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Nonlocal controllability of mild solutions for neutral evolution equations with state-dependent delay in Fréchet spaces

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In this paper, we prove the controllability of mild solutions of neutral functional evolution equations with state-dependent delay and nonlocal conditions. We establish the non local controllability of mild solutions under certain conditions by combining Avramescu's nonlinear alternative for the sum of compact and contraction operators in Fréchet spaces with semigroup theory.

Key words: neutral evolution equations, mild solution, controllability, infinite delay, state-dependent delay, nonlocal conditions, fixed point, nonlinear alternative, semigroup theory, Fréchet spaces

1. Introduction

In this paper, we study the controllability of mild solutions defined on the semi-infinite real interval $J := [0, +\infty)$, for a class of first order neutral functional evolution equations with infinite state-dependent delay and nonlocal conditions in a real Banach space $(E, |\cdot|)$.

In Section 3, we consider the following nonlocal neutral functional differential evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y(t) - g(t, y_{\rho(t,y_t)})\right] - A(t)y(t) = Cu(t) + f(t, y_{\rho(t,y_t)}), \quad a.e.\ t \in J, \quad (1)$$

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$$y(t) = \phi(t) - h_t(y), \quad t \in (-\infty, 0], \tag{2}$$

where $\mathcal{B}$ is an abstract phase space which will be defined later; $f,\ g\colon J \times \mathcal{B} \to E$, $\phi \in \mathcal{B}$, $\rho\colon J \times \mathcal{B} \to \mathbb{R}$ and $h_t\colon \mathcal{B} \to E$ are given functions; the control function $u(\cdot)$ is given in $L^2(J, E)$ is the Banach space of admissible control function; C is a bounded linear operator from E into E and $\{A(t)\}_{t \in J}$ is a family of linear closed (not necessarily bounded) operators from E into E which generates an evolution system of operators $\{U(t, s)\}_{(t,s) \in J \times J}$ for $s \leqslant t$.

For any continuous function y and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta) \quad \text{for } \theta \leqslant 0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t \leqslant 0$ up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.

Then in Section 4, we present an example to illustrate the previous abstract theory obtained.

Partial and neutral functional differential equations are used in the evolution modelling of physical, biological and economic systems while describing their behavior over a range of time. The response of these systems depends not only on their current state but also on their past history.

In recent decades, many authors have applied semigroup theory, fixed point arguments, degree theory and non-compactness measures to study the existence and uniqueness of mild, strong and classical solutions of semilinear functional differential equations. See the books of Ahmed [3], Pazy [28] and Wu [30] for more details on these theories. Furthermore, the concept of phase space $\mathcal{B}$ plays an important role in both qualitative and quantitative theoretical studies of infinite delays. A common choice is a semi-norm space satisfying the corresponding axioms, introduced by Hale and Kato in [23].

Numerous studies have been conducted on the controllability of linear and nonlinear systems represented by ODEs in a finite-dimensional space. Many authors extend the concept of controllability to infinite dimensional systems in a Banach space with unbounded operators. The controllability problems can be transformed into fixed point problems, as demonstrated by Quinn and Carmichael in [29]. Fu examined the controllability of two types of abstract neutral functional differential equations with unbounded delay in [21, 22]. Benchohra *et al.* investigated several classes of functional differential equations and inclusions using fixed point arguments and provided some controllability results in [7].

Baghli *et al.* examined the existence, uniqueness, and controllability of mild solutions for several evolution problems with finite and infinite delay in [2,10–15]. However, in recent years, complex cases have arisen in modelling where the delay depends on an unknown function. These equations are often referred to as state-dependent delay equations. We refer readers to the work of Abada *et al.* [1] and Baghli *et al.* in [6, 9, 16]. More recently, Baghli and Mebarki proved the existence of mild solutions for the class of neutral type integro-differential evolution inclusions constraints with infinite state-dependent delay in [27].

The concept of nonlocal conditions was introduced by Byszewski to extend the classical constraint-based problems in the papers [18, 19]. Nonlocal conditions can be used to describe certain physical phenomena. Nonlocal conditions are used in physics because they are more efficient than classical initial conditions. Furthermore, due to the accuracy of nonlocal conditions, they are largely involved in the boundary value problems. Several papers have studied the existence of solutions for differential equations with nonlocal conditions over the last few years. We refer the reader to the papers [17, 26].

We will extend in this paper the previous controllability results obtained by Baghli *et al.* precisely in [6] to our nonlocal neutral problem (1)–(2). We use Avramescu's nonlinear alternative [8] in combination with semigroup theory [3, 28] to establish sufficient conditions for the existence of mild controllable solutions of the sum of compact operators and contraction maps in Fréchet spaces.

2. Preliminaries

In this section, we introduce notations, definitions and theorems which are used throughout this paper.

Let $C(J, E)$ be the continuous functions space from J into E and $B(E)$ be the all bounded linear operators space from E into E, with the norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y\colon J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. Let $L^1(J, E)$ be the Banach space of measurable functions $y\colon J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)|\,\mathrm{d}t.$$

Let X be a Fréchet space with a semi-norms family $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the semi-norms family $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \|x\|_3 \leqslant \cdots \quad \text{for every } x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leqslant \overline{M}_n \quad \text{for all } y \in Y.$$

In what follows, we assume that $\{A(t)\}_{t\geqslant 0}$ is a closed densely defined linear unbounded operators family on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 1. *A family $\{U(t,s)\}_{(t,s)\in\Delta}$ of bounded linear operators $U(t,s)\colon J\times J\to E$ where $(t,s)\in\Delta := \{(t,s)\in J\times J\colon s\leqslant t\}$ is called an evolution system if the following properties are satisfied:*

1. *$U(t,t) = I$ where I is the identity operator in E,*
2. *$U(t,s)U(s,\tau) = U(t,\tau)$ for $\tau\leqslant s\leqslant t$,*
3. *$U(t,s)\in B(E)$, where for every $(t,s)\in\Delta$ and for each $y\in E$, the mapping $(t,s)\to U(t,s)y$ is continuous.*

In this paper we use the axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [23] and follow the terminology used by Hino, Murakami and Naito in [25]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfying the following axioms:

(A_1) If $y\colon(-\infty,b)\to E$, $b>0$, is continuous on $[0,b]$ and $y_0\in\mathcal{B}$, then for every $t\in[0,b)$ the following conditions hold:

- (i) $y_t\in\mathcal{B}$;
- (ii) There exists a positive constant $\mathcal{D}$ such that $|y(t)|\leqslant\mathcal{D}\|y_t\|_{\mathcal{B}}$;
- (iii) There exist two functions $K(\cdot),\ M(\cdot)\colon\mathbb{R}^+\to\mathbb{R}^+$ independent of $y(t)$ with K continuous and M locally bounded such that:

$$\|y_t\|_{\mathcal{B}}\leqslant K(t)\sup\{|y(s)|\colon 0\leqslant s\leqslant t\}+M(t)\|y_0\|_{\mathcal{B}}.$$

Denote $K_b=\sup\{K(t)\colon t\in[0,b]\}$ and $M_b=\sup\{M(t)\colon t\in[0,b]\}$.

(A_2) For the function $y(\cdot)$ in (A_1), y_t is a $\mathcal{B}$-valued continuous function on $[0,b]$.

(A_3) The space $\mathcal{B}$ is complete.

Remark 1.

1. *Condition (ii) in (A_1) is equivalent to $|\phi(0)|\leqslant\mathcal{D}\|\phi\|_{\mathcal{B}}$ for every $\phi\in\mathcal{B}$.*
2. *Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi,\psi\in\mathcal{B}$ can check $\|\phi-\psi\|_{\mathcal{B}}=0$ without necessarily $\phi(\theta)=\psi(\theta)$ for all $\theta\leqslant 0$.*
3. *From the equivalence in the first remark, we can see that for all $\phi,\psi\in\mathcal{B}$ such that $\|\phi-\psi\|_{\mathcal{B}}=0$. This implies necessarily that $\phi(0)=\psi(0)$.*

Here are some examples of phase spaces. For more details we refer the reader to the book by Hino *et al.* [25].

Example 1. Let
BC denote the bounded continuous functions space defined from $\mathbb{R}^-$ to E;
BUC denote the bounded uniformly continuous functions space defined from $\mathbb{R}^-$ to E;

$C^\infty := \left\{\phi \in BC\colon \lim_{\theta\to-\infty} \phi(\theta) \text{ exist in } E\right\}$;

$C^0 := \left\{\phi \in BC\colon \lim_{\theta\to-\infty} \phi(\theta) = 0\right\}$, endowed with the uniform norm $\|\phi\| = \sup_{\theta\leqslant 0} |\phi(\theta)|$.

We have that the spaces BUC, C^∞ and C^0 satisfy assumptions (A_1)–(A_3). However, BC satisfy axioms (A_1), (A_3) not axiom (A_2).

