, d_n\right] \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \operatorname{diag}\left[d_1^{-1}, \dots, d_n^{-1}\right] \\ &= \begin{bmatrix} a_{11} & a_{12}\dfrac{d_1}{d_2} & \dots & a_{1n}\dfrac{d_1}{d_n} \\ a_{21}\dfrac{d_2}{d_1} & a_{22} & \dots & a_{2n}\dfrac{d_2}{d_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\dfrac{d_n}{d_1} & a_{n2}\dfrac{d_n}{d_2} & \dots & a_{nn} \end{bmatrix}. \end{aligned} \tag{5}$$

Now let us consider the Metzler matrix

$$A = A_1 + kA_2 \in M_n, \tag{6}$$

where $A_1 \in M_n$, $A_2 \in \Re^{n \times n}$ and $k > 0$ is a scalar.

Knowing the matrices A_1 and A_2 find a nonsingular matrix D such that the scalar k takes the maximal value $k_{\max} > 0$ for which the matrix

$$\overline{A} = DAD^{-1} = D(A_1 + kA_2)D^{-1} = DA_1D^{-1} + kDA_2D^{-1} \tag{7}$$

is asymptotically stable.

To find the $k_{\max}$ the following theorem can be used.

Theorem 3 *The positive system (1) is asymptotically stable if the sum of entries of each column (row) of the matrix A is negative.*

Example 1 Consider the matrix (6) for

$$A_1 = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.4 \end{bmatrix}. \tag{8}$$

In this case the matrix (6) has the form

$$A = A_1 + kA_2 = \begin{bmatrix} -3+0.2k & 1+0.3k \\ 2+0.3k & -4+0.4k \end{bmatrix} \tag{9}$$

and using Theorem 3 we obtain for columns:

column 1: $-1 + 0.5k < 0 \to k < 2$, column 2: $-3 + 0.7k < 0 \to k < 4.286$ and for rows:

row 1: $-2 + 0.5k < 0 \to k < 4$, row 2: $-2 + 0.7k < 0 \to k < 2.857$.

Therefore, $k_{\max}$ for which the matrix (9) is asymptotically stable is $k_{\max} < 2$.

If we apply the approach based on the matrix $D = \operatorname{diag}[d_1, d_2]$ then we obtain

$$\overline{A} = D(A_1 + kA_2)D^{-1} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} -3+0.2k & 1+0.3k \\ 2+0.3k & -4+0.4k \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} -3+0.2k & (1+0.3k)\dfrac{d_1}{d_2} \\ (2+0.3k)\dfrac{d_2}{d_1} & -4+0.4k \end{bmatrix}. \tag{10}$$

Using Theorem 3 we obtain for example for $d_1 = 0.8$, $d_2 = 0.6$ for

column 1: $-3 + 0.2k + (2 + 0.3k)\dfrac{d_2}{d_1} < 0 \;\to\; k < 3.329$,

column 2: $-4 + 0.4k + (1 + 0.3k)\dfrac{d_1}{d_2} < 0 \;\to\; k < 3.333$ for

row 1: $-3 + 0.2k + (1 + 0.3k)\dfrac{d_1}{d_2} < 0 \;\to\; k < 2.778$,

row 2: $-4 + 0.4k + (2 + 0.3k)\dfrac{d_2}{d_1} < 0 \;\to\; k < 4$.

Therefore, in this case the matrix (10) is asymptotically stable for $k_{\max} < 2.778$.

In general case applying the known optimization methods we may find $k_{\max}$ for systems with some additional restrictions on the stability of the systems.

Now let us consider the autonomous discrete-time linear system

$$x_{k+1} = \overline{A}x_k\,, \qquad k = 0, 1, \dots, \tag{11}$$

where $x_k \in \Re^n$ is the state vector and $\overline{A} \in \Re^{n\times n}$ is the state matrix.

The discrete-time system (11) is called asymptotically stable if $\lim\limits_{k\to 1} x_k = 0$ for all initial conditions $x_0 \in \Re^n$.

Theorem 4 [1, 5, 10] *The discrete-time system (11) is asymptotically stable if and only if* $|z_l| < 1$ *for* $l = 1, \dots, n$ *where* z_l *are the eigenvalues of the matrix* $\overline{A}$ *of the system.*

In a similar way for the discrete-time system we may also defined in the complex plane z the circles $\overline{C}_i$ and $\overline{C}'_j$.

Theorem 5 *The linear discrete-time system (11) is asymptotically stable if all the circles* $\overline{C}_i$ *and* $\overline{C}'_j$ *are located inside the unit circle (with the center in the point* $(0,\ 0)$ *and radius equal 1).*

Note that the considerations can be extended to continuous-time bounded linear systems.

