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Pontryagin's maximum principle for the Roesser model with a fractional Caputo derivative

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In this paper, we study the modern mathematical theory of the optimal control problem associated with the fractional Roesser model and described by Caputo partial derivatives, where the functional is given by the Riemann-Liouville fractional integral. In the formulated problem, a new version of the increment method is applied, which uses the concept of an adjoint integral equation. Using the Banach fixed point principle, we prove the existence and uniqueness of a solution to the adjoint problem. Then the necessary and sufficient optimality condition is derived in the form of the Pontryagin's maximum principle. Finally, the result obtained is illustrated by a concrete example.

Key words: fractional optimal control, Pontryagin's maximum principle, Caputo derivative, Roesser model

1. Introduction

Modern optimal control theory is an important branch of mathematics that optimizes some objective functions and has many applications in science, engineering, and operations research. The scope of its application is constantly expanding, starting from the study of economic models and ending with mathematical models in physics. The rapid development of the theory of control of systems with lumped parameters is largely associated with the use of Pontryagin's maximum principle [32]. Further, various necessary conditions for optimality of the first and higher order are obtained for various systems [11, 24, 26, 27, 38, 39].

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Interest in the study of the problem of optimal control does not fade away and is reinforced by new applied problems. Opportunities in the study of systems described by equations with fractional derivatives have opened a new layer of unexplored problems in the management of such systems.

It is known that materials with memory and hereditary effects are better modeled by fractional order models than by integer ones [8, 19, 31, 34]. Optimal control theory is a field of mathematics that has been developed for many years, but fractional optimal control theory is a completely new field of mathematics. Fractional optimal control problem can be defined with respect to various definitions of fractional derivatives. But the most important types of fractional derivatives are the Riemann-Liouville and Caputo fractional derivatives.

In the modern theory of optimal control, there are already many works containing the formulation and results of the study of various optimal control problems for systems of fractional order. Main directions of research here are related to necessary optimality conditions (see, e.g. [2, 17] and the references therein) and numerical methods for constructing optimal controls (see e.g., [33] and the references therein). Pontryain's maximum principle for fractional optimal control problems proved in [6, 16, 44–46].

For effective control of many systems, real control objects must be considered as objects with distributed parameters, i.e. objects whose state at each moment of time is characterized by functions. Optimal control problems arise for systems whose state is described by differential equations in partial derivatives, functional-differential or integral equations-distributed or, in other words, infinite-dimensional systems. Optimal control problems described by hyperbolic equations considered in various papers and various necessary optimality conditions were obtained [13, 40–42].

In the analysis of many physical phenomena, researchers consider partial differential equations in two dimensions using fractional (noninteger) order partial differential operators. The most interesting areas where this approach has found application are two-dimensional models describing the dynamics of atomic dislocation in crystals [9], anomalous diffusion processes [28], and nonlinear cable equations used in electrophysiology [7]. Recently, fractional order partial differential equations with local and nonlocal conditions have been intensively studied [1, 25, 43].

The control of distributed systems described by fractional-order equations is a promising direction in the development of control theory, including from the point of view of mathematical modelling and applied problems, but there are still few works on this topic, see, for example, [14, 18]. The paper [18] investigates the optimal control problem associated with the fractional Roesser model described by Riemann-Liouville partial derivatives. The existence theorem for optimal solutions and the maximum principle for such a problem with inhomogeneous boundary conditions are proved.

The works [5, 10, 12, 15, 20–23, 30, 36, 37, 47], focuses on different order time-invariant and time-varying finite-dimensional systems, covering both continuous and discrete-time topics.

In [10] the class of nonlinear neutral fractional-integro-differential inclusions with infinite delay in Banach spaces is of interest. On the basis of Martelli's fixed point theorem, a theorem on the existence of mild solutions of fractional-integro-differential inclusions is obtained. The paper [15] studies two types of problems (the initial problem and the nonlocal Cauchy problem) for differential equations of fractional order with Hilfer's ψ-derivative in the multidimensional case. On the basis of the equivalence relation, new and general results are established for the existence of differential equations of fractional order with Hilfer ψ-operators of several variables in the space of weighted continuous functions for the nonlocal Cauchy problem. The work [37] considers such qualitative properties as uniform stability, asymptotic stability, and Mittag-Leffler stability of a trivial solution and boundedness of nonzero solutions of a system of nonlinear fractional integrodelay differential equations with Caputo fractional derivative, multiple kernels, and multiple delays are investigated. The paper [30] considers a semilinear functional-differential equation of fractional order in a Banach space under the assumption that its linear part is a generator of a noncompact semigroup. It is assumed that the nonlinearity satisfies the regularity condition expressed in terms of noncompactness measures. The theory of condensing mappings is used to obtain results about local and global existence. The same approach is applied to a neutral functional differential equation. The paper [4] discusses a new class of singular fractional systems in a multidimensional state space described by the Roesser continuous-time models. Necessary and sufficient conditions for asymptotic stability and feasibility are established using linear matrix inequalities. The article [3] is devoted to the global practical problem of stabilization with Mittag-Leffler feedback for a class of uncertain systems of fractional order, which is a wider class of nonlinearities than the Lipschitz ones. The article [47] considers an optimal control problem in which the dynamic system is controlled by a nonlinear Caputo fractional state equation. First, a linearized maximum principle is obtained, then the concept of a quasi-singular control is introduced, and on this basis an analogue of the Legendre-Clebsch conditions is obtained. When the analogue of the Legendre-Clebsch condition degenerates, a high-order necessary optimality condition is obtained. The article [29] discusses the problem of linear-quadratic (LQ) optimization for irregular singular systems of fractional order. The purpose of this article is to find pairs of control states that satisfy the dynamic constraint in the form of irregular singular systems of fractional order, such that the objective functional LQ is minimized. In [35], positive linear-fractional systems with continuous time are considered. Positive fractional systems without delays and positive fractional systems with a single control delay are studied.

As far as we know, the necessary optimality condition of the Pontryagin's maximum principle type for Roesser processes described by Caputo-fractional partial derivatives has not yet been studied. Therefore, this paper is devoted to the derivation of a necessary and sufficient optimality condition of the Pontryagin maximum principle type for an optimal control problem with distributed parameters. In this paper, we study the optimal control problem in which the state of the controlled object is described by Caputo fractional partial derivatives. First, the investigated problem is reduced to an equivalent integral equation and, using the Banach principle, the existence and uniqueness of a solution to the problem for each fixed admissible control is proved. The posed problem of optimal control is investigated using a new version of the increment method, in which the concept of a conjugate equation of an integral form is essentially used. The adjoint equation is the sum of partial operators of fractional integration with weight. Using commutation of fractional integrals with power functions, one can go directly from weighted fractional integrals to nonweighted fractional integrals. Then, using the fixed point of the Banach principle, we prove the existence and uniqueness of a solution to the adjoint problem, after which a necessary and sufficient optimality condition is derived in the form of the Pontryagin's maximum principle.

The rest of the paper is organized as follows. In Section 2, the definitions and basic properties of partial fractional order integrals and derivatives are recalled, and also some preliminary results are proved. Section 3 gives a formulation of the optimal control problem related to the fractional Roesser model described by partial Caputo derivatives. Section 4 is devoted to the proof of the theorem on the existence and uniqueness of the trajectory. Section 5 calculates the increment of the functional, proves the existence and uniqueness of a solution to the conjugate equation, and proves the Pontryagin's maximum principle. An example is given to illustrate the result.

2. Notations, definitions, and preliminary results

In this section, we give some definitions and basic concepts of partial fractional integrals and derivatives (for details, see [1, 19, 34]).

Let numbers r, n_i, $i = 1, 2$, $n \in \mathbb{N}$ be fixed. Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ be the spaces of n-dimensional vectors and $(n \times n)$-matrices. By $\|\cdot\|$, we denote a norm in $\mathbb{R}^n$ and the corresponding norm in $\mathbb{R}^{n\times n}$. Let numbers x_i^0, $X_i \in \mathbb{R}$, $x_i^0 < X_i$, $i = 1, 2$ be fixed, $G_i = (x_i^0, X_i)$, $i = 1, 2$, $G = G_1 \times G_2$, and let X be one of the spaces $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$.

Let $L^1(G,\ X)$ the Lebesgue space of summable functions defined on G with values in X, endowd with its usual norm $\|\cdot\|_{L^1}$. $L^\infty(G,\ X)$ the Lebesgue space of essentially bounded functions defined on G with values in X, endowed with its usual norm $\|\varphi(\cdot,\ \cdot)\|_{[x_1^0,\, X_1]\times[x_2^0,\, X_2]} = \operatorname*{ess\,sup}_{(x_1,x_2)\in[x_1^0,\, X_1]\times[x_2^0,\, X_2]} \|\varphi(x_1,\ x_2)\|$. $C(G,\ X)$ the space of continuous functions on G with values in X, endowed with the uniform norm $\|\cdot\|_G$.