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi)\colon (s,\phi) \in J\times\mathcal{B},\ \rho(s,\phi) \leqslant 0\}.$$

We always assume that $\rho\colon J\times\mathcal{B}\to\mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:
(H_ϕ) The function $t\to\phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi\colon \mathcal{R}(\rho^-)\to(0,+\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leqslant \mathcal{L}^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t\in\mathcal{R}(\rho^-).$$

Remark 2. *Continuous and bounded functions verified frequently the condition (H_ϕ) (see [25]).*

Lemma 1. *[24] If $y\colon(-\infty,b]\to E$ is a function such that $y_0=\phi\in\mathcal{B}$, then*

$$\|y_s\|_{\mathcal{B}} \leqslant (M_b+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)|;\ \theta\in[0,\max\{0,s\}]\},\ s\in\mathcal{R}(\rho^-)\cup J$$

where $\mathcal{L}^\phi = \sup_{t\in\mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Proposition 1. *[6] From (H_ϕ), (A_1) and Lemma 1, for all $t\in[0,n]$ and $n\in\mathbb{N}$ we have*

$$\|y_{\rho(t,y_t)}\|_{\mathcal{B}} \leqslant K_n|y(t)| + \left(M_n+\mathcal{L}^\phi\right)\|\phi\|_{\mathcal{B}}.$$

Definition 2. *A function $f\colon J\times\mathcal{B}\to E$ is said to be an L^1_{loc}-Carathéodory function if it satisfies:*

(i) for each $t\in J$ the function $f(t,\cdot)\colon\mathcal{B}\to E$ is continuous;

(ii) for each $y\in\mathcal{B}$ the function $f(\cdot,y)\colon J\to E$ is measurable;

(iii) for every positive integer q there exists $\vartheta_q\in L^1_{\mathrm{loc}}(J,\mathbb{R}^+)$ such that

$$|f(t,y)| \leqslant \vartheta_q(t) \quad \textit{for all } \|y\|_{\mathcal{B}}\leqslant q \textit{ and almost each } t\in J.$$

Definition 3. *A function $f: X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that:*

$$\|f(x) - f(y)\|_n \leqslant k_n \|x - y\|_n \quad \textit{for all } x, y \in X.$$

Now we present the nonlinear alternative used in this paper given by Avramescu in Fréchet spaces which is an extension of the alternative provided in Banach spaces by Burton and Kirk. We refer to [8] and the references therein.

Theorem 1. *Nonlinear Alternative of Avramescu*

Let X be a Fréchet space and let $A, B: X \to X$ be two operators satisfying:

(1) A is a compact operator,

(2) B is a contraction.

Then either one of the following statements holds:

(C1) The operator $A + B$ has a fixed point;

(C2) The set $\left\{x \in X,\ x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right)\right\}$ is unbounded for some $\lambda \in (0,1)$.

3. Semilinear neutral evolution equations

We present in this section our main controllability results for problem (1)–(2). Before stating and proving this result, we give the definition of mild solutions for the nonlocal problem (1)–(2) and we define the concept of controllability for that problem.

Definition 4. *We say that the continuous function $y(\cdot): \mathbb{R} \to E$ is a mild solution of (1)–(2) if $y(t) = \phi(t) - h_t(y)$ for all $t \in (-\infty, 0]$ and y satisfies the following integral equation*

$$\begin{aligned} y(t) = {} & U(t,0)\left[\phi(0) - h_0(y) - g(0,\phi)\right] + g(t, y_{\rho(t,y_t)}) \\ & + \int_0^t U(t,s)A(s)g(s, y_{\rho(s,y_s)})\,\mathrm{d}s + \int_0^t U(t,s)Cu(s)\,\mathrm{d}s \\ & + \int_0^t U(t,s)f(s, y_{\rho(s,y_s)})\,\mathrm{d}s \quad \textit{for each } t \in J. \end{aligned} \tag{3}$$

Definition 5. *The neutral evolution problem (1)–(2) is said to be nonlocally controllable if for every initial function $\phi \in \mathcal{B}$, $y^\star \in E$ and $n \in \mathbb{N}$, there is some control $u \in L^2([0,n], E)$ such that the mild solution $y(\cdot)$ of (1)–(2) satisfies the terminal condition*

$$y(n) + h_n(y) = y^\star. \tag{4}$$

Let us introduce the following hypotheses which are assumed hereafter:

$(H1)$ $U(t,s)$ is compact for $t-s>0$ and there exist two constants $\widehat{M}\geqslant 1, \overline{M}_0>0$ such that

$$\|U(t,s)\|_{B(E)}\leqslant \widehat{M} \qquad \text{for every } (t,s)\in\Delta$$

and

$$\|A^{-1}(t)\|_{B(E)}\leqslant \overline{M}_0 \qquad \text{for all } t\in J.$$

$(H2)$ For all $R>0$, there exists $l_R\in L^1_{\text{loc}}(J,\mathbb{R}^+)$ such that

$$|f(t,u)-f(t,v)|\leqslant l_R(t)\|u-v\|_{\mathcal{B}}$$

for all $u,v\in\mathcal{B}$ with $\|u\|_{\mathcal{B}}\leqslant R$ and $\|v\|_{\mathcal{B}}\leqslant R$.

$(H3)$ For each $n\in\mathbb{N}$, the linear operator $W\colon L^2([0,n],E)\to E$ is defined by

$$Wu=\int_0^n U(n,s)Cu(s)\,\mathrm{d}s,$$

has a pseudo invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2([0,n],E)/\ker W$ and there exists positive constants $\widetilde{M}$ and $\widetilde{M}_1$ such that

$$\|C\|_{B(E)}\leqslant\widetilde{M} \qquad \textit{and} \qquad \|\widetilde{W}^{-1}\|\leqslant\widetilde{M}_1.$$

$(H4)$ For each $n\in\mathbb{N}$, there exists a constant $\sigma_n>0$ such that

$$|h_t(u)-h_t(v)|\leqslant\sigma_n\|u-v\|_{\mathcal{B}}$$

for all $u,v\in\mathcal{B}$ with $\|u\|_{\mathcal{B}}\leqslant n$ and $\|v\|_{\mathcal{B}}\leqslant n$.

$(H6)$ The function g is completely continuous and for any bounded set $Q\subset\mathcal{B}$ the set $\{t\to g(t,y_{\rho(t,y_t)})\colon\ y\in Q\}$ is equicontinuous in $C(J,E)$.

Remark 3.

1. *For the construction of $\widetilde{W}^{-1}$, see the paper of Quinn and Carmichael [29].*
2. *By condition $(H2)$, we deduce that*

   $(H2)'$ *There exists a function $p\in L^1_{\text{loc}}(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi\colon\mathbb{R}^+\to(0,+\infty)$ and such that*

$$|f(t,u)|\leqslant p(t)\psi(\|u\|_{\mathcal{B}}) \quad \textit{for a.e. } t\in J \textit{ and each } u\in\mathcal{B}.$$

3. *We get from condition $(H4)$ that*

$(H4)'$ *There exists* $\widehat{\sigma}_n > 0$ *such that*

$$|h_t(u)| \leqslant \widehat{\sigma}_n$$

for each $t \in J$ *and* $u \in \mathcal{B}$ *with* $\|u\|_{\mathcal{B}} \leqslant n$.

4. $(H5)$ *There exists a constant* $0 < L < \dfrac{1}{\overline{M}_0 K_n}$ *such that*

$$|A(t)g(t,\phi)| \leqslant L\left(\|\phi\|_{\mathcal{B}} + 1\right) \quad \textit{for all } t \in J \textit{ and } \phi \in \mathcal{B}.$$

From (H_ϕ) and Proposition 1, we state the following property:

Corollary 1. *For any function* $y\colon (-\infty, b] \to E$ *such that* $y(t) = \phi(t) - h_t(y)$ *for* $t \leqslant 0$, *then for each* $t \in [0,n]$ *and* $n \in \mathbb{N}$ *we have*

$$\|y_{\rho(t,y_t)}\|_{\mathcal{B}} \leqslant K_n|y(t)| + \left(M_n + \mathcal{L}_h^{\phi}\right)(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n),$$

for $\rho \in \mathcal{R}(\rho^-)$, *and* $\mathcal{L}_h^{\phi} = \sup\limits_{t \in \mathcal{R}(\rho^-)} \mathcal{L}_h^{\phi}(t)$ *with* $\mathcal{L}_h^{\phi}(t) = \mathcal{L}^{\phi(t)-h_t(y)}$.

Consider the following space

$$B_{+\infty} = \{y\colon \mathbb{R} \to E\colon\ y|_{[0,T]} \text{ continuous for } T > 0 \text{ and } y_0 \in \mathcal{B}\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0,T]$.

Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by:

$$\|y\|_n := \sup\{e^{-\tau L_n^*(t)}|y(t)|\colon t \in [0,n]\}$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)\,\mathrm{d}s$, $\bar{l}_n(t) = K_n\widehat{M}l_n(t)$ and l_n is the function from $(H2)$. Then $B_{+\infty}$ is a Fréchet space with those family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

Theorem 2. *Assume that* (H_ϕ) *and* $(H1)$–$(H6)$ *hold and moreover for each* $n \in \mathbb{N}^*$, *there exists a constant* $M_n^\star > 0$ *such that*

$$\frac{M_n^\star}{\alpha_n + K_n\widehat{M}\dfrac{\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1}{1 - \overline{M}_0 L K_n}\left[M_n^\star + \psi\left(M_n^\star\right)\right]\|\zeta\|_{L^1}} > 1, \tag{5}$$

with $\zeta(t) = \max(L, p(t))$ *and*

$$\alpha_n = \left(K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}_h^{\phi}\right)\|\phi\|_{\mathcal{B}} + \left(K_n\widehat{M} + M_n + \mathcal{L}_h^{\phi}\right)\widehat{\sigma}_n + \frac{K_n\xi_n}{1 - \overline{M}_0 L K_n},$$

where

$$\xi_n = \xi_n\left(y^\star, \phi, \widehat{\sigma}_n\right)$$

$$= \Big[\Big(\widehat{M}+1\Big)\overline{M}_0 L + \widehat{M}Ln\Big]\Big(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\Big) + \widehat{M}\widetilde{M}\widetilde{M}_1 n\Big(1+\overline{M}_0 L K_n\Big)|y^\star|$$
$$+ \Big[\Big(\widehat{M}+M_n+\mathcal{L}_h^{\phi}\Big)\overline{M}_0 L\Big(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\Big) + \widehat{M}\mathcal{D}\Big(\widehat{M}\widetilde{M}\widetilde{M}_1 n + \overline{M}_0 L K_n\Big)\Big]\|\phi\|_{\mathcal{B}}$$
$$+ \Big[\Big(M_n+\mathcal{L}_h^{\phi}\Big)\overline{M}_0 L\Big(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\Big) + \widehat{M}\widetilde{M}\widetilde{M}_1 n\Big(\widehat{M}+1+\overline{M}_0 L K_n\Big)$$
$$+ \widehat{M}\overline{M}_0 L K_n\Big]\widehat{\sigma}_n\,.$$

Then the neutral functional evolution equation with infinite state-dependent delay (1)–(2) *is nonlocally controllable on* $\mathbb{R}$.