3. Convex linear combinations of continuous-time linear systems

Consider the pair of real matrices of the same dimension

$$A_1,\ A_2 \in \Re^{n\times n} \tag{12a}$$

and the real number q satisfying the condition

$$0 \leqslant q \leqslant 1. \tag{12b}$$

Definition 1 *The real matrix*

$$A(q) = (1-q)A_1 + qA_2\,, \tag{13}$$

where q *satisfies the condition (12b), is called the convex linear combination of the matrices (12a).*

Theorem 6 *The convex linear combination (13) of the asymptotic stable matrices (12a) is also asymptotically stable matrix for all value of* q *satisfying the condition (12b).*

Proof. By Gershgorin Theorem the circles corresponding to all values of q satisfying the condition (12b) are located between the circle corresponding to $q = 0$ and to the circle corresponding to $q = 1$. By assumption the circles corresponding to $q = 0$ and to $q = 1$ are located in the left-hand side of the complex plane since the matrices (12a) are asymptotically stable. Therefore, the convex linear combination (13) is asymptotically stable for all values of q satisfying the condition (12b). □

Example 2 Consider the convex linear combination of the asymptotically stable matrices

$$A_1 = \begin{bmatrix} -3 & -2 \\ 1 & -3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}. \tag{14}$$

The convex linear combination of the matrices (14) for $q = 0.4$ has the form

$$\begin{aligned} A(0.4) &= (1-0.4)A_1 + 0.4A_2 = 0.6\begin{bmatrix} -3 & |-2| \\ 1 & -3 \end{bmatrix} + 0.4\begin{bmatrix} -3 & |-2| \\ |-2| & -4 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ 1.4 & -3.4 \end{bmatrix}. \end{aligned} \tag{15}$$

The circle corresponding to the matrix (15) is located in the left-hand side of the complex plane (Fig. 1). The matrix (15) is asymptotically stable.

C_{r1}, C_{r2} – circle related to row 1 (centre $(-3, 0)$ radius 2) and row 2 (centre $(-3.4, 0)$ radius 1.4) of the matrix A respectively; C_{c1}, C_{c2} – circle related to column 1 (centre $(-3, 0)$ radius 1.4) and column 2 (centre $(-3.4, 0)$ radius 2) of the matrix A respectively.

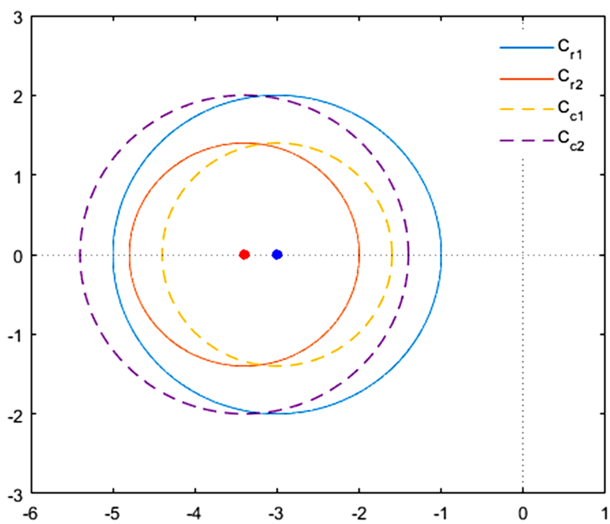

Figure 1: Illustration to Example 2

4. Convex linear combinations of discrete-time linear systems

The convex linear combination of two matrices of discrete-time linear system (11) is defined in a similar way as for continuous-time linear systems.

Definition 2 *The real matrix*

$$\overline{A}(q) = (1-q)\overline{A}_1 + q\overline{A}_2 \tag{16a}$$

is called the convex linear combination of the matrices

$$\overline{A}_1,\ \ \overline{A}_2 \in \Re^{n\times n}, \tag{16b}$$

where q satisfies the condition (12b).

Theorem 7 *The convex linear combination (16) of the asymptotically stable matrices (16b) is also asymptotically stable matrix for all values of q satisfying the condition (12b).*

Proof is similar to the proof of Theorem 6.

Example 3 *Consider the convex linear combination of the asymptotically stable matrices*

$$\overline{A}_1 = \begin{bmatrix} 0.4 & 0.3 \\ -0.2 & 0.5 \end{bmatrix}, \qquad \overline{A}_2 = \begin{bmatrix} 0.5 & -0.3 \\ -0.3 & 0.4 \end{bmatrix} \tag{17}$$

of the discrete-time linear system (11).