Definition 1. *Let $\alpha_1 \in (0,\ 1)$, and $u(\cdot) \in L^1(G)$. The left-sided and right-sided partial Riemann-Liouville fractional integrals of the order α_1 of $u(\cdot)$ with respect to x_1 are defined by the expressions*

$$\left(I^{\alpha_1}_{x_1^0+}u\right)(x_1,\ x_2) = \frac{1}{\Gamma(\alpha_1)}\int_{x_1^0}^{x_1}(x_1 - s_1)^{\alpha_1-1}\,u\,(s_1,\, x_2)\,\mathrm{d}s_1\,,$$

$$\left(I^{\alpha_1}_{X_1-}u\right)(x_1,\ x_2) = \frac{1}{\Gamma(\alpha_1)}\int_{x_1}^{X_1}(s_1 - x_1)^{\alpha_1-1}\,u\,(s_1,\, x_2)\,\mathrm{d}s_1\,,$$

for almost all $x \in G$, where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined as

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}\mathrm{d}t, \qquad \xi > 0.$$

Analogously, we define the integrals

$$\left(I^{\alpha_2}_{x_2^0+}u\right)(x_1,\ x_2) = \frac{1}{\Gamma(\alpha_2)}\int_{x_2^0}^{x_2}(x_2 - s_2)^{\alpha_2-1}\,u\,(x_1,\ s_2)\,\mathrm{d}s_2,$$

$$\left(I^{\alpha_2}_{X_2-}u\right)(x_1,\ x_2) = \frac{1}{\Gamma(\alpha_2)}\int_{x_2}^{X_2}(s_2 - x_2)^{\alpha_2-1}\,u\,(x_1,\ s_2)\,\mathrm{d}s_2,$$

for almost all $x \in G$.

The Beta function is defined by the Euler integral of the first kind:

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t, \qquad \alpha,\ \beta > 0.$$

The function is connected with the Gamma functions by the relation $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Definition 2. *The left-sided partial Riemann-Liouville fractional derivative of order* $\alpha_1 \in (0,\ 1)$ *of* $u(\cdot)$ *with respect to* x_1 *is defined by the expression*

$$\left(D^{\alpha_1}_{x_1^0+}u\right)(x_1,\ x_2) = \frac{\partial}{\partial x_1}\left(I^{1-\alpha_1}_{x_1^0+}u\right)(x_1,\ x_2),$$

for almost all $x \in G$.

Similarly, we define the derivative

$$\left(D^{\alpha_2}_{x_2^0+}u\right)(x_1,\ x_2) = \frac{\partial}{\partial x_2}\left(I^{1-\alpha_2}_{x_2^0+}u\right)(x_1,\ x_2),$$

for almost all $x \in G$.

Definition 3. *The left-sided partial Caputo fractional derivative (regularized derivative)of order* $\alpha_1 \in (0,\ 1)$ *of* $u(\cdot)$ *with respect to* x_1 *is defined by expression*

$$\begin{aligned}\left(D^{\alpha_1}_{1}u\right)(x_1,\ x_2) &\equiv \left({}^{c}D^{\alpha_1}_{x_1^0+}u\right)(x_1,\ x_2)\\ &= \frac{\partial}{\partial x_1}\left(I^{1-\alpha_1}_{x_1^0+}\left(u(\cdot,\ x_2) - u\left(x_1^0,\ x_2\right)\right)\right)(x_1, x_2)\\ &= \frac{1}{\Gamma(1-\alpha_1)}\frac{\partial}{\partial x_1}\int\limits_{x_1^0}^{x_1}(x_1 - s_1)^{-\alpha_1}\left(u\left(s_1,\ x_2\right) - u(x_1^0,\ x_2)\right)\mathrm{d}s_1\,.\end{aligned}$$

Similarly, we define the derivative

$$\begin{aligned}\left(D^{\alpha_2}_{2}u\right)\ (x_1,\ x_2) &\equiv \left({}^{c}D^{\alpha_2}_{x_2^0+}u\right)(x_1,\ x_2)\\ &= \frac{\partial}{\partial x_2}\left(I^{1-\alpha_2}_{x_2^0+}\left(u\left(x_1,\ \cdot\right) - u(x_1,\ x_2^0)\right)\right)(x_1,\ x_2)\\ &= \frac{1}{\Gamma(1-\alpha_2)}\frac{\partial}{\partial x_2}\int\limits_{x_2^0}^{x_2}(x_2 - s_2)^{-\alpha_2}\left(u\left(x_1, s_2\right) - u(x_1,\ x_2^0)\right)ds_2\,,\end{aligned}$$

where $\alpha_2 \in (0,\ 1)$.

By $AC_\infty^\alpha(G, \mathbb{R}^{n_1+n_2}) = AC_\infty^{\alpha_1}(G, \mathbb{R}^{n_1}) \times AC_\infty^{\alpha_2}(G, \mathbb{R}^{n_2})$ *we denote the set of all functions* $z = (z_1, z_2)\colon G \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *such that*

$$z_1(x) = \psi_2(x_2) + \frac{1}{\Gamma(\alpha_1)} \int\limits_{x_1^0}^{x_1} (x_1 - \tau_1)^{\alpha_1 - 1} \varphi_1(\tau_1, x_2) \mathrm{d}\tau_1, \quad x \in G \text{ a.e.},$$

$$z_2(x) = \psi_1(x_1) + \frac{1}{\Gamma(\alpha_2)} \int\limits_{x_2^0}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1} \varphi_2(x_1, \tau_2) \mathrm{d}\tau_2, \quad x \in G \text{ a.e.},$$

where $\psi_2(\cdot) \in L^\infty(G_2, \mathbb{R}^{n_1})$, $\psi_1(\cdot) \in L^\infty(G_1, \mathbb{R}^{n_2})$, $\varphi_1(\cdot) \in L^\infty(G, \mathbb{R}^{n_1})$, $\varphi_2(\cdot) \in L^\infty(G, \mathbb{R}^{n_2})$.

Proposition 1. *For any* $\psi_i(\cdot) \in L^\infty(G_i, \mathbb{R}^{n_j})$, $i, j = 1, 2$, $i \neq j$, *and* $\varphi_i(\cdot) \in L^\infty(G, \mathbb{R}^{n_i})$, $i = 1, 2$ *the functions* $z_i(\cdot) \in AC_\infty^{\alpha_i}(G, \mathbb{R}^{n_i})$, $i = 1, 2$ *are correctly defined for almost all* $x \in G$. *The inequalities*

$$\left\| z_1(x_1^2, x_2) - z_1(x_1^1, x_2) \right\| \leqslant \frac{2}{\Gamma(\alpha_1 + 1)} \|\varphi_1(\cdot, x_2)\|_{[x_1^0, X_1]} (x_1^2 - x_1^1)^{\alpha_1}, \ \ x_2 \in G_2 \text{ a.e.},$$

$$\left\| z_2(x_1, x_2^2) - z_2(x_1, x_2^1) \right\| \leqslant \frac{2}{\Gamma(\alpha_2 + 1)} \|\varphi_2(x_1, \cdot)\|_{[x_2^0, X_2]} (x_2^2 - x_2^1)^{\alpha_2}, \ \ x_1 \in G_1 \text{ a.e.},$$

are valid for $x_1^1 < x_1^2$, $x_2^1 < x_2^2$, $x_1^1, x_1^2 \in G_1$, $x_2^1, x_2^2 \in G_2$.

In particular

a) $z_1(\cdot, x_2) \in C(G_1, \mathbb{R}^{n_1})$ *and* $z_1(x_1^0, x_2) = \psi_2(x_2)$ *for almost all* $x_2 \in G_2$,

b) $z_2(x_1, \cdot) \in C(G_2, \mathbb{R}^{n_2})$ *and* $z_2(x_1, x_2^0) = \psi_1(x_1)$ *for almost all* $x_1 \in G_1$.