Proof. We transform the problem (1)–(2) into a fixed-point problem. For that, let us consider the operator $N\colon B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leqslant 0; \\ U(t,0)\,[\phi(0) - h_0(y) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) & \\ \quad + \displaystyle\int_0^t U(t,s)A(s)g(s,y_{\rho(s,y_s)})\,\mathrm{d}s + \int_0^t U(t,s)Cu_y(s)\,\mathrm{d}s & \\ \quad + \displaystyle\int_0^t U(t,s)f(s,y_{\rho(s,y_s)})\,\mathrm{d}s, & \text{if } t\in J. \end{cases}$$

Let's first introduce the following proposition:

Proposition 2. *From the inequalities* (3) *and* (4) *and the hypotheses* $(H1)$, $(H2)'$, $(H3)$, $(H4)'$ *and* $(H5)$, *for all* $t \in [0,n]$ *and* $n \in \mathbb{N}$ *we have*

$$\begin{aligned} |u_y(t)| \leqslant{}& \widetilde{M}_1\Big[|y^\star| + \widehat{M}(\mathcal{D}+\overline{M}_0 L)\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)(\overline{M}_0 L + \widehat{\sigma}_n) + \widehat{M}Ln\Big] \\ &+ \widetilde{M}_1\overline{M}_0 L\|y_{\rho(n,y_n)}\|_{\mathcal{B}} + \widetilde{M}_1\widehat{M}L\int_0^n \|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}}\,\mathrm{d}\tau \\ &+ \widetilde{M}_1\widehat{M}\int_0^n p(\tau)\psi(\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}})\,\mathrm{d}\tau. \end{aligned} \tag{6}$$

Proof. Using hypothesis $(H3)$, for an arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = \widetilde{W}^{-1}\Big[y^\star - h_n(y) - U(n,0)\big(\phi(0) - h_0(y) - g(0,\phi)\big) - g(n, y_{\rho(n,y_n)}) - \int_0^n U(n,s)A(s)g(s, y_{\rho(s,y_s)})\,\mathrm{d}s - \int_0^n U(n,s)f(s, y_{\rho(s,y_s)})\,\mathrm{d}s\Big](t).$$

By the hypotheses $(H1)$, $(H3)$, $(H4)'$, $(H5)$ and using Remark 1, we get

$$\begin{aligned}
|u_y(t)| &\leqslant \|\widetilde{W}^{-1}\|_{B(E)}\Big[|y^\star| + |h_n(y)| + \|U(t,0)\|_{B(E)}\big(|\phi(0)| + |h_0(y)| + |g(0,\phi)|\big) \\
&\quad + \big|g(n, y_{\rho(n,y_n)})\big| + \int_0^n \|U(n,\tau)\|_{B(E)}\,\big|A(\tau)g(\tau, y_{\rho(\tau,y_\tau)})\big|\,\mathrm{d}\tau \\
&\quad + \int_0^n \|U(n,\tau)\|_{B(E)}\,\big|f(\tau, y_{\rho(\tau,y_\tau)})\big|\,\mathrm{d}\tau\Big] \\
&\leqslant \widetilde{M}_1\Big[|y^\star| + \widehat{\sigma}_n + \widehat{M}\Big(\mathcal{D}\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n + \|A^{-1}(0)\|_{B(E)}\,|A(0)g(0,\phi)|\Big) \\
&\quad + \|A^{-1}(n)\|_{B(E)}\,\big|A(n)g(n, y_{\rho(n,y_n)})\big| + \widehat{M}\int_0^n \big|A(\tau)g(\tau, y_{\rho(\tau,y_\tau)})\big|\,\mathrm{d}\tau \\
&\quad + \widehat{M}\int_0^n \big|f(\tau, y_{\rho(\tau,y_\tau)})\big|\,\mathrm{d}\tau\Big] \\
&\leqslant \widetilde{M}_1\Big[|y^\star| + \widehat{\sigma}_n + \widehat{M}\big(\mathcal{D}\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n + \overline{M}_0L(\|\phi\|_{\mathcal{B}}+1)\big) + \overline{M}_0L(\|y_{\rho(n,y_n)}\|_{\mathcal{B}}+1) \\
&\quad + \widehat{M}\int_0^n L(\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}} + 1)\,\mathrm{d}\tau + \widehat{M}\int_0^n \big|f(\tau, y_{\rho(\tau,y_\tau)})\big|\,\mathrm{d}\tau\Big] \\
&\leqslant \widetilde{M}_1\Big[|y^\star| + \widehat{M}(\mathcal{D} + \overline{M}_0L)\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)(\overline{M}_0L + \widehat{\sigma}_n) + \widehat{M}Ln\Big] \\
&\quad + \widetilde{M}_1\overline{M}_0L\|y_{\rho(n,y_n)}\|_{\mathcal{B}} + \widetilde{M}_1\widehat{M}L\int_0^n \|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}}\,\mathrm{d}\tau + \widetilde{M}_1\widehat{M}\int_0^n \big|f(\tau, y_{\rho(\tau,y_\tau)})\big|\,\mathrm{d}\tau.
\end{aligned}$$

Applying $(H2)'$, we get

$$|u_y(t)| \leqslant \widetilde{M}_1\Big[|y^\star| + \widehat{M}(\mathcal{D} + \overline{M}_0L)\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)(\overline{M}_0L + \widehat{\sigma}_n) + \widehat{M}Ln\Big]$$

$$+ \widetilde{M}_1 \overline{M}_0 L \|y_{\rho(n,y_n)}\|_{\mathcal{B}} + \widetilde{M}_1 \widehat{M} L \int_0^n \|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}} \mathrm{d}\tau$$
$$+ \widetilde{M}_1 \widehat{M} \int_0^n p(\tau)\psi(\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}}) \mathrm{d}\tau.$$

□

Using the previous proposition, we will prove that the operator N has a fixed point $y(\cdot)$ which is the mild solution of the nonlocal neutral evolution equation (1)–(2).

For $\phi \in \mathcal{B}$, we define the function $x(\cdot) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leqslant 0; \\ U(t,0)[\phi(0) - h_0(y)], & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi - h_0(y)$.

For each function $z \in B_{+\infty}$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0, & \text{if } t \leqslant 0; \\ z(t), & \text{if } t \in J. \end{cases}$$

If $y(\cdot)$ satisfies (3), we can decompose it as $y(t) = z(t) + x(t)$ for $t \in J$, which implies $y_t = z_t + x_t$, for every $t \in J$. The function $z(\cdot)$ satisfies $z_0 = 0$ and for $t \in J$, we get

$$z(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t,0)g(0,\phi)$$
$$+ \int_0^t U(t,s)A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \mathrm{d}s + \int_0^t U(t,s)Cu_{z+x}(s) \mathrm{d}s$$
$$+ \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \mathrm{d}s.$$

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}.$$

We define for $t \in J$ the operators $F,\ G\colon B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\,\mathrm{d}s + \int_0^t U(t,s)Cu_{z+x}(s)\,\mathrm{d}s$$

and

$$G(z)(t) = \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\,\mathrm{d}s.$$

It is obvious that operator N has a fixed point if and only if $F+G$ has a fixed point, so it turns to prove that $F+G$ has a fixed point. The proof will be given in five steps.

Step 1: F is continuous.

Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $B^0_{+\infty}$ such that $z_k \to z$ in $B^0_{+\infty}$. By the hypotheses $(H1)$ and $(H3)$, we get for every $t \in [0,n]$

$$\begin{aligned}
&|F(z_k)(t) - F(z)(t)| \\
&\quad\leqslant |g(t, z_{k\rho(t,z_{kt}+x_t)} + x_{\rho(t,z_{kt}+x_t)}) - g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| \\
&\qquad + \int_0^t \|U(t,s)\|_{B(E)} \\
&\qquad \times \left|A(s)g(s, z_{k\rho(s,z_{ks}+x_s)} + x_{\rho(s,z_{ks}+x_s)}) - A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\right| \mathrm{d}s \\
&\qquad + \int_0^t \|U(t,s)\|_{B(E)}\|C\|_{B(E)}|u_{z_k+x}(s) - u_{z+x}(s)|\mathrm{d}s \\
&\quad\leqslant |g(t, z_{k\rho(t,z_{kt}+x_t)} + x_{\rho(t,z_{kt}+x_t)}) - g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| \\
&\qquad + \widehat{M}\int_0^t \Big|A(s)g(s, z_{k\rho(s,z_{ks}+x_s)} + x_{\rho(s,z_{ks}+x_s)}) - A(s)g(s, z_{\rho(s,z_s+x_s)} \\
&\qquad + x_{\rho(s,z_s+x_s)})\Big|\mathrm{d}s + \widehat{M}\widetilde{M}\int_0^t |u_{z_k+x}(s) - u_{z+x}(s)|\mathrm{d}s.
\end{aligned}$$

Using the hypotheses $(H1)$, $(H3)$ and $(H4)$, we get

$$\begin{aligned}
&|u_{z_k+x}(s) - u_{z+x}(s)| \\
&\quad \leqslant \|\widetilde{W}^{-1}\|_{B(E)} \Bigg[|h_n(z_k + x) - h_n(z + x)| + \|U(s,0)\|_{B(E)} |h_0(z_k + x) - h_0(z + x)| \\
&\qquad + \left| g(n, z_{k\rho(n,z_{kn}+x_n)} + x_{\rho(n,z_{kn}+x_n)}) - g(n, z_{\rho(n,z_n+x_n)} + x_{\rho(n,z_n+x_n)}) \right| \\
&\qquad + \int_0^n \|U(n,\tau)\|_{B(E)} \left| A(\tau) g(\tau, z_{k\rho(\tau,z_{k\tau}+x_\tau)} + x_{\rho(\tau,z_{k\tau}+x_\tau)}) \right. \\
&\qquad \left. - A(\tau) g(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}) \right| \mathrm{d}\tau \\
&\qquad + \int_0^n \|U(n,\tau)\|_{B(E)} \left| f(\tau, z_{k\rho(\tau,z_{k\tau}+x_\tau)} + x_{\rho(\tau,z_{k\tau}+x_\tau)}) \right. \\
&\qquad \left. - f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}) \right| \mathrm{d}\tau \Bigg] \\
&\quad \leqslant \widetilde{M}_1 \Bigg[\sigma_n \|z_k - z\|_{\mathcal{B}} + \widehat{M} \sigma_n \|z_k - z\|_{\mathcal{B}} \\
&\qquad + \left| g(n, z_{k\rho(n,z_{kn}+x_n)} + x_{\rho(n,z_{kn}+x_n)}) - g(n, z_{\rho(n,z_n+x_n)} + x_{\rho(n,z_n+x_n)}) \right| \\
&\qquad + \widehat{M} \int_0^n \left| A(\tau) g(\tau, z_{k\rho(\tau,z_{k\tau}+x_\tau)} + x_{\rho(\tau,z_{k\tau}+x_\tau)}) \right. \\
&\qquad \left. - A(\tau) g(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}) \right| \mathrm{d}\tau \\
&\qquad + \widehat{M} \int_0^n \left| f(\tau, z_{k\rho(\tau,z_{k\tau}+x_\tau)} + x_{\rho(\tau,z_{k\tau}+x_\tau)}) - f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}) \right| \mathrm{d}\tau \Bigg].
\end{aligned}$$

Then

$$\begin{aligned}
&|F(z_k)(t) - F(z)(t)| \\
&\quad \leqslant \left| g(t, z_{k\rho(t,z_{kt}+x_t)} + x_{\rho(t,z_{kt}+x_t)}) - g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) \right| \\
&\qquad + \widehat{M} \int_0^t \left| A(s) g(s, z_{k\rho(s,z_{ks}+x_s)} + x_{\rho(s,z_{ks}+x_s)}) \right. \\
&\qquad \left. - A(s) g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \right| \mathrm{d}s + \widehat{M}\widetilde{M}\widetilde{M}_1 n(\widehat{M} + 1)\sigma_n \|z_k - z\|_{\mathcal{B}}
\end{aligned}$$

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$$+ \widehat{M}\widetilde{M}\widetilde{M}_1 n \left|g(n, z_{k\rho(n,z_{kn}+x_n)} + x_{\rho(n,z_{kn}+x_n)}) - g(n, z_{\rho(n,z_n+x_n)} + x_{\rho(n,z_n+x_n)})\right|$$
$$+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n \int_0^n \left|A(s)g(s, z_{k\rho(s,z_{ks}+x_s)} + x_{\rho(s,z_{ks}+x_s)})\right.$$
$$\left. - A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\right| \mathrm{d}s$$
$$+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n \int_0^n \left|f(s, z_{k\rho(s,z_{ks}+x_s)} + x_{\rho(s,z_{ks}+x_s)})\right.$$
$$\left. -f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\right| \mathrm{d}s.$$

From the hypothesis $(H6)$ and since f is continuous, we obtain by the Lebesgue dominated convergence theorem

$$|F(z_k)(t) - F(z)(t)| \to 0 \quad \text{as} \quad k \to +\infty.$$

Thus F is continuous.

Step 2: F maps bounded sets of $B^0_{+\infty}$ into bounded sets.

Indeed, it is enough to show that for any $d > 0$, there exists a positive constant ℓ such that for each $z \in B_d = \{z \in B^0_{+\infty} : \|z\|_n \leqslant d\}$ one has $\|F(z)\|_n \leqslant \ell$.

Let $z \in B_d$. By the hypotheses $(H1)$ and $(H3)$, we have for each $t \in [0, n]$

$$|F(z)(t)| \leqslant |g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + \|U(t,0)\|_{B(E)}|g(0,\phi)|$$
$$+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\mathrm{d}s$$
$$+ \int_0^t \|U(t,s)\|_{B(E)}\|C\|_{B(E)}|u_{z+x}(s)|\mathrm{d}s$$
$$\leqslant \|A^{-1}(t)\|_{B(E)}\left|A(t)g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})\right|$$
$$+ \widehat{M}\|A^{-1}(0)\|_{B(E)}|A(0)g(0,\phi)|$$
$$+ \widehat{M}\int_0^t |A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\mathrm{d}s + \widehat{M}\widetilde{M}\int_0^t |u_{z+x}(s)|\mathrm{d}s.$$

Using $(H1)$, $(H5)$ and the inequality (6), we get

$$
\begin{aligned}
|F(z)(t)| &\leqslant \overline{M}_0 L\left(\|z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}}+1\right)+\widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}}+1)\\
&+\widehat{M}\int_0^t L\left(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+1\right)\mathrm{d}s\\
&+\widehat{M}\widetilde{M}\int_0^t \widetilde{M}_1\Bigg[|y^\star|+\widehat{M}(\mathcal{D}+\overline{M}_0 L)\|\phi\|_{\mathcal{B}}+(\widehat{M}+1)(\overline{M}_0 L+\widehat{\sigma}_n)+\widehat{M}Ln\\
&+\overline{M}_0 L\|z_{\rho(n,z_n+x_n)}+x_{\rho(n,z_n+x_n)}\|_{\mathcal{B}}+\widehat{M}L\int_0^n\|z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}}\mathrm{d}\tau\\
&+\widehat{M}\int_0^n p(\tau)\psi(\|z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}})\mathrm{d}\tau\Bigg]\mathrm{d}s\\
&\leqslant \overline{M}_0 L\|z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}}+\widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}}+(\widehat{M}+1)\overline{M}_0 L\\
&+\widehat{M}L\int_0^t\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\mathrm{d}s+\widehat{M}Ln\\
&+\widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|y^\star|+\widehat{M}(\mathcal{D}+\overline{M}_0 L)\|\phi\|_{\mathcal{B}}+(\widehat{M}+1)(\overline{M}_0 L+\widehat{\sigma}_n)+\widehat{M}Ln\right]\\
&+\widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0 Ln\|z_{\rho(n,z_n+x_n)}+x_{\rho(n,z_n+x_n)}\|_{\mathcal{B}}\\
&+\widehat{M}^2\widetilde{M}\widetilde{M}_1 Ln\int_0^n\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\mathrm{d}s\\
&+\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\mathrm{d}s\\
&\leqslant \left[(\widehat{M}+1)\overline{M}_0 L+\widehat{M}Ln\right]\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\right)+\widehat{M}\widetilde{M}\widetilde{M}_1 n|y^\star|\\
&+\widehat{M}\left[\overline{M}_0 L(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1)+\widehat{M}\widetilde{M}\widetilde{M}_1 n\mathcal{D}\right]\|\phi\|_{\mathcal{B}}+\widehat{M}\widetilde{M}\widetilde{M}_1 n(\widehat{M}+1)\widehat{\sigma}_n\\
&+\widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0 Ln\|z_{\rho(n,z_n+x_n)}+x_{\rho(n,z_n+x_n)}\|_{\mathcal{B}}+\overline{M}_0 L\|z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}}\\
&+\widehat{M}L\int_0^t\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\mathrm{d}s+\widehat{M}^2\widetilde{M}\widetilde{M}_1 Ln\int_0^n\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\mathrm{d}s\\
&+\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\mathrm{d}s.
\end{aligned}
$$