The convex linear combination of the matrices (17) for $q = 0.3$ has the form

$$\begin{aligned}\overline{A}(0.3) &= (1-0.3)\overline{A}_1 + 0.3\overline{A}_2 \\ &= 0.7\begin{bmatrix} 0.4 & 0.3 \\ |-0.2| & 0.5 \end{bmatrix} + 0.3\begin{bmatrix} 0.5 & |-0.3| \\ |-0.3| & 0.4 \end{bmatrix} = \begin{bmatrix} 0.43 & 0.03 \\ 0.23 & 0.47 \end{bmatrix}.\end{aligned} \tag{18}$$

The circle corresponding to the matrix (18) is located inside the unite circle (Fig. 2). Therefore, the matrix (18) is asymptotically stable.

C_{r1}, C_{r2} – circle related to row 1 (centre (0.43, 0) radius 0.03) and row 2 (centre (0.47, 0) radius 0.23) of the matrix $\overline{A}$ respectively; C_{c1}, C_{c2} – circle related to column 1 (centre (0.43, 0) radius 0.23) and column 2 (centre (0.47, 0) radius 0.03) of the matrix $\overline{A}$ respectively.

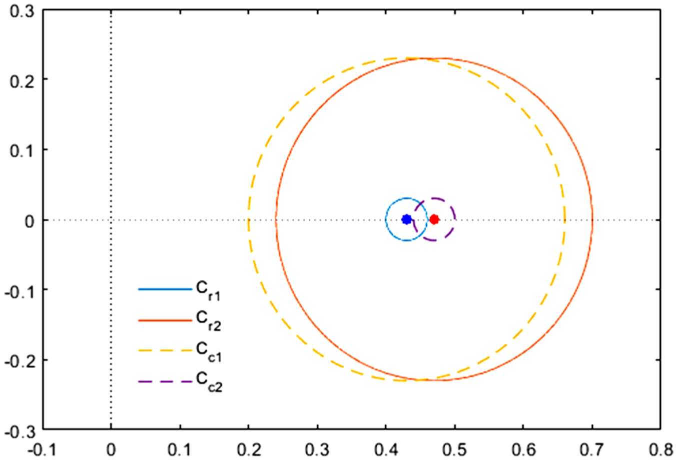

Figure 2: Illustration to Example 3

5. Positive linear systems

In this section the results of the last two sections will be extended to the positive continuous-time and discrete-time linear systems. Consider the autonomous continuous-time linear system (1)

Definition 3 *[6, 7, 9] The system (1) is called positive if $x(t) \in \Re_+^n$, $t \geqslant 0$ for any initial conditions $x_0 = x(0) \in \Re_+^n$.*

Theorem 8 *[6, 7, 9] The system (1) is positive if and only if its matrix A is the Metzler matrix.*

Definition 4 *[6, 7, 9] The positive system (1) is called asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for all $x(0) \in \Re_+^n$.*

Theorem 9 *[6, 7, 9] The positive system (1) is asymptotically stable if and only if one of the equivalent conditions is satisfied:*

1. *All coefficient of the characteristic polynomial*

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{19}$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \ldots, n-1$;

2. *There exists strictly positive vector $\lambda^T = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix}^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that*

$$A\lambda < 0 \quad or \quad A^T\lambda < 0. \tag{20}$$

For the positive system (1) the equalities (2b) and (2c) take the forms

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\,, \quad i = 1, \ldots, n \tag{21a}$$

$$R'_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} a_{ij}\,, \quad j = 1, \ldots, n \tag{21b}$$

since the off-diagonal entries of the Metzler matrices are not-negative.

Therefore, for positive systems the Theorem 6 takes the form

Theorem 10 *The convex linear combination (13) of Metzler matrices $A_1,\ A_2 \in M_n$ is also Metzler matrix and it is asymptotically stable if the matrices are asymptotically stable.*

Now let us consider the autonomous discrete-time linear systems (11).

Definition 5 *[7, 9] The fractional system (11) is called (internally) positive if $x_i \in \Re^n_+$, $i \in Z_+ = 0, 1, \ldots$ for any initial conditions $x_0 \in \Re^n_+$.*

Theorem 11 *[7, 9] The system (11) is positive if and only if*

$$A \in \Re^{n \times n}_{+}. \tag{22}$$

Definition 6 *The positive system (11) is called asymptotically stable if*

$$\lim_{i \to \infty} x_i = 0 \quad \textit{for all} \ \ x_0 \in \Re^n_+\,. \tag{23}$$

Theorem 12 [7] *The positive system (11) is asymptotically stable if and only if one of the equivalent conditions is satisfied:*

1. *All coefficient of the characteristic polynomial*

$$p_A(z) = \det\left[I_n(z+1) - \overline{A}\right] = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \tag{24}$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \ldots, n-1$.