Proposition 2. *For any* $z(\cdot) = (z_1(\cdot), z_2(\cdot)) \in AC_\infty^\alpha(G, \mathbb{R}^{n_1+n_2})$, *the values* $({}^cD^{\alpha_1}_{x_1^0+} z_1)(x_1, x_2)$, $({}^cD^{\alpha_2}_{x_2^0+} z_2)(x_1, x_2)$ *are correctly defined for almost all* $(x_1, x_2) \in G$. *Moreover, the inclusions* $({}^cD^{\alpha_1}_{x_1^0+} z_1)(\cdot) \in L^\infty(G, \mathbb{R}^{n_1})$, $({}^cD^{\alpha_2}_{x_2^0+} z_2)(\cdot) \in L^\infty(G, \mathbb{R}^{n_2})$ *holds and*

$$\left(I^{\alpha_1}_{x_1^0+} ({}^cD^{\alpha_1}_{x_1^0+} z_1) \right)(x_1, x_2) = z_1(x_1, x_2) - z_1(x_1^0, x_2), \quad x_2 \in G_2 \text{ a.e.},$$

$$\left(I^{\alpha_2}_{x_2^0+} ({}^cD^{\alpha_2}_{x_2^0+} z_2) \right)(x_1, x_2) = z_2(x_1, x_2) - z_2(x_1, x_2^0), \quad x_1 \in G_1 \text{ a.e.}.$$

3. Problem statement

The investigation object of the present paper is optimal control problem where system is driven by the linear Roesser equation with a fractional Caputo derivative of the order $\alpha = (\alpha_1,\ \alpha_2)$, $\alpha_i \in (0,\ 1), i = 1,\ 2$:

$$\begin{aligned}\left(D_1^{\alpha_1} z_1\right)(x) &= a_1(x)z_1(x) + a_2(x)z_2(x) + f_1\left(x,\ u(x)\right), \quad x \in G \ \text{ a.e.},\\ \left(D_2^{\alpha_2} z_2\right)(x) &= b_1(x)z_1(x) + b_2(x)z_2(x) + f_2\left(x,\ u(x)\right), \quad x \in G \ \text{ a.e.}\end{aligned} \tag{1}$$

The boundary conditions for (1) are given in the following form

$$\begin{aligned} z_1\left(x_1^0,\ x_2\right) &= \varphi_{10}(x_2), \quad x_2 \in G_2\,,\\ z_2\left(x_1,\ x_2^0\right) &= \varphi_{01}(x_1), \quad x_1 \in G_1 \ \text{ a.e.},\end{aligned} \tag{2}$$

where $D_i^{\alpha_i}, i = 1,\ 2$ are the partial α_i-order Caputo derivatives, $z_1(x_1,\ x_2) \in \mathbb{R}^{n_1}$, $z_2(x_1,\ x_2) \in \mathbb{R}^{n_2}$ are state vectors, the functions $a_i\colon G \longrightarrow \mathbb{R}^{n_1 \times n_i}$, $b_i\colon G \longrightarrow \mathbb{R}^{n_2 \times n_i}$, $i = 1,\ 2$, $\varphi_{10}\colon G_2 \longrightarrow \mathbb{R}^{n_1}$, $\varphi_{01}\colon G_1 \longrightarrow \mathbb{R}^{n_2}$ are essentially bounded, $u(\cdot)$ is r-dimensional measurable and bounded vector function of controlling effects on the G, $f_1(x,\ u)$ $(f_2(x,\ u))$ is n_1 (n_2)-dimensional vector function defined on $G \times \mathbb{R}^r$ and satisfying conditions:

1) measurable and bounded by x in G for all $u \in \mathbb{R}^r$,
2) $f_i(x,\ u), i = 1,\ 2$ are continuous by u in $\mathbb{R}^r$ for almost all $x \in G$.

It is assumed that almost everywhere on G the controlling effects satisfy the boundedness of the type of the inclusion:

$$u(x) \in V, \tag{3}$$

V is a nonempty bounded set in $\mathbb{R}^r$.

As a solution of the problem (1)-(2) corresponding to the fixed control function $u(\cdot)$, we consider the function $z(\cdot) = (z_1(\cdot),\ z_2(\cdot)) \in AC_\infty^{\alpha_1}(G,\ \mathbb{R}^{n_1}) \times AC_\infty^{\alpha_2}(G,\ \mathbb{R}^{n_2})$ satisfies differential equations (1) and condition (2) for almost every $x \in G$, $x_1 \in G_1$, $x_2 \in G_2$, respectively.

The goal of the optimal control problem is the minimization of the functional

$$J(u) = \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_1^0}^{X_1} \int_{x_2^0}^{X_2} (X_1 - x_1)^{\beta_1 - 1} (X_2 - x_2)^{\beta_2 - 1} \times \left[c_1'(x) z_1(x) + c_2'(x) z_2(x) + f_0(x, u(x)) \right] \mathrm{d}x + \frac{1}{\Gamma(\beta_1)} \int_{x_1^0}^{X_1} (X_1 - x_1)^{\beta_1 - 1} \, d_1'(x_1) z_2 \left(x_1, X_2\right) \mathrm{d}x_1 + \frac{1}{\Gamma(\beta_2)} \int_{x_2^0}^{X_2} (X_2 - x_2)^{\beta_2 - 1} \, d_2'(x_2) z_1 \left(X_1, x_2\right) \mathrm{d}x_2 \tag{4}$$

determined in the solutions of problem (1)–(2) for admissible control satisfying the condition (3).

Here $c_1(\cdot) \in L^\infty(G, \mathbb{R}^{n_1})$, $c_2(\cdot) \in L^\infty(G, \mathbb{R}^{n_2})$, $d_1(\cdot) \in L^\infty(G_1, \mathbb{R}^{n_2})$, $d_2(\cdot) \in L^\infty(G_2, \mathbb{R}^{n_1})$, scalar function $\varphi_0(x, u)$ defined on $G \times V$ and satisfying conditions 1) measurable and bounded by x in G for all $u \in V$, 2) $\varphi_0(x, u)$ is continuous by u in V for almost all $x \in G$, $\alpha_i \leqslant \beta_i < 1 + \alpha_i$, $i = 1, 2$.
The admissible control together with corresponding solutions of the problem (1)–(2) is called an admissible process. The admissible process $(u(\cdot), z(\cdot))$ being the solution of problem (1)–(4), i.e. minimizing the functional (4) at constraints (1)–(3) is said to be an optimal process, while $u(\cdot)$ – an optimal control.

4. Existence and uniqueness of the solution of the problem (1)–(4)

Solutions of system (1)–(2) on admissible controls we will consider a vector of functions of the form

$$\begin{aligned} z_1(x) &= \varphi_{10}(x_2) + \frac{1}{\Gamma(\alpha_1)} \int_{x_1^0}^{x_1} (x_1 - \tau_1)^{\alpha_1 - 1} \varphi_1 \left(\tau_1, \, x_2\right) \mathrm{d}\tau_1 \,, \\ z_2(x) &= \varphi_{01}(x_1) + \frac{1}{\Gamma(\alpha_2)} \int_{x_2^0}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1} \varphi_2 \left(x_1, \, \tau_2\right) \mathrm{d}\tau_2 \,, \end{aligned} \tag{5}$$

for which equations (1) and initial conditions (2) are satisfied almost everywhere on G, where $\varphi_1(\cdot) \in L^\infty(G, \mathbb{R}^{n_1})$, $\varphi_2(\cdot) \in L^\infty(G, \mathbb{R}^{n_2})$. It is obvious that this solution $z(\cdot) = (z_1(\cdot),\ z_2(\cdot))$ belongs to the set $AC^{\alpha}_{\infty}(G, \mathbb{R}^{n_1+n_2}) \cong AC^{\alpha_1}_{\infty}(G, \mathbb{R}^{n_1}) \times AC^{\alpha_2}_{\infty}(G, \mathbb{R}^{n_2})$. It is easy to see that the existence of a solution to system (1)–(2) in the set $AC^{\alpha}_{\infty}(G, \mathbb{R}^{n_1+n_2})$, corresponding to a control u, is equivalent to the existence of a solution to system

$$\begin{aligned} \varphi_1(x) &= \frac{a_1(x)}{\Gamma(\alpha_1)} \int_{x_1^0}^{x_1} (x_1 - \tau_1)^{\alpha_1 - 1} \varphi_1(\tau_1, x_2)\, d\tau_1 \\ &\quad + \frac{a_2(x)}{\Gamma(\alpha_2)} \int_{x_2^0}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1} \varphi_2(x_1, \tau_2)\, d\tau_2 + m_1(x,\ u(x)), \quad x \in G \text{ a.e.}, \\ \varphi_2(x) &= \frac{b_1(x)}{\Gamma(\alpha_1)} \int_{x_1^0}^{x_1} (x_1 - \tau_1)^{\alpha_1 - 1} \varphi_1(\tau_1, x_2)\, d\tau_1 \\ &\quad + \frac{b_2(x)}{\Gamma(\alpha_2)} \int_{x_2^0}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1} \varphi_2(x_1, \tau_2)\, d\tau_2 + m_2(x,\ u(x)), \quad x \in G \text{ a.e.}, \end{aligned} \tag{6}$$

in the set $\varphi(\cdot) = (\varphi_1(\cdot),\ \varphi_2(\cdot)) \in L^{\infty}(G, \mathbb{R}^{n_1+n_2})$, where $m_1(x,\ u(x)) = f_1(x,\ u(x)) - a_1(x)\varphi_{10}(x_2) - a_2(x)\varphi_{01}(x_1)$, $m_2(x,\ u(x)) = f_2(x,\ u(x)) - b_1(x)\varphi_{10}(x_2) - b_2(x)\varphi_{01}(x_1)$.