Using Corollary 1 and the fact that $x_0 = \phi - h_0(z+x)$, we get

$$\begin{aligned}
&\left\| z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)} \right\|_{\mathcal{B}} \\
&\qquad \leqslant \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \\
&\qquad \leqslant K_n|z(s)| + (M_n + \mathcal{L}_h^{\phi})\|z_0\|_{\mathcal{B}} + K_n|x(s)| + (M_n + \mathcal{L}_h^{\phi})\|x_0\|_{\mathcal{B}} \\
&\qquad \leqslant K_n|z(s)| + K_n\|U(s,0)\|_{B(E)}\left(|\phi(0)| + |h_0(z+x)|\right) \\
&\qquad\quad + (M_n + \mathcal{L}_h^{\phi})\left(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) \\
&\qquad \leqslant K_n|z(s)| + K_n\widehat{M}\left(\mathcal{D}\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) + (M_n + \mathcal{L}_h^{\phi})\left(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) \\
&\qquad \leqslant K_n|z(s)| + (K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}_h^{\phi})\|\phi\|_{\mathcal{B}} \\
&\qquad\quad + (K_n\widehat{M} + M_n + \mathcal{L}_h^{\phi})\widehat{\sigma}_n .
\end{aligned}$$

Set

$$c_n := \left(K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}_h^{\phi}\right)\|\phi\|_{\mathcal{B}} + \left(K_n\widehat{M} + M_n + \mathcal{L}_h^{\phi}\right)\widehat{\sigma}_n .$$

Then we obtain

$$\left\| z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)} \right\|_{\mathcal{B}} \leqslant K_n|z(s)| + c_n . \tag{7}$$

Since $z \in B_d$, then we have

$$\left\| z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)} \right\|_{\mathcal{B}} \leqslant K_n d + c_n := \delta_n . \tag{8}$$

Note that we have

$$\begin{aligned}
&\left\| z_{\rho(n,z_n+x_n)} + x_{\rho(n,z_n+x_n)} \right\|_{\mathcal{B}} \\
&\qquad \leqslant K_n|y(n)| + (M_n + \mathcal{L}_h^{\phi})\|y_0\|_{\mathcal{B}} \\
&\qquad \leqslant K_n|y^{\star} - h_n(y)| + (M_n + \mathcal{L}_h^{\phi})\left(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) \\
&\qquad \leqslant K_n|y^{\star}| + K_n\widehat{\sigma}_n + (M_n + \mathcal{L}_h^{\phi})\left(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) \\
&\qquad \leqslant K_n|y^{\star}| + (M_n + \mathcal{L}_h^{\phi})\|\phi\|_{\mathcal{B}} + (K_n + M_n + \mathcal{L}_h^{\phi})\widehat{\sigma}_n . \qquad (9)
\end{aligned}$$

We get, using the nondecreasing character of ψ, for each $t \in [0, n]$

$$
\begin{aligned}
|F(z)(t)| &\leqslant \Big[(\widehat{M}+1)\overline{M}_0 L + \widehat{M}Ln\Big]\Big(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\Big) + \widehat{M}\widetilde{M}\widetilde{M}_1 n|y^\star| \\
&+ \widehat{M}\Big[\overline{M}_0 L(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1) + \widehat{M}\widetilde{M}\widetilde{M}_1 n\mathcal{D}\Big]\|\phi\|_{\mathcal{B}} + \widehat{M}\widetilde{M}\widetilde{M}_1 n(\widehat{M}+1)\widehat{\sigma}_n \\
&+ \widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0 Ln\Big[K_n|y^\star| + (M_n + \mathcal{L}_h^\phi)\|\phi\|_{\mathcal{B}} + (K_n + M_n + \mathcal{L}_h^\phi)\widehat{\sigma}_n\Big] \\
&+ \overline{M}_0 L(K_n|z(t)| + c_n) + \widehat{M}L\int_0^t (K_n|z(s)| + c_n)\,\mathrm{d}s \\
&+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 Ln\int_0^n (K_n|z(s)| + c_n)\,\mathrm{d}s \\
&+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(K_n|z(s)| + c_n)\,\mathrm{d}s \\
&\leqslant \Big[(\widehat{M}+1)\overline{M}_0 L + \widehat{M}Ln\Big]\Big(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\Big) + \widehat{M}\widetilde{M}\widetilde{M}_1 n\Big(1+\overline{M}_0 LK_n\Big)|y^\star| \\
&+ \widehat{M}\Big[\overline{M}_0 L(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1) + \widehat{M}\widetilde{M}\widetilde{M}_1\mathcal{D}n + \widetilde{M}\widetilde{M}_1\overline{M}_0 Ln(M_n+\mathcal{L}_h^\phi)\Big]\|\phi\|_{\mathcal{B}} \\
&+ \widehat{M}\widetilde{M}\widetilde{M}_1 n\Big[\widehat{M} + 1 + \overline{M}_0 L(K_n + M_n + \mathcal{L}_h^\phi)\Big]\widehat{\sigma}_n + \overline{M}_0 L\delta_n + \widehat{M}Ln\delta_n \\
&+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n^2 L\delta_n + \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\psi(\delta_n)\|p\|_{L^1}\,.
\end{aligned}
$$

So

$$
\begin{aligned}
|F(z)(t)| &\leqslant \Big[(\widehat{M}+1)\overline{M}_0 L + \widehat{M}Ln\Big]\Big(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\Big) + \widehat{M}\widetilde{M}\widetilde{M}_1 n\Big(1+\overline{M}_0 LK_n\Big)|y^\star| \\
&+ \widehat{M}\Big[\overline{M}_0 L(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1) + \widehat{M}\widetilde{M}\widetilde{M}_1\mathcal{D}n + \widetilde{M}\widetilde{M}_1\overline{M}_0 Ln(M_n+\mathcal{L}_h^\phi)\Big]\|\phi\|_{\mathcal{B}} \\
&+ \widehat{M}\widetilde{M}\widetilde{M}_1 n\Big[\widehat{M} + 1 + \overline{M}_0 L(K_n + M_n + \mathcal{L}_h^\phi)\Big]\widehat{\sigma}_n \\
&+ L\Big[\overline{M}_0 + \widehat{M}n(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1)\Big]\delta_n + \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\psi(\delta_n)\|p\|_{L^1} := \ell_n\,.
\end{aligned}
$$

Thus there exists a positive number ℓ_n such that

$$\|F(z)\|_n \leqslant \ell_n.$$

Hence $F(B_d) \subset B_{\ell_n}$.

Step 3: F maps bounded sets into equicontinuous sets of $B^0_{+\infty}$.

We consider B_d as in Step 2 and we show that $F(B_d)$ is equicontinuous. Let $\tau_1,\ \tau_2 \in J$ with $\tau_2 > \tau_1$ and $z \in B_d$. Then

$$\begin{aligned}
&|F(z)(\tau_2) - F(z)(\tau_1)| \\
&\quad \leqslant \left| g(\tau_2, z_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})} + x_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})} + x_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})}) \right| \\
&\qquad + \|U(\tau_2,0) - U(\tau_1,0)\|_{B(E)} \|A^{-1}(0)\|_{B(E)} |A(0)g(0,\phi)| \\
&\qquad + \int_0^{\tau_1} \|U(\tau_2,s) - U(\tau_1,s)\|_{B(E)} |A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \mathrm{d}s \\
&\qquad + \int_{\tau_1}^{\tau_2} \|U(\tau_2,s)\|_{B(E)} |A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \mathrm{d}s \\
&\qquad + \int_0^{\tau_1} \|U(\tau_2,s) - U(\tau_1,s)\|_{B(E)} \|C\|_{B(E)} |u_{z+x}(s)| \mathrm{d}s \\
&\qquad + \int_{\tau_1}^{\tau_2} \|U(\tau_2,s)\|_{B(E)} \|C\|_{B(E)} |u_{z+x}(s)| \mathrm{d}s.
\end{aligned}$$

By the inequalities (6), (8) and (9) and using the nondecreasing character of ψ, we get

$$\begin{aligned}
|u_{z+x}(s)| &\leqslant \widetilde{M}_1 \left[|y^\star| + \widehat{M}(\mathcal{D} + \overline{M}_0 L)\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)(\overline{M}_0 L + \widehat{\sigma}_n) + \widehat{M} L n \right] \\
&\quad + \widetilde{M}_1 \overline{M}_0 L \|z_{\rho(n,z_n+x_n)} + x_{\rho(n,z_n+x_n)}\|_{\mathcal{B}} \\
&\quad + \widetilde{M}_1 \widehat{M} L \int_0^n \|z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}} \mathrm{d}\tau \\
&\quad + \widetilde{M}_1 \widehat{M} \int_0^n p(\tau)\psi(\|z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}}) \mathrm{d}\tau \\
&\leqslant \widetilde{M}_1 \left[|y^\star| + \widehat{M}(\mathcal{D} + \overline{M}_0 L)\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)(\overline{M}_0 L + \widehat{\sigma}_n) + \widehat{M} L n \right] \\
&\quad + \widetilde{M}_1 \overline{M}_0 L \left[K_n |y^\star| + (M_n + \mathcal{L}_h^\phi)\|\phi\|_{\mathcal{B}} + (K_n + M_n + \mathcal{L}_h^\phi)\widehat{\sigma}_n \right] \\
&\quad + \widetilde{M}_1 \widehat{M} L n \delta_n + \widehat{M} \widetilde{M}_1 \psi(\delta_n) \|p\|_{L^1}.
\end{aligned}$$