2. *There exists strictly positive vector $\lambda^T =$ $big[\lambda_1 \ \ \cdots \ \ \lambda_n]^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that*

$$[\overline{A} - I_n]\lambda < 0, \qquad \lambda^T[\overline{A} - I_n] < 0. \tag{25}$$

Theorem 13 [7] *The positive system (11) is asymptotically stable if the sum of entries of each column (row) of the matrix* $\overline{A}$ *is less than one.*

For positive linear discrete-time systems all entries of the matrices (16b) are non-negative. From Definition 2 it follows that the convex linear combination (16a) has also all nonnegative entries. Therefore, for positive linear discrete-time systems we have the following.

Theorem 14 *The convex linear combination (16a) of non-negative matrices* $\overline{A}_1,\ \overline{A}_2 \in \Re_+^{n\times n}$ *is also non-negative matrix and it is asymptotically stable if the matrices are asymptotically stable.*

6. Concluding remarks

The asymptotic stability of the convex linear combination of continuous-time and discrete-time linear systems has been analyzed. Using the Gershgorin theorem it is shown that the convex linear combination of the linear asymptotically stable continuous-time (Theorem 6) and discrete-time linear systems (Theorem 7) is also asymptotically stable. It is shown that the above thesis is also valid (and even simpler) for positive linear systems (Theorems 8 and 13). The considerations can extended to the fractional linear systems and to the descriptor standard and fractional linear systems. An open problem is an extension of these considerations to the different fractional orders linear systems.

References

[1] E. Antsaklis and A. Michel: *Linear Systems*. Birkhauser, Boston, 2006.

[2] L. Farina and S. Rinaldi: *Positive Linear Systems: Theory and Applications*. J. Wiley & Sons, New York, 2000.

[3] S. Gerschgorin: Über die Abgrenzung der Eigenwerte einer Matrix. *Izvestiya Akademii Nauk SSSR*. Ser. 7, Otdeleniye fiziko-matematicheskikh nauk. 6 (in German), 6: (1931), 749–754.

[4] T. Kaczorek: Global stability of nonlinear feedback systems with fractional positive linear parts. *International Journal of Applied Mathematics and Computer Science*, **30**(3), (2020), 493–500. DOI: 10.34768/amcs-2020-0036.

[5] T. Kaczorek: *Linear Control Systems*, 2. Research Studies Press LTD., J. Wiley, New York 1992.

[6] T. Kaczorek: *Positive 1D and 2D Systems*. London, UK: Springer-Verlag, 2002.

[7] T. Kaczorek: *Selected Problems of Fractional Systems Theory*. Berlin, Germany: Springer-Verlag, 2011.

[8] T. Kaczorek and K. Borawski: *Descriptor Systems of Integer and Fractional Orders*. Springer Nature Switzerland, 2021.

[9] T. Kaczorek and K. Rogowski: *Fractional Linear Systems and Electrical Circuits*. In Studies in Systems, Decision and Control, 13. Springer, 2015.

[10] T Kailath: *Linear systems*. Prentice Hall, Englewood Cliffs, New York, 1980.

[11] R. Kalman: Mathematical description of linear systems. *SIAM Journal of Control*, **1**(2), 152–192, (1963). DOI: 10.1137/0301010.

[12] R. Kalman: On the general theory of control systems. *Proceedings of the First International Congress on Automatic Control*, London, UK, Butterworth, (1960), 481–493.

[13] J. Klamka: *Controllability of Dynamical Systems*. Kluwer, Academic Publ., Dordrecht, 1991.

[14] H. Rosenbrock: *State-space and Multivariable Theory*. New York, USA, J. Wiley, 1970.

[15] A. Ruszewski: *Practical stability and asymptotic stability of interval fractional discretetime linear state-space system*. In: R. Szewczyk et al. (Eds.) Recent Advances in Automation, Robotics and Measuring Techniques. Advances in Intelligent Systems and Computing, 267 (2014), 217–227. DOI: 10.1007/978-3-319-05353-0_22.

[16] A. Ruszewski: Robust stability of a class of an uncertain fractional discrete-time linear state-space system. In: R. Szewczyk et al. (Eds.) Innovations in Automation, Robotics and Measurement Techniques. Advances in Intelligent Systems and Computing, 550 (2017), 195–203. DOI: 10.1007/978-3-319-54042-9_18.

[17] Ł. Sajewski: Stabilization of positive descriptor fractional continuous-time linear system with two different fractional orders by decentralized controller. 21st International Conference on System Theory, Control and Computing (ICSTCC), Sinaia, Romania, (2017).

[18] Ł. Sajewski: Stabilization of positive descriptor fractional discrete-time linear system with two different fractional orders by decentralized controller. *Bulletin of the Polish Academy of Sciences Technical Sciences*, **65**(5), (2017), 709–714. DOI: 10.1515/bpasts-2017-0076.

[19] S.M. Zak: *Systems and Control*. New York, Oxford University Press 2003.