Applying the Banach contraction principle, we shall prove a theorem on the existence and uniqueness of a solution $z(\cdot) = (z_1(\cdot), z_2(\cdot))$ of system (1)–(2) for any control $u(\cdot) \in L^{\infty}(G, \mathbb{R}^r)$.

Theorem 1. *Let the above conditions are met for given tasks (1)–(4). Then problem (1)–(2) possesses a unique solution* $z(\cdot) = (z_1(\cdot),\ z_2(\cdot)) \in AC^{\alpha}_{\infty}(G,\ \mathbb{R}^{n_1+n_2})$ *corresponding to any control* $u(\cdot) \in L^{\infty}(G,\ \mathbb{R}^r)$.

Proof. Let us consider the Banach space $L^{\infty,e}(G, \mathbb{R}^{n_1+n_2})$ of essential bounded functions $\Phi(\cdot) = (\Phi_1(\cdot),\ \Phi_2(\cdot))$ with the Bielecki norm

$$\begin{aligned} \|\Phi(\cdot)\|_{\infty,e} &= \|\Phi_1(\cdot)\|_{\infty,e} + \|\Phi_2(\cdot)\|_{\infty,e} = \operatorname*{esssup}_{x \in G} \left(\|\Phi_1(x)\|\, e^{-(x_1 - x_1^0)k} e^{-(x_2 - x_2^0)k} \right) \\ &\quad + \operatorname*{esssup}_{x \in G} \left(\|\Phi_2(x)\|\, e^{-(x_1 - x_1^0)k} e^{-(x_2 - x_2^0)k} \right), \end{aligned}$$

where $k > 0$ is a fixed constant.

It is easy to see that this norm is equivalent to the classical one and consequently, $L^{\infty,e}(G,\ \mathbb{R}^{n_1+n_2})$ with the Bielecki norm is complete. To prove this theorem it suffices to show that for any control $u(\cdot) \in L^{\infty,e}(G, \mathbb{R}^r)$ there exists a unique fixed point of the operator $A = (A_1,\ A_2) : L^{\infty,e}\left(G,\ \mathbb{R}^{n_1+n_2}\right) \to L^{\infty,e}\left(G,\ \mathbb{R}^{n_1+n_2}\right)$,

$$\begin{aligned}(A_1\varphi)(x) &= \frac{a_1(x)}{\Gamma(\alpha_1)}\int_{x_1^0}^{x_1}(x_1-\tau_1)^{\alpha_1-1}\varphi_1(\tau_1,\ x_2)\,\mathrm{d}\tau_1\\ &+ \frac{a_2(x)}{\Gamma(\alpha_2)}\int_{x_2^0}^{x_2}(x_2-\tau_2)^{\alpha_2-1}\varphi_2(x_1,\ \tau_2)\,\mathrm{d}\tau_2 + m_1(x,\ u(x)),\quad x\in G\ \text{a.e.},\\ (A_2\varphi)(x) &= \frac{b_1(x)}{\Gamma(\alpha_1)}\int_{x_1^0}^{x_1}(x_1-\tau_1)^{\alpha_1-1}\varphi_1(\tau_1, x_2)\,\mathrm{d}\tau_1\\ &+ \frac{b_2(x)}{\Gamma(\alpha_2)}\int_{x_2^0}^{x_2}(x_2-\tau_2)^{\alpha_2-1}\varphi_2(x_1,\ \tau_2)\,\mathrm{d}\tau_2 + m_2(x,\ u(x)),\quad x\in G\ \text{a.e.}\end{aligned} \tag{7}$$

Under the conditions imposed on the tasks it follows that $(A\varphi)(\cdot) \in L^{\infty,e}(G,\ \mathbb{R}^{n_1+n_2})$. Hence, the operator A is correctly defined. Let us show that this operator is a contraction. Let us fixed $\varphi(\cdot) = (\varphi_1(\cdot),\ \varphi_2(\cdot))$, $\psi(\cdot) = (\psi_1(\cdot),\ \psi_2(\cdot)) \in L^{\infty,e}\left(G,\ \mathbb{R}^{n_1+n_2}\right)$. For almost all $x \in G$, we derive

$$\begin{aligned}\|(A_1\varphi)(x) - (A_1\psi)(x)\| &\leqslant \frac{a_1}{\Gamma(\alpha_1)}\int_{x_1^0}^{x_1}(x_1-\tau_1)^{\alpha_1-1}\,\|\varphi_1(\tau_1, x_2) - \psi_1(\tau_1, x_2)\|\,\mathrm{d}\tau_1\\ &+ \frac{a_2}{\Gamma(\alpha_2)}\int_{x_2^0}^{x_2}(x_2-\tau_2)^{\alpha_2-1}\,\|\varphi_2(x_1, \tau_2) - \psi_2(x_1, \tau_2)\|\,\mathrm{d}\tau_2,\end{aligned}$$

$$\begin{aligned}\|(A_2\varphi)(x) - (A_2\psi)(x)\| &\leqslant \frac{b_1}{\Gamma(\alpha_1)}\int_{x_1^0}^{x_1}(x_1-\tau_1)^{\alpha_1-1}\,\|\varphi_1(\tau_1, x_2) - \psi_1(\tau_1, x_2)\|\,\mathrm{d}\tau_1\\ &+ \frac{b_2}{\Gamma(\alpha_2)}\int_{x_2^0}^{x_2}(x_2-\tau_2)^{\alpha_2-1}\,\|\varphi_2(x_1, \tau_2) - \psi_2(x_1, \tau_2)\|\,\mathrm{d}\tau_2,\end{aligned}$$

where $a_i = \operatorname*{ess\,sup}_{x\in G}\|a_i(x)\|$, $b_i = \operatorname*{ess\,sup}_{x\in G}\|b_i(x)\|$, $i = 1,\ 2$.

Since

$$\begin{aligned}\|\varphi_1(\xi_1,\,x_2)-\psi_1(\xi_1,\,x_2)\| &\leqslant \|\varphi_1(\xi_1,\,x_2)-\psi_1(\xi_1,\,x_2)\|\,e^{-(\xi_1-x_1^0)k}e^{-(x_2-x_2^0)k}\\ &\quad\times e^{(\tau_1-x_1^0)k}e^{(x_2-x_2^0)k}\\ &\leqslant \|\varphi_1(\cdot)-\psi_1(\cdot)\|_{\infty,e}\,e^{(\tau_1-x_1^0)k}e^{(x_2-x_2^0)k},\\ &x_1^0\leqslant\xi_1\leqslant\tau_1\leqslant X_1,\quad x_2^0\leqslant x_2\leqslant X_2\,,\end{aligned}$$

and

$$\begin{aligned}\|\varphi_2\,(x_1,\,\xi_2)-\psi_2(x_1,\,\xi_2)\| &\leqslant \|\varphi_2(x_1,\,\xi_2)-\psi_2(x_1,\,\xi_2)\|\,e^{-(x_1-x_1^0)k}e^{-(\xi_2-x_2^0)k}\\ &\quad\times e^{(x_1-x_1^0)k}e^{(\tau_2-x_2^0)k}\\ &\leqslant \|\varphi_2(\cdot)-\psi_2(\cdot)\|_{\infty,e}\,e^{(x_1-x_1^0)k}e^{(\tau_2-x_2^0)k},\\ &x_1^0\leqslant x_1\leqslant X_1,\quad x_2^0\leqslant\xi_2\leqslant\tau_2\leqslant X_2\,,\end{aligned}$$