So

$$|u_{z+x}(s)| \leqslant \widetilde{M}_1 \left(1 + \overline{M}_0 L K_n\right) |y^\star| + \widetilde{M}_1 \left[\widehat{M}\mathcal{D} + \overline{M}_0 L(\widehat{M} + M_n + \mathcal{L}_h^\phi)\right] \|\phi\|_\mathcal{B}$$
$$+ \widetilde{M}_1 \left[\widehat{M} + 1 + \overline{M}_0 L(K_n + M_n + \mathcal{L}_h^\phi)\right] \widehat{\sigma}_n + \widetilde{M}_1 L \left[\overline{M}_0(\widehat{M} + 1) + \widehat{M}n\right]$$
$$+ \widehat{M}\widetilde{M}_1 Ln\delta_n + \widehat{M}\widetilde{M}_1 \psi(\delta_n)\|p\|_{L^1} := \omega_n . \tag{10}$$

By the hypothesis $(H1)$, $(H3)$, $(H5)$ and the inequality (10), we have

$$|F(z)(\tau_2) - F(z)(\tau_1)|$$
$$\leqslant \left| g(\tau_2, z_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})}) \right|$$
$$+ \overline{M}_0 L \left(\|\phi\|_\mathcal{B} + 1\right) \|U(\tau_2, 0) - U(\tau_1, 0)\|_{B(E)}$$
$$+ \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} L \left(\|z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)}\|_\mathcal{B} + 1\right) \mathrm{d}s$$
$$+ \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} L \left(\|z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)}\|_\mathcal{B} + 1\right) \mathrm{d}s$$
$$+ \widetilde{M}\omega_n \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \mathrm{d}s + \widetilde{M}\omega_n \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} \mathrm{d}s.$$

Using the inequality (8), we get

$$|F(z)(\tau_2) - F(z)(\tau_1)|$$
$$\leqslant \left| g(\tau_2, z_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})}) \right|$$
$$+ \overline{M}_0 L \left(\|\phi\|_\mathcal{B} + 1\right) \|U(\tau_2, 0) - U(\tau_1, 0)\|_{B(E)}$$
$$+ L(\delta_n + 1) \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \mathrm{d}s + L(\delta_n + 1) \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} \mathrm{d}s$$
$$+ \widetilde{M}\omega_n \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \mathrm{d}s + \widetilde{M}\omega_n \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} \mathrm{d}s.$$

Note that $|F(z)(\tau_2) - F(z)(\tau_1)|$ tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$ independently of $z \in B_d$.

The right hand side of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$ as a consequence of $(H6)$ and the fact that $U(t,s)$ is a strongly continuous and compact operator for $t > s$ implies the continuity in the uniform operator topology (see [4, 28]).

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.

Let $t \in J$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_d$ we define

$$\begin{aligned} F_\epsilon(z)(t) &= g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t,0)g(0,\phi) \\ &\quad + \int_0^{t-\epsilon} U(t,s)A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\,\mathrm{d}s \\ &\quad + \int_0^{t-\epsilon} U(t,s)Cu_{z+x}(s)\,\mathrm{d}s \\ &= g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t,t-\epsilon)U(t-\epsilon,0)g(0,\phi) \\ &\quad + U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\,\mathrm{d}s \\ &\quad + U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)Cu_{z+x}(s)\,\mathrm{d}s. \end{aligned}$$

Note that the set

$$\Big\{ g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t-\epsilon,0)g(0,\phi) + \int_0^{t-\epsilon} U(t-\epsilon,s)A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\,\mathrm{d}s + \int_0^{t-\epsilon} U(t-\epsilon,s)Cu_{z+x}(s)\,\mathrm{d}s : z \in B_d \Big\}$$

is bounded.

Since $U(t,s)$ is a compact operator and by hypothesis $(H6)$, we conclude that the set $Z_\epsilon(t) = \{F_\epsilon(z)(t) : z \in B_d\}$ is precompact in E for every ϵ sufficiently small, $0 < \epsilon < t$. Moreover using the inequalities (8), (10) and the hypotheses

$(H3)$ and $(H5)$, we obtain

$$\begin{aligned}|F(z)(t)-F_\epsilon(z)(t)| &\leqslant \int_{t-\epsilon}^{t} \|U(t,s)\|_{B(E)}|A(s)g(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|\mathrm{d}s\\ &\quad+\int_{t-\epsilon}^{t}\|U(t,s)\|_{B(E)}\|C\|_{B(E)}|u_{z+x}(s)|\mathrm{d}s\\ &\leqslant \int_{t-\epsilon}^{t}\|U(t,s)\|_{B(E)}L\left(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+1\right)\mathrm{d}s\\ &\quad+\widetilde{M}\omega_n\int_{t-\epsilon}^{t}\|U(t,s)\|_{B(E)}\mathrm{d}s\\ &\leqslant L(\delta_n+1)\int_{t-\epsilon}^{t}\|U(t,s)\|_{B(E)}\mathrm{d}s+\widetilde{M}\omega_n\int_{t-\epsilon}^{t}\|U(t,s)\|_{B(E)}\mathrm{d}s.\end{aligned}$$

Then

$$|F(z)(t)-F_\epsilon(z)(t)|\to 0 \quad as\ \epsilon\to 0.$$

Therefore there are precompact sets arbitrary close to the set $\{F(z)(t)\colon z\in B_d\}$. Hence the set $\{F(z)(t)\colon z\in B_d\}$ is precompact in E. So we deduce from Steps 1, 2 and 3 that F is a continuous compact operator.

Step 4: G is a contraction.

Indeed, consider $z,\ \overline{z}\in B^0_{+\infty}$. By the hypotheses $(H1)$ and $(H2)$ for each $t\in[0,n]$ and $n\in\mathbb{N}$

$$\begin{aligned}&|G(z)(t)-G(\overline{z})(t)|\\ &\leqslant\int_0^t\|U(t,s)\|_{B(E)}\left|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})\right|\mathrm{d}s\\ &\leqslant\int_0^t\widehat{M}l_n(s)\|z_{\rho(s,z_s+x_s)}-\overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}\mathrm{d}s.\end{aligned}$$

Using inequality (7), we obtain

$$\begin{aligned}|G(z)(t)-G(\overline{z})(t)| &\leqslant \int_0^t \widehat{M} l_n(s) K_n |z(s)-\overline{z}(s)|\,\mathrm{d}s \\ &\leqslant \int_0^t \overline{l}_n(s)|z(s)-\overline{z}(s)|\,\mathrm{d}s \\ &\leqslant \int_0^t \left[\overline{l}_n(s) e^{\tau L_n^*(s)}\right] e^{-\tau L_n^*(s)}|z(s)-\overline{z}(s)|\,\mathrm{d}s \\ &\leqslant \int_0^t \left[\frac{e^{\tau L_n^*(s)}}{\tau}\right]' \mathrm{d}s \|z-\overline{z}\|_n \\ &\leqslant \frac{1}{\tau} e^{\tau L_n^*(t)} \|z-\overline{z}\|_n.\end{aligned}$$

Therefore,

$$\|G(z)-G(\overline{z})\|_n \leqslant \frac{1}{\tau}\|z-\overline{z}\|_n.$$

So, the operator G is a contraction for all $n \in \mathbb{N}$.

Step 5: For applying Theorem 1, we must check (C2): i.e. it remains to show that the following set is bounded

$$\Gamma = \left\{z \in B^0_{+\infty} \colon z = \lambda F(z) + \lambda G\left(\frac{z}{\lambda}\right) \text{ for some } 0 < \lambda < 1\right\}.$$

Let $z \in \Gamma$. Then, by the hypotheses $(H1)$, $(H2)'$, $(H3)$, $(H5)$ and the inequality (6), we have for each $t \in [0, n]$

$$\begin{aligned}|z(t)| \leqslant{}& \lambda \left|g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})\right| + \lambda \|U(t,0)\|_{B(E)} |g(0,\phi)| \\ &+ \lambda \int_0^t \|U(t,s)\|_{B(E)} \left|A(s) g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\right| \mathrm{d}s \\ &+ \lambda \int_0^t \|U(t,s)\|_{B(E)} \|C\|_{B(E)} |u_{z+x}(s)|\,\mathrm{d}s\end{aligned}$$

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$$+\lambda\int_0^t \|U(t,s)\|_{B(E)}\left|f\left(s,\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right)\right|\mathrm{d}s$$

$$\leqslant \lambda\|A^{-1}(t)\|_{B(E)}\left|A(t)g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})\right|$$

$$+\lambda\|U(t,0)\|_{B(E)}\|A^{-1}(0)\|_{B(E)}\,|A(0)g(0,\phi)|$$

$$+\lambda\int_0^t \|U(t,s)\|_{B(E)}\left|A(s)g(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\right|\mathrm{d}s$$

$$+\lambda\int_0^t \|U(t,s)\|_{B(E)}\|C\|_{B(E)}|u_{z+x}(s)|\mathrm{d}s$$

$$+\lambda\int_0^t \|U(t,s)\|_{B(E)}\left|f\left(s,\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right)\right|\mathrm{d}s$$

$$\leqslant \lambda\overline{M}_0L\left(\|z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}}+1\right)+\lambda\widehat{M}\overline{M}_0L(\|\phi\|_{\mathcal{B}}+1)$$

$$+\lambda\widehat{M}L\int_0^t\left(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+1\right)\mathrm{d}s$$

$$+\lambda\widehat{M}\widetilde{M}\int_0^t\widetilde{M}_1\Big[|y^\star|+\widehat{M}(\mathcal{D}+\overline{M}_0L)\|\phi\|_{\mathcal{B}}+(\widehat{M}+1)(\overline{M}_0L+\widehat{\sigma}_n)+\widehat{M}Ln$$

$$+\overline{M}_0L\|z_{\rho(n,z_n+x_n)}+x_{\rho(n,z_n+x_n)}\|_{\mathcal{B}}+\widehat{M}L\int_0^n\|z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}}\mathrm{d}\tau$$

$$+\widehat{M}\int_0^n p(\tau)\psi(\|z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}})\mathrm{d}\tau\Big]\mathrm{d}s$$

$$+\lambda\widehat{M}\int_0^t p(s)\psi\left(\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}}\right)\mathrm{d}s.$$

Then

$$\begin{aligned}|z(t)| \leqslant \lambda &\left[(\widehat{M}+1)\overline{M}_0 L + \widehat{M}Ln\right]\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right) + \lambda\widehat{M}\widetilde{M}\widetilde{M}_1 n|y^\star| \\
&+ \lambda\widehat{M}\left[\overline{M}_0 L(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1) + \widehat{M}\widetilde{M}\widetilde{M}_1 \mathcal{D}n\right]\|\phi\|_{\mathcal{B}} + \lambda\widehat{M}\widetilde{M}\widetilde{M}_1 n(\widehat{M}+1)\widehat{\sigma}_n \\
&+ \lambda\widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0 Ln\|z_{\rho(n,z_n+x_n)} + x_{\rho(n,z_n+x_n)}\|_{\mathcal{B}} \\
&+ \lambda\overline{M}_0 L\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} \\
&+ \lambda\widehat{M}L\int_0^t \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\,\mathrm{d}s \\
&+ \lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1 Ln\int_0^n \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\,\mathrm{d}s \\
&+ \lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\,\mathrm{d}s \\
&+ \lambda\widehat{M}\int_0^t p(s)\psi\left(\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}}\right)\mathrm{d}s.\end{aligned}$$

Using Corollary 1 and inequality (7), we obtain

$$\begin{aligned}\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}} &\leqslant \left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}\right\|_{\mathcal{B}} + \|x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}} \\
&\leqslant \frac{K_n|z(s)|}{\lambda} + (M_n + \mathcal{L}_h^\phi)\|z_0\|_{\mathcal{B}} + K_n|x(s)| + (M_n + \mathcal{L}_h^\phi)\|x_0\|_{\mathcal{B}} \\
&\leqslant \frac{K_n|z(s)|}{\lambda} + K_n\|U(s,0)\|_{B(E)}(|\phi(0)| + |h_0(z+x)|) \\
&\quad + (M_n + \mathcal{L}_h^\phi)\left(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) \\
&\leqslant \frac{K_n|z(s)|}{\lambda} + K_n\widehat{M}\left(\mathcal{D}\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) + (M_n + \mathcal{L}_h^\phi)\left(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) \\
&\leqslant \frac{K_n|z(s)|}{\lambda} + (K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}_h^\phi)\|\phi\|_{\mathcal{B}} \\
&\quad + (K_n\widehat{M} + M_n + \mathcal{L}_h^\phi)\widehat{\sigma}_n\,.\end{aligned}$$

Then, we get

$$\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}} \leqslant \frac{K_n|z(s)|}{\lambda} + c_n. \tag{11}$$

By inequalities (7), (9) and the previous one and the nondecreasing character of ψ, we obtain

$$\begin{aligned}|z(t)| \leqslant \lambda &\left[(\widehat{M}+1)\overline{M}_0L+\widehat{M}Ln\right]\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)+\lambda\widehat{M}\widetilde{M}\widetilde{M}_1n|y^\star| \\ &+\lambda\widehat{M}\left[\overline{M}_0L(\widehat{M}\widetilde{M}\widetilde{M}_1n+1)+\widehat{M}\widetilde{M}\widetilde{M}_1n\mathcal{D}\right]\|\phi\|_{\mathcal{B}}+\lambda\widehat{M}\widetilde{M}\widetilde{M}_1n(\widehat{M}+1)\widehat{\sigma}_n \\ &+\lambda\widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0Ln\left[K_n|y^\star|+(M_n+\mathcal{L}_h^\phi)\|\phi\|_{\mathcal{B}}+(K_n+M_n+\mathcal{L}_h^\phi)\widehat{\sigma}_n\right] \\ &+\lambda\overline{M}_0L(K_n|z(t)|+c_n)+\lambda\widehat{M}L\int_0^t(K_n|z(s)|+c_n)\,\mathrm{d}s \\ &+\lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1Ln\int_0^n(K_n|z(s)|+c_n)\,\mathrm{d}s+\lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1n\int_0^n p(s)\psi(K_n|z(s)|+c_n)\,\mathrm{d}s \\ &+\lambda\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n|z(s)|}{\lambda}+c_n\right)\mathrm{d}s.\end{aligned}$$

Then

$$\begin{aligned}|z(t)| \leqslant \lambda &\left[(\widehat{M}+1)\overline{M}_0L+\widehat{M}Ln\right]\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)+\lambda\widehat{M}\widetilde{M}\widetilde{M}_1n\left(1+\overline{M}_0LK_n\right)|y^\star| \\ &+\lambda\widehat{M}\left[\overline{M}_0L(\widehat{M}\widetilde{M}\widetilde{M}_1n+1)+\widetilde{M}\widetilde{M}_1n\left(\widehat{M}\mathcal{D}+\overline{M}_0L(M_n+\mathcal{L}_h^\phi)\right)\right]\|\phi\|_{\mathcal{B}} \\ &+\lambda\widehat{M}\widetilde{M}\widetilde{M}_1n\left[\widehat{M}+1+\overline{M}_0L(K_n+M_n+\mathcal{L}_h^\phi)\right]\widehat{\sigma}_n+\lambda\overline{M}_0Lc_n+\lambda\overline{M}_0LK_n|z(t)| \\ &+\lambda\widehat{M}L\int_0^t(K_n|z(s)|+c_n)\,\mathrm{d}s+\lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1Ln\int_0^n(K_n|z(s)|+c_n)\,\mathrm{d}s \\ &+\lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1n\int_0^n p(s)\psi(K_n|z(s)|+c_n)\,\mathrm{d}s+\lambda\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n|z(s)|}{\lambda}+c_n\right)\mathrm{d}s.\end{aligned}$$

Set

$$\begin{aligned}\xi_n := &\left[(\widehat{M}+1)\overline{M}_0L+\widehat{M}Ln\right]\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)+\widehat{M}\widetilde{M}\widetilde{M}_1n\left(1+\overline{M}_0LK_n\right)|y^\star| \\ &+\widehat{M}\left[\overline{M}_0L(\widehat{M}\widetilde{M}\widetilde{M}_1n+1)+\widetilde{M}\widetilde{M}_1n\left(\widehat{M}\mathcal{D}+\overline{M}_0L(M_n+\mathcal{L}_h^\phi)\right)\right]\|\phi\|_{\mathcal{B}} \\ &+\widehat{M}\widetilde{M}\widetilde{M}_1n\left[\widehat{M}+1+\overline{M}_0L(K_n+M_n+\mathcal{L}_h^\phi)\right]\widehat{\sigma}_n+\overline{M}_0Lc_n.\end{aligned}$$

We consider the function $\widetilde{u}(t) := \sup_{\theta\in[0,t]} |z(\theta)|$. Then by the nondecreasing character of ψ, we get for $t \in [0,n]$

$$\begin{aligned}
&(1-\overline{M}_0 L K_n)\widetilde{u}(t)\\
&\quad\leqslant \lambda\xi_n + \lambda\widehat{M}L\int_0^t (K_n\widetilde{u}(s)+c_n)\,\mathrm{d}s + \lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1 Ln\int_0^n (K_n\widetilde{u}(s)+c_n)\,\mathrm{d}s\\
&\qquad + \lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(K_n\widetilde{u}(s)+c_n)\,\mathrm{d}s + \lambda\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)\mathrm{d}s.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\frac{K_n\widetilde{u}(t)}{\lambda}+c_n \leqslant c_n + \frac{K_n\xi_n}{1-\overline{M}_0LK_n} + \frac{K_n\widehat{M}}{1-\overline{M}_0LK_n}\Bigg[& L\int_0^t (K_n\widetilde{u}(s)+c_n)\,\mathrm{d}s\\
&+ \widehat{M}\widetilde{M}\widetilde{M}_1 nL\int_0^n (K_n\widetilde{u}(s)+c_n)\,\mathrm{d}s + \widehat{M}\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(K_n\widetilde{u}(s)+c_n)\,\mathrm{d}s\\
&+\int_0^t p(s)\psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)\mathrm{d}s\Bigg].
\end{aligned}$$