then for a.e., $x\in G$ we obtain

$$\begin{aligned}&\|(A_1\varphi)\,(x)-(A_1\psi)\,(x)\|\\ &\quad\leqslant\left(\frac{a_1}{\Gamma(\alpha_1)}\int_{x_1^0}^{x_1}(x_1-\tau_1)^{\alpha_1-1}\,e^{(\tau_1-x_1^0)k}\,\mathrm{d}\tau_1e^{(x_2-x_2^0)k}\right)\|\varphi_1(\cdot)-\psi_1(\cdot)\|_{\infty,e}\\ &\quad+\left(\frac{a_2}{\Gamma(\alpha_2)}\int_{x_2^0}^{x_2}(x_2-\tau_2)^{\alpha_2-1}\,e^{(\tau_2-x_2^0)k}\,\mathrm{d}\tau_2e^{(x_1-x_1^0)k}\right)\|\varphi_2(\cdot)-\psi_2(\cdot)\|_{\infty,e}\,,\end{aligned}$$

$$\begin{aligned}&\|(A_2\varphi)\,(x)-(A_2\psi)\,(x)\|\\ &\quad\leqslant\left(\frac{b_1}{\Gamma(\alpha_1)}\int_{x_1^0}^{x_1}(x_1-\tau_1)^{\alpha_1-1}\,e^{(\tau_1-x_1^0)k}\,\mathrm{d}\tau_1e^{(x_2-x_2^0)k}\right)\|\varphi_1(\cdot)-\psi_1(\cdot)\|_{\infty,e}\\ &\quad+\left(\frac{b_2}{\Gamma(\alpha_2)}\int_{x_2^0}^{x_2}(x_2-\tau_2)^{\alpha_2-1}\,e^{(\tau_2-x_2^0)k}\,\mathrm{d}\tau_2e^{(x_1-x_1^0)k}\right)\|\varphi_2(\cdot)-\psi_2(\cdot)\|_{\infty,e}\,.\end{aligned}$$

Taking the inequality

$$\frac{1}{\Gamma(\alpha_i)}\int_{x_i^0}^{x_i}(x_i-\tau_i)^{\alpha_i-1}\,e^{(\tau_i-x_i^0)k}\,\mathrm{d}\tau_i\leqslant e^{(x_i-x_i^0)k}k^{-\alpha_i},\quad i=1,\,2$$

into account, we have

$$\|(A_1\varphi)(\cdot) - (A_1\psi)(\cdot)\|_{\infty,e} \leqslant \frac{a_1}{k^{\alpha_1}}\|\varphi_1(\cdot) - \psi_1(\cdot)\|_{\infty,e} + \frac{a_2}{k^{\alpha_2}}\|\varphi_2(\cdot) - \psi_2(\cdot)\|_{\infty,e},$$

$$\|(A_2\varphi)(\cdot) - (A_2\psi)(\cdot)\|_{\infty,e} \leqslant \frac{b_1}{k^{\alpha_1}}\|\varphi_1(\cdot) - \psi_1(\cdot)\|_{\infty,e} + \frac{b_2}{k^{\alpha_2}}\|\varphi_2(\cdot) - \psi_2(\cdot)\|_{\infty,e}.$$

Denoting $a = \max\{a_1, a_2, b_1, b_2\}$, $\alpha = \min\{\alpha_1, \alpha_2\}$, from these inequalities we obtain

$$\|(A\varphi)(\cdot) - (A\psi)(\cdot)\|_{\infty,e} \leqslant \frac{2a}{k^{\alpha}}\|\varphi(\cdot) - \psi(\cdot)\|_{\infty,e}.$$

Thus due to the choice of the number k, the operator A is a contraction. By the Banach contraction principle, this operator has a unique fixed point, which is a unique solution $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot)) \in L^{\infty,e}(G, \mathbb{R}^{n_1+n_2})$ of the system (6). Hence, problem (1)–(2) has a unique solution. □

5. Necessary and sufficient conditions for optimality

The aim of this section is to derive the necessary optimality conditions for problem (1)–(4). To obtain the necessary and sufficient conditions for optimality, first we find increment of the functional (4). Let $(u(\cdot), z(\cdot))$ be a fixed admissible process in problem (1)-(4). Along with this process, consider another admissible process $(\overline{u}(\cdot) = u(\cdot) + \Delta u(\cdot), \overline{z}(\cdot) = z(\cdot) + \Delta z(\cdot))$. Then the increment $\Delta z(\cdot)$ of the state $z(\cdot)$ is a solution to the system of linear partial differential equation with a fractional Caputo derivative of the order $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \in (0, 1)$, $i = 1, 2$:

$$\left(D_1^{\alpha_1}\Delta z_1\right)(x) = a_1(x)\Delta z_1(x) + a_2(x)\Delta z_2(x) + \Delta f_1(x, u(x)), \quad x \in G \text{ a.e.},$$

$$\left(D_2^{\alpha_2}\Delta z_2\right)(x) = b_1(x)\Delta z_1(x) + b_2(x)\Delta z_2(x) + \Delta f_2(x, u(x)), \quad x \in G \text{ a.e.},$$

with the following conditions

$$\Delta z_1\left(x_1^0, x_2\right) = 0, \quad x_2 \in G_2 \quad \text{a.e.},$$

$$\Delta z_2\left(x_1, x_2^0\right) = 0, \quad x_1 \in G_1 \quad \text{a.e.},$$

where $\Delta f_i(x, u(x)) = f_i(x, u(x) + \Delta u(x)) - f_i(x, u(x))$, $i = 1, 2$.

We introduce some nontrivial vector functions $\psi_1 : G \to \mathbb{R}^{n_1}$ and $\psi_2 : G \to \mathbb{R}^{n_2}$. Then the increment of objective functional (4) may be represented as

$$\Delta J(u) = \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_1^0}^{X_1} \int_{x_2^0}^{X_2} (X_1 - x_1)^{\beta_1 - 1} (X_2 - x_2)^{\beta_2 - 1}$$

$$\times \left[c_1'(x)\Delta z_1(x) + c_2'(x)\Delta z_2(x) + \Delta f_0(x,\ u(x)) \right] \mathrm{d}x$$

$$+ \ \frac{1}{\Gamma(\beta_1)} \int_{x_1^0}^{X_1} (X_1 - x_1)^{\beta_1 - 1} \, d_1'(x_1)\Delta z_2(x_1,\ X_2)\,\mathrm{d}x_1$$

$$+ \ \frac{1}{\Gamma(\beta_2)} \int_{x_2^0}^{X_2} (X_2 - x_2)^{\beta_2 - 1} \, d_2'(x_2)\Delta z_1(X_1,\ x_2)\,\mathrm{d}x_2$$

$$+ \ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1^0}^{X_1} \int_{x_2^0}^{X_2} (X_1 - x_1)^{\alpha_1 - 1} (X_2 - x_2)^{\alpha_2 - 1}$$

$$\times \Big\{ \psi_1'(x) \Big[\left(D_1^{\alpha_1} \Delta z_1 \right)(x) - a_1(x)\Delta z_1(x) - a_2(x)\Delta z_2(x)$$

$$- \ \Delta f_1(x,\ u(x)) \Big] + \psi_2'(x) \Big[\left(D_2^{\alpha_2} \Delta z_2 \right)(x)$$

$$- \ b_1(x)\Delta z_2(x) - b_2(x)\Delta z_2(x) - \Delta f_2(x,\ u(x)) \Big] \Big\} \mathrm{d}x.$$

Using relation

$$\Delta z_1(x) = \frac{1}{\Gamma(\alpha_1)} \int_{x_1^0}^{x_1} (x_1 - \tau_1)^{\alpha_1 - 1} \left(D_1^{\alpha_1} \Delta z_1 \right)(\tau_1, x_2)\,\mathrm{d}\tau_1 \,,$$

$$\Delta z_2(x) = \frac{1}{\Gamma(\alpha_2)} \int_{x_2^0}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1} \left(D_2^{\alpha_2} \Delta z_2 \right)(x_1, \tau_2)\,\mathrm{d}\tau_2 \,,$$

we get

$$\Delta J(u) = \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_1^0}^{X_1} \int_{x_2^0}^{X_2} (X_1 - x_1)^{\beta_1 - 1} (X_2 - x_2)^{\beta_2 - 1}$$

$$\times \left[c_1'(x) \frac{1}{\Gamma(\alpha_1)} \int_{x_1^0}^{x_1} (x_1 - \tau_1)^{\alpha_1 - 1} \left(D_1^{\alpha_1} \Delta z_1 \right) (\tau_1, x_2) \, \mathrm{d}\tau_1 + c_2'(x) \right.$$

$$\left. \times \frac{1}{\Gamma(\alpha_2)} \int_{x_2^0}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1} \left(D_2^{\alpha_2} \Delta z_2 \right) (x_1, \tau_2) \, \mathrm{d}\tau_2 + \Delta f_0(x, \ u(x)) \right] \mathrm{d}x$$

$$+ \frac{1}{\Gamma(\beta_1)} \int_{x_1^0}^{X_1} (X_1 - x_1)^{\beta_1 - 1} d_1'(x_1) \frac{1}{\Gamma(\alpha_2)} \int_{x_2^0}^{X_2} (X_2 - x_2)^{\alpha_2 - 1} \left(D_2^{\alpha_2} \Delta z_2 \right) (x_1, x_2) \, \mathrm{d}x_2 \, \mathrm{d}x_1$$

$$+ \frac{1}{\Gamma(\beta_2)} \int_{x_2^0}^{X_2} (X_2 - x_2)^{\beta_2 - 1} d_2'(x_2) \frac{1}{\Gamma(\alpha_1)} \int_{x_1^0}^{X_1} (X_1 - x_1)^{\alpha_1 - 1} \left(D_1^{\alpha_1} \Delta z_1 \right) (x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

$$+ \ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1^0}^{X_1} \int_{x_2^0}^{X_2} (X_1 - x_1)^{\alpha_1 - 1} (X_2 - x_2)^{\alpha_2 - 1}$$

$$\times \left\{ \psi_1'(x) \left[\left(D_1^{\alpha_1} \Delta z_1 \right) (x) - a_1(x) \frac{1}{\Gamma(\alpha_1)} \int_{x_1^0}^{x_1} (x_1 - \tau_1)^{\alpha_1 - 1} \left(D_1^{\alpha_1} \Delta z_1 \right) (\tau_1, x_2) \, \mathrm{d}\tau_1 \right. \right.$$

$$\left. - a_2(x) \frac{1}{\Gamma(\alpha_2)} \int_{x_2^0}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1} \left(D_2^{\alpha_2} \Delta z_2 \right) (x_1, \tau_2) \, \mathrm{d}\tau_2 - \Delta f_1 \left(x, \ u(x) \right) \right]$$

$$+ \ \psi_2'(x) \left[\left(D_2^{\alpha_2} \Delta z_2 \right) (x) - b_1(x) \frac{1}{\Gamma(\alpha_1)} \int_{x_1^0}^{x_1} (x_1 - \tau_1)^{\alpha_1 - 1} \left(D_1^{\alpha_1} \Delta z_1 \right) (\tau_1, x_2) \, \mathrm{d}\tau_1 \right.$$

$$\left. \left. - b_2(x) \frac{1}{\Gamma(\alpha_2)} \int_{x_2^0}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1} \left(D_2^{\alpha_2} \Delta z_2 \right) (x_1, \tau_2) d\tau_2 - \Delta f_2 \left(x, \ u(x) \right) \right] \right\} \mathrm{d}x.$$

Using the Dirichlet formulas for permutation, the order of integration and after simple transformations we have

$$\Delta J(u) = -\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int\limits_{x_1^0}^{X_1} \int\limits_{x_2^0}^{X_2} (X_1 - x_1)^{\alpha_1 - 1}(X_2 - x_2)^{\alpha_2 - 1} \Delta_u H(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2$$

$$+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int\limits_{x_1^0}^{X_1} \int\limits_{x_2^0}^{X_2} (X_1 - x_1)^{\alpha_1 - 1}(X_2 - x_2)^{\alpha_2 - 1} \Bigg\{ \Bigg[\psi_1'(x) - \frac{(X_1 - x_1)^{1-\alpha_1}}{\Gamma(\alpha_1)}$$

$$\times \int\limits_{x_1}^{X_1} (X_1 - \tau_1)^{\alpha_1 - 1}(\tau_1 - x_1)^{\alpha_1 - 1} \big(\psi_1'(\tau_1, x_2) a_1(\tau_1, x_2)$$

$$+ \psi_2'(\tau_1, x_2) b_1(\tau_1, x_2)\big)\, \mathrm{d}\tau_1 + \frac{\Gamma(\alpha_2)(X_1 - x_1)^{1-\alpha_1}(X_2 - x_2)^{\beta_2 - \alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)}$$

$$\times \int\limits_{x_1}^{X_1} (X_1 - \tau_1)^{\beta_1 - 1}(\tau_1 - x_1)^{\alpha_1 - 1} c_1'(\tau_1, x_2)\, \mathrm{d}\tau_1$$

$$+ \frac{\Gamma(\alpha_2)(X_2 - x_2)^{\beta_2 - \alpha_2}}{\Gamma(\beta_2)} d_2'(x_2) \Bigg] \left(D_1^{\alpha_1} \Delta z_1 \right)(x_1, x_2)$$

$$+ \Bigg[\psi_2'(x) - \frac{(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\alpha_2)} \int\limits_{x_2}^{X_2} (X_2 - \tau_2)^{\alpha_2 - 1}(\tau_2 - x_2)^{\alpha_2 - 1}$$

$$\times \left(\psi_1'(x_1, \tau_2) a_2(x_1, \tau_2) + \psi_2'(x_1, \tau_2) b_2(x_1, \tau_2) \right) \mathrm{d}\tau_2$$

$$+ \frac{\Gamma(\alpha_1)(X_1 - x_1)^{\beta_1 - \alpha_1}(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int\limits_{x_2}^{X_2} (X_2 - \tau_2)^{\beta_2 - 1}(\tau_2 - x_2)^{\alpha_2 - 1} c_2'(x_1, \tau_2)\, \mathrm{d}\tau_2$$

$$+ \frac{\Gamma(\alpha_1)(X_1 - x_1)^{\beta_1 - \alpha_1}}{\Gamma(\beta_1)} d_1'(x_1) \Bigg] \left(D_2^{\alpha_2} \Delta z_2 \right)(x_1, x_2) \Bigg\} \mathrm{d}x_1\, \mathrm{d}x_2\,, \tag{8}$$

where

$$H(x) = \psi_1'(x) f_1(x, u(x)) + \psi_2'(x) f_2(x, u(x))$$
$$- \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\,(X_1 - x_1)^{\beta_1 - \alpha_1}\,(X_2 - x_2)^{\beta_2 - \alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} f_0(x, u(x))\,.$$

In order to simplify this expression, we will use the arbitrariness of vector-functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ to define the conjugate system. For this let

$$\psi_1(x) = \frac{(X_1 - x_1)^{1-\alpha_1}}{\Gamma(\alpha_1)} \int\limits_{x_1}^{X_1} (X_1 - \tau_1)^{\alpha_1 - 1} (\tau_1 - x_1)^{\alpha_1 - 1} \left[a_1'(\tau_1, x_2)\psi_1(\tau_1, x_2) + b_1'(\tau_1, x_2)\psi_2(\tau_1, x_2)\right] \mathrm{d}\tau_1 - \frac{\Gamma(\alpha_2)\,(X_1 - x_1)^{1-\alpha_1}\,(X_2 - x_2)^{\beta_2 - \alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \int\limits_{x_1}^{X_1} (X_1 - \tau_1)^{\beta_1 - 1} (\tau_1 - x_1)^{\alpha_1 - 1} c_1(\tau_1, x_2)\,\mathrm{d}\tau_1 - \frac{\Gamma(\alpha_2)\,(X_2 - x_2)^{\beta_2 - \alpha_2}}{\Gamma(\beta_2)} d_2(x_2), \quad x \in G \text{ a.e.}, \tag{9}$$

$$\psi_2(x) = \frac{(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\alpha_2)} \int\limits_{x_2}^{X_2} (X_2 - \tau_2)^{\alpha_2 - 1} (\tau_2 - x_2)^{\alpha_2 - 1} \left[a_2'(x_1, \tau_2)\psi_1(x_1, \tau_2) + b_2'(x_1, \tau_2)\psi_2(x_1, \tau_2)\right] \mathrm{d}\tau_2 - \frac{\Gamma(\alpha_1)(X_1 - x_1)^{\beta_1 - \alpha_1}(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \int\limits_{x_2}^{X_2} (X_2 - \tau_2)^{\beta_2 - 1} (\tau_2 - x_2)^{\alpha_2 - 1} c_2(x_1, \tau_2)\,\mathrm{d}\tau_2 - \frac{\Gamma(\alpha_1)\,(X_1 - x_1)^{\beta_1 - \alpha_1}}{\Gamma(\beta_1)} d_1(x_1), \quad x \in G \text{ a.e.} \tag{10}$$

Taking into account these equalities in (8), for the functional increment we obtain the formula

$$\Delta J(u) = -\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \times \int\limits_{x_1^0}^{X_1} \int\limits_{x_2^0}^{X_2} (X_1 - x_1)^{\alpha_1 - 1} (X_2 - x_2)^{\alpha_2 - 1} \Delta_u H(x_1, x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \,. \tag{11}$$

The problem (9), (10) is said to be a conjugated problem.