Set

$$\alpha_n := c_n + \frac{K_n\xi_n}{1-\overline{M}_0LK_n}.$$

By the nondecreasing character of ψ and for $\lambda < 1$, we get

$$\begin{aligned}
\frac{K_n\widetilde{u}(t)}{\lambda}+c_n \leqslant \alpha_n + \frac{K_n\widehat{M}}{1-\overline{M}_0LK_n}\Bigg[& L\int_0^t \left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)\mathrm{d}s\\
&+ \widehat{M}\widetilde{M}\widetilde{M}_1 nL\int_0^n \left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)\mathrm{d}s\\
&+ \widehat{M}\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)\mathrm{d}s + \int_0^t p(s)\psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)\mathrm{d}s\Bigg].
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) := \sup\left\{\frac{K_n\widetilde{u}(s)}{\lambda} + c_n : 0 \leqslant s \leqslant t\right\}, \quad t \in J.$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = \dfrac{K_n\widetilde{u}(t^\star)}{\lambda} + c_n$. If $t^\star \in [0,n]$, by the previous inequality, we have for $t \in [0,n]$

$$\mu(t) \leqslant \alpha_n + \frac{K_n\widehat{M}}{1-\overline{\overline{M}}_0LK_n}\Bigg[L\int_0^t \mu(s)\,\mathrm{d}s + \widehat{M}\widetilde{M}\widetilde{M}_1nL\int_0^n \mu(s)\,\mathrm{d}s + \widehat{M}\widetilde{M}\widetilde{M}_1n\int_0^n p(s)\psi(\mu(s))\,\mathrm{d}s + \int_0^t p(s)\psi(\mu(s))\,\mathrm{d}s\Bigg]$$

$$\leqslant \alpha_n + K_n\widehat{M}\frac{\widehat{M}\widetilde{M}\widetilde{M}_1n+1}{1-\overline{\overline{M}}_0LK_n}\left[L\int_0^n \mu(s)\,\mathrm{d}s + \int_0^n p(s)\psi(\mu(s))\,\mathrm{d}s\right].$$

Set $\zeta(t) := \max(L, p(t))$ for $t \in [0,n]$, then

$$\mu(t) \leqslant \alpha_n + K_n\widehat{M}\frac{\widehat{M}\widetilde{M}\widetilde{M}_1n+1}{1-\overline{\overline{M}}_0LK_n}\int_0^n \zeta(s)\,[\mu(s)+\psi(\mu(s))]\,\mathrm{d}s.$$

Consequently,

$$\frac{\|z\|_n}{\alpha_n + K_n\widehat{M}\dfrac{\widehat{M}\widetilde{M}\widetilde{M}_1n+1}{1-\overline{\overline{M}}_0LK_n}\left[\|z\|_n + \psi\left(\|z\|_n\right)\right]\|\zeta\|_{L^1}} \leqslant 1.$$

Then by the condition (5), there exists a constant $M_n^\star$ such that $\mu(t) \leqslant M_n^\star$. Since $\|z\|_n \leqslant \mu(t)$, we have $\|z\|_n \leqslant M_n^\star$, which means that the set $\varGamma$ is bounded, i.e. the statement (C2) in Theorem 1 does not hold.

From Avramescu's nonlinear alternative [8], we deduce that (C1) holds i.e. the operator $F+G$ has a fixed-point $z^\star$. Thus, there exists at least a fixed point $y^\star(t) = z^\star(t) + x(t),\ t \in \mathbb{R}$ of the operator N, which is a mild solution of the nonlocal problem (1)–(2). Thus the neutral evolution system (1)–(2) is nonlocally controllable on $\mathbb{R}$.

4. Example

To illustrate the previous results, we consider the neutral functional differential equation

$$
\begin{cases}
\dfrac{\partial}{\partial t}\left[v(t,\xi)-\displaystyle\int_{-\infty}^{0} a_3(s-t)v\left(s-\rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\eta)|v(t,\eta)|^2 d\eta\right),\xi\right)\mathrm{d}s\right] \\
\quad = \dfrac{\partial^2 v}{\partial \xi^2}(t,\xi)+a_0(t,\xi)v(t,\xi)+d(\xi)u(t) \\
\qquad +\displaystyle\int_{-\infty}^{0} a_1(s-t)v\left[s-\rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\eta)|v(t,\eta)|^2 d\eta\right),\xi\right]\mathrm{d}s, \\
\qquad\qquad t \geqslant 0,\ \xi\in[0,\pi], \\
v(t,0)=v(t,\pi)=0, \qquad t\geqslant 0, \\
v(\theta,\xi)+\displaystyle\sum_{i=1}^{p} c_i v(\theta+t_i,\xi)=v_0(\theta,\xi), \qquad \theta\leqslant 0,\ \xi\in[0,\pi],
\end{cases}
\tag{12}
$$

where $a_0\colon \mathbb{R}^+\times[0,\pi]\to\mathbb{R}$ is a given function such that $a_0(\cdot,\xi)$ is continuous and $a_0(t,\cdot)$ is uniformly Hölder continuous in t (see [20]); $a_1, a_3\colon \mathbb{R}^-\to\mathbb{R}$; $\rho_1\colon\mathbb{R}^+\to\mathbb{R}$; $\rho_2\colon\mathbb{R}\to\mathbb{R}$; $a_2\colon[0,\pi]\to\mathbb{R}$ and $v_0\colon\mathbb{R}^-\times[0,\pi]\to\mathbb{R}$ are continuous functions. c_i, $i=1,\cdots,p$, are given constants and $0<t_1<\cdots<t_p<+\infty$.

Let $E=L^2([0,\pi],\mathbb{R})$, $u(\cdot)\colon\mathbb{R}^+\to E$ is a given control and $d\colon[0,\pi]\to E$ is a continuous function.

Consider the operator $A\colon D(A)\subset E\to E$ given by $Aw=w''$ with domain

$$D(A):=\{w\in E\colon w''\in E, w(0)=w(\pi)=0\}.$$

Thus A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geqslant 0}$ on E. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n\in\mathbb{N}$, and corresponding normalized eigenfunctions give by

$$z_n(\xi)=\sqrt{\frac{2}{\pi}}\sin(n\xi).$$

In addition, $\{z_n\colon n\in\mathbb{N}\}$ is an orthonormal basis of E and

$$T(t)x=\sum_{n=1}^{\infty} e^{-n^2 t}(x,z_n)z_n \quad x\in E, t\geqslant 0.$$

It follows from this representation that $T(t)$ is compact for every $t > 0$ and that

$$\|T(t)\| \leqslant e^{-t} \quad \text{for every } t \geqslant 0.$$

On the domain $D(A)$, we define the operators $A(t)\colon D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi).$$

By assuming that $a_0(\cdot)$ is continuous and that $a_0(t,\xi) \leqslant -\delta_0$ ($\delta_0 > 0$) for every $t \in \mathbb{R}$, $\xi \in [0,\pi]$, it follows that the system

$$u'(t) = A(t)u(t) \quad t \geqslant s,$$

$$u(s) = x \in E,$$

has an associated evolution family given by

$$U(t,s)x(\xi) = \left[T(t-s)\exp\left(\int_s^t a_0(\tau,\xi)\,\mathrm{d}\tau\right)x\right](\xi).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that

$$\|U(t,s)\| \leqslant e^{-(1+\delta_0)(t-s)} \text{ for every } (t,s) \in \Delta.$$

Set $\mathcal{B} = BUC(\mathbb{R}^-, E)$ the space of bounded uniformly continuous functions defined from $\mathbb{R}^-$ to E endowed with the uniform norm

$$\|\phi\| = \sup_{\theta\in\mathbb{R}^-} |\phi(\theta)|.$$

Theorem 3. *Let $\phi \in \mathcal{B}$. Assume that condition (H_ϕ) holds and the functions $[0,\pi]\colon \mathbb{R}^+ \to E$, $\rho_1\colon \mathbb{R}^+ \to \mathbb{R}$, $\rho_2\colon \mathbb{R} \to \mathbb{R}$, $a_1, a_3\colon \mathbb{R}^- \to \mathbb{R}$, $a_2\colon [0,\pi] \to \mathbb{R}$ and $v_0 : \mathbb{R}^- \times [0,\pi] \to \mathbb{R}$ are continuous. Then the neutral differential equation (12) is nonlocally controllable on $\mathbb{R}$.*

Proof. From the assumptions and for $C \in B(\mathbb{R}, E)$, we have that

$$y(t)(\xi) = v(t,\xi), \qquad t \in \mathbb{R},\ \ \xi \in [0,\pi],$$

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)\,\mathrm{d}s, \qquad t \geqslant 0,\ \ \xi \in [0,\pi],$$

$$g(t,\psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)\,\mathrm{d}s, \qquad t \geqslant 0,\ \ \xi \in [0,\pi],$$

$$\rho(t,\psi)(\xi) = t - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\eta)|\psi(0,\xi)|^2 d\eta\right), \qquad t \geqslant 0, \ \ \xi \in [0,\pi],$$

$$Cu(t)(\xi) = d(\xi)u(t), \qquad t \geqslant 0, \ \ \xi \in [0,\pi], \ \ u \in \mathbb{R}, \ \ d(\xi) \in E,$$

$$h_t(v)(\xi) = \sum_{i=1}^{p} c_i v(t+t_i,\xi), \qquad t \leqslant 0, \ \ \xi \in [0,\pi]$$

and

$$\phi(t)(\xi) = v_0(t,\xi), \qquad t \leqslant 0, \ \ \xi \in [0,\pi],$$

are well defined which permit to transform system (12) into the abstract system (1)–(2). Moreover, the functions f and g are bounded linear operators. Then, the nonlocal controllability of mild solutions can be deduced from a direct application of Theorem 2 and the conclusion of our theorem hold.

From Remark 2, we have the following result.

Corollary 2. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then the system (12) is nonlocally controllable on $\mathbb{R}$.*

5. Conclusion

In this paper, the neutral evolution equations with state-dependent delay and when the conditions are nonlocal have been studied. By using Avramescu's nonlinear alternative in combination with semigroup theory to get sufficient conditions for the existence of a fixed point of an appropriate corresponding operator, which is the sum of compact operators and contraction maps in Fréchet spaces, we have established the existence of a mild solution and its controllability over the whole reel line.

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