Lemma 1. *For any $a(\cdot) \in L^{\infty}(G, \mathbb{R}^n)$ the function*

$$l_1(x) = \frac{\Gamma(\alpha_2)\,(X_1 - x_1)^{1-\alpha_1}\,(X_2 - x_2)^{\beta_2-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \int_{x_1}^{X_1} (X_1 - \tau_1)^{\beta_1-1}\,(\tau_1 - x_1)^{\alpha_1-1}\,a(\tau_1, x_2)\,\mathrm{d}\tau_1\,, \quad x \in G \tag{12}$$

$$\left(l_2(x) = \frac{\Gamma(\alpha_1)\,(X_1 - x_1)^{\beta_1-\alpha_1}\,(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \int_{x_2}^{X_2} (X_2 - \tau_2)^{\beta_2-1}\,(\tau_2 - x_2)^{\alpha_2-1}\,a(x_1, \tau_2)\,\mathrm{d}\tau_2, \quad x \in G \right)$$

essentially bounded and continuous with respect to variable x_1 (x_2).

Proof. We first prove the essentially boundedness of the function $l_1(\cdot)$. Passing to the norm $\mathbb{R}^n$ in (12), we obtain

$$\|l_1(x)\| \leqslant \frac{\Gamma(\alpha_2)\,(X_1 - x_1)^{1-\alpha_1}\,(X_2 - x_2)^{\beta_2-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \int_{x_1}^{X_1} (X_1 - \tau_1)^{\beta_1-1}\,(\tau_1 - x_1)^{\alpha_1-1}\,\mathrm{d}\tau_1\,\|a(\cdot)\|_{L^\infty}\,, \quad x \in G.$$

Now we change the variables of the right-hand side of inequality

$$\|l_1(x)\| \leqslant \frac{\Gamma(\alpha_2)\,(X_1 - x_1)^{1-\alpha_1}\,(X_2 - x_2)^{\beta_2-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \int_0^1 z^{\alpha_1-1}(1-z)^{\beta_1-1}\,(X_1 - x_1)^{\alpha_1+\beta_1-1}\,\mathrm{d}z\,\|a(\cdot)\|_{L^\infty}$$
$$\leqslant \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\beta_2)\Gamma(\alpha_1+\beta_1)} \left(X_1 - x_1^0\right)^{\beta_1} \left(X_2 - x_2^0\right)^{\beta_2-\alpha_2} \|a(\cdot)\|_{L^\infty}\,, \quad x \in G.$$

By the scheme from [34, Lemma 3.2], we can prove the equality

$$l_1(x) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\,(X_1 - x_1)^{\beta_1-\alpha_1}\,(X_2 - x_2)^{\beta_2-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \frac{1}{\Gamma(\alpha_1)} \int_{x_1}^{X_1} (\tau_1 - x_1)^{\alpha_1-1} \left(a\,(\tau_1, x_2) + \left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1} a\,(\cdot,\, x_2)\right)(\tau_1,\, x_2) \right) \mathrm{d}\tau_1\,,$$

where

$$\left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1}a(\cdot,x_2)\right)(x_1,\ x_2)=\frac{(1-\beta_1)\sin\alpha_1\pi}{\pi}$$
$$\times\int_{x_1}^{X_1}K_{\alpha_1,\beta_1}\left(X_1-x_1,\ X_1-\tau_1\right)a(\tau_1,x_2)\,\mathrm{d}\tau_1\,,$$

$$K_{\alpha_1,\beta_1}(\xi,\eta)=\eta^{\beta_1-1}\int_0^1 z^{\alpha_1}(1-z)^{-\alpha_1}\left(\eta+z(\xi-\eta)\right)^{-\beta_1}\mathrm{d}z,\quad \xi>\eta>0.$$

The following estimate holds for $\left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1}a(\cdot,x_2)\right)(x_1,\ x_2)$:

$$\left\|\left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1}a(\cdot,x_2)\right)(x_1,\ x_2)\right\|\leqslant\frac{|1-\beta_1|\sin\alpha_1\pi}{(X_1-x_1)^{\beta_1}\pi}$$
$$\times\int_{x_1}^{X_1}(X_1-\tau_1)^{\beta_1-1}\left\|a(\tau_1,x_2)\right\|\mathrm{d}\tau_1\int_0^1 z^{\alpha_1-\beta_1}(1-z)^{-\alpha_1}\mathrm{d}z$$
$$\leqslant\frac{|1-\beta_1|\sin\alpha_1\pi}{\beta_1\pi}B\left(\alpha_1-\beta_1+1,\ 1-\alpha_1\right)\|a(\cdot)\|_{L^\infty}. \tag{13}$$

Now we prove the continuous function $l_1(\cdot)$ with respect to variable x_1. Let $x_1^1,\ x_1^2\in\left[x_1^0,\ X_1\right]$ and $x_1^1<x_1^2$. Then

$$\left\|l_1\left(x_1^1,x_2\right)-l_1\left(x_1^2,x_2\right)\right\|\leqslant\frac{\Gamma(\alpha_2)(X_2-x_2)^{\beta_2-\alpha_2}\left((X_1-x_1^1)^{\beta_1-\alpha_1}-(X_1-x_1^2)^{\beta_1-\alpha_1}\right)}{\Gamma(\beta_1)\Gamma(\beta_2)}$$
$$\times\left\|\int_{x_1^1}^{X_1}\left(\tau_1-x_1^1\right)^{\alpha_1-1}\left(a(\tau_1,x_2)+\left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1}a(\cdot,\ x_2)\right)(\tau_1,\ x_2)\right)\mathrm{d}\tau_1\right\|$$
$$+\frac{\Gamma(\alpha_2)\left(X_2-x_2\right)^{\beta_2-\alpha_2}\left(X_1-x_1^2\right)^{\beta_1-\alpha_1}}{\Gamma(\beta_1)\Gamma(\beta_2)}$$
$$\times\left\|\int_{x_1^1}^{X_1}\left(\tau_1-x_1^1\right)^{\alpha_1-1}\left(a\left(\tau_1,x_2\right)+\left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1}a\left(\cdot,\ x_2\right)\right)(\tau_1,\ x_2)\right)\mathrm{d}\tau_1\right.$$

$$- \int_{x_1^2}^{X_1} \left(\tau_1 - x_1^2\right)^{\alpha_1 - 1} \left(a(\tau_1, x_2) + \left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1} a\,(\cdot,\, x_2)\right)(\tau_1,\, x_2)\right) \mathrm{d}\tau_1 \Bigg\|$$

$$\leqslant \left(X_1 - x_1^0\right)^{\alpha_1} M_{\alpha_1,\beta_1} \|a\|_{L^\infty} \left(x_1^2 - x_1^1\right)^{\beta_1 - \alpha_1} \frac{\Gamma(\alpha_2)\left(X_2 - x_2^0\right)^{\beta_2 - \alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)}$$

$$+ \frac{\Gamma(\alpha_2)\left(X_2 - x_2\right)^{\beta_2 - \alpha_2}\left(X_1 - x_1^2\right)^{\beta_1 - \alpha_1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \left(\Bigg\| \int_{x_1^2}^{X_1} \left((\tau_1 - x_1^1)^{\alpha_1 - 1} - (\tau_1 - x_1^2)^{\alpha_1 - 1} \right) \right.$$

$$\left(a(\tau_1, x_2) + \left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1} a(\cdot, x_2)\right)(\tau_1, x_2)\right) \mathrm{d}\tau_1 \Bigg\|$$

$$\left. + \Bigg\| \int_{x_1^1}^{x_1^2} \left(\tau_1 - x_1^1\right)^{\alpha_1 - 1} \left(a\,(\tau_1, x_2) + \left(\mathbb{R}_{X_1-}^{\alpha_1,\beta_1} a(\cdot,\, x_2)\right)(\tau_1,\, x_2)\right) \mathrm{d}\tau_1 \Bigg\| \right)$$

$$\leqslant \frac{\Gamma(\alpha_2)\left(X_2 - x_2^0\right)^{\beta_2 - \alpha_2}}{\Gamma(\beta_2)} \left\{ \frac{1}{\Gamma(\beta_1)} \left(X_1 - x_1^0\right)^{\alpha_1} M_{\alpha_1,\beta_1} \|a\|_{L^\infty} \left(x_1^2 - x_1^1\right)^{\beta_1 - \alpha_1} \right.$$

$$+ (X_1 - x_1^2)^{\beta_1 - \alpha_1} \left[\left| (x_1^2 - x_1^1)^{\alpha_1} - ((X_1 - x_1^1)^{\alpha_1} - (X_1 - x_1^2)^{\alpha_1}) \right| + (x_1^2 - x_1^1)^{\alpha_1} \right]$$

$$\left. \times M_{\alpha_1,\beta_1} ||a||_{L^\infty} \right\}$$

$$\leqslant \frac{\Gamma(\alpha_2)\left(X_2 - x_2^0\right)^{\beta_2 - \alpha_2}}{\Gamma(\beta_2)} \left[\frac{1}{\Gamma(\beta_1)} (X_1 - x_1^0)^{\alpha_1} M_{\alpha_1,\beta_1} \|a\|_{L^\infty} (x_1^2 - x_1^1)^{\beta_1 - \alpha_1} \right.$$

$$\left. + 2\left(X_1 - x_1^2\right)^{\beta_1 - \alpha_1} M_{\alpha_1,\beta_1} \|a\|_{L^\infty} \left(x_1^2 - x_1^1\right)^{\alpha_1} \right] = \frac{\Gamma(\alpha_2)\left(X_2 - x_2^0\right)^{\beta_2 - \alpha_2}}{\Gamma(\beta_2)}$$

$$\times \left[\frac{1}{\Gamma(\beta_1)} \left(X_1 - x_1^0\right)^{\alpha_1} \left(x_1^2 - x_1^1\right)^{\beta_1 - \alpha_1 - \gamma} + 2\left(X_1 - x_1^2\right)^{\beta_1 - \alpha_1} \left(x_1^2 - x_1^1\right)^{\alpha_1 - \gamma} \right]$$

$$\times M_{\alpha_1,\beta_1} \|a\|_{L^\infty} \left(x_1^2 - x_1^1\right)^{\gamma} \leqslant M\left(x_1^2 - x_1^1\right)^{\gamma},$$

where $M_{\alpha_1,\beta_1} = \dfrac{1}{\alpha_1 \Gamma(\beta_1)} \left(1 + \dfrac{|1 - \beta_1| \sin \alpha_1 \pi}{\beta_1 \pi} B\,(\alpha_1 - \beta_1 + 1,\ 1 - \alpha_1)\right),$

$$\gamma = \min\left\{\beta_1 - \alpha_1,\ \alpha_1\right\},$$

$$M = \frac{3\Gamma(\alpha_2)\left(X_2 - x_2^0\right)^{\beta_2-\alpha_2}}{\Gamma(\beta_2)}(X_1 - x_1^0)^{\beta_1-\gamma} M_{\alpha_1,\beta_1} \|a\|_{L^\infty}. \qquad \square$$

Using Lemma 1 we can prove the following Lemma.

Lemma 2. *For any $\Phi_1(\cdot)$, $C_1(\cdot) \in L^\infty(G, \mathbb{R}^{n_1})$, $(\Phi_2(\cdot), C_2(\cdot) \in L^\infty(G, \mathbb{R}^{n_2}))$ and $d_2(\cdot) \in L^\infty(G_2, \mathbb{R}^{n_1})$, $(d_1(\cdot) \in L^\infty(G_1, \mathbb{R}^{n_2}))$ the function*

$$\psi_1(x) = \frac{(X_1 - x_1)^{1-\alpha_1}}{\Gamma(\alpha_1)} \int_{x_1}^{X_1} (X_1 - \tau_1)^{\alpha_1-1} (\tau_1 - x_1)^{\alpha_1-1} \Phi_1(\tau_1, x_2) \, d\tau_1$$
$$- \frac{\Gamma(\alpha_2)(X_1 - x_1)^{1-\alpha_1}(X_2 - x_2)^{\beta_2-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_1}^{X_1} (X_1 - \tau_1)^{\beta_1-1} (\tau_1 - x_1)^{\alpha_1-1} C_1(\tau_1, x_2) \, d\tau_1$$
$$- \frac{\Gamma(\alpha_2)}{\Gamma(\beta_2)}(X_2 - x_2)^{\beta_2-\alpha_2} d_2(x_2),$$
$$\Bigg(\psi_2(x) = \frac{(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\alpha_2)} \int_{x_2}^{X_2} (X_2 - \tau_2)^{\alpha_2-1} (\tau_2 - x_2)^{\alpha_2-1} \Phi_2(x_1, \tau_2) \, d\tau_2$$
$$- \frac{\Gamma(\alpha_1)(X_1 - x_1)^{\beta_1-\alpha_1}(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_2}^{X_2} (X_2 - \tau_2)^{\beta_2-1} (\tau_2 - x_2)^{\alpha_2-1} C_2(x_1, \tau_2) \, d\tau_2$$
$$- \frac{\Gamma(\alpha_1)(X_1 - x_1)^{\beta_1-\alpha_1}}{\Gamma(\beta_1)} d_1(x_1)\Bigg),$$

essentially bounded and continuous with respect to variable x_1 (x_2).

Lemma 3. *System of integral equations (9), (10) has a unique solution in the space $L^{\infty,e}(G, \mathbb{R}^{n_1+n_2})$.*

Proof. Denote by $L^{\infty,e}(G, \mathbb{R}^{n_1+n_2}) = L^{\infty,e}(G, \mathbb{R}^{n_1}) \times L^{\infty,e}(G, \mathbb{R}^{n_2})$ the space of essentially bounded functions $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$, defined on the G with the Bielecki norm

$$\begin{aligned}\|\varphi(\cdot)\|_{\infty,e} &= \|\varphi_1(\cdot)\|_{\infty,e} + \|\varphi_2(\cdot)\|_{\infty,e} \\ &= \operatorname*{esssup}_{x\in G}\left(\|\varphi_1(x)\| e^{-(X_1-x_1)k} e^{-(X_2-x_2)k}\right) \\ &\quad + \operatorname*{esssup}_{x\in G}\left(\|\varphi_2(x)\| e^{-(X_1-x_1)k} e^{-(X_2-x_2)k}\right).\end{aligned}$$

Obviously, $L^{\infty,e}\left(G,\ \mathbb{R}^{n_1+n_2}\right)$ is a Banach space. Using the right-hand sides of the system of integral equations (9), (10) we introduce the following operators

$$(K_1\psi)(x) = \frac{(X_1 - x_1)^{1-\alpha_1}}{\Gamma(\alpha_1)} \int_{x_1}^{X_1} (X_1 - \tau_1)^{\alpha_1 - 1} (\tau_1 - x_1)^{\alpha_1 - 1}$$

$$\times \left[a_1'\left(\tau_1, x_2\right)\psi_1\left(\tau_1, x_2\right) + b_1'\left(\tau_1, x_2\right)\psi_2\left(\tau_1, x_2\right)\right] \mathrm{d}\tau_1$$

$$- \frac{\Gamma(\alpha_2)(X_1 - x_1)^{1-\alpha_1}(X_2 - x_2)^{\beta_2 - \alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_1}^{X_1} (X_1 - \tau_1)^{\beta_1 - 1}(\tau_1 - x_1)^{\alpha_1 - 1} c_1(\tau_1, x_2)\,\mathrm{d}\tau_1$$

$$- \frac{\Gamma(\alpha_2)}{\Gamma(\beta_2)} (X_2 - x_2)^{\beta_2 - \alpha_2} d_2(x_2), \quad x \in G, \quad \text{a.e.}$$

$$(K_2\psi)(x) = \frac{(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\alpha_2)} \int_{x_2}^{X_2} (X_2 - \tau_2)^{\alpha_2 - 1} (\tau_2 - x_2)^{\alpha_2 - 1}$$

$$\times \left[a_2'\left(x_1, \tau_2\right)\psi_1\left(x_1, \tau_2\right) + b_2'\left(x_1, \tau_2\right)\psi_2\left(x_1, \tau_2\right)\right] \mathrm{d}\tau_2$$

$$- \frac{\Gamma(\alpha_1)(X_1 - x_1)^{\beta_1 - \alpha_1}(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_2}^{X_2} (X_2 - \tau_2)^{\beta_2 - 1}(\tau_2 - x_2)^{\alpha_2 - 1} c_2(x_1, \tau_2)\,\mathrm{d}\tau_2$$

$$- \frac{\Gamma(\alpha_1)}{\Gamma(\beta_1)} (X_1 - x_1)^{\beta_1 - \alpha_1} d_1(x_1), \quad x \in G, \quad \text{a.e.},$$

where $\psi(\cdot) = (\psi_1(\cdot),\ \psi_2(\cdot)) \in L^{\infty,e}\left(G,\ \mathbb{R}^{n_1+n_2}\right)$.

Lemma 2 implies that $(K\psi)(\cdot) = \big((K_1\psi)(\cdot),\ (K_2\psi)(\cdot)\big) \in L^{\infty,e}\left(G, \mathbb{R}^{n_1+n_2}\right)$. Hence, the operator K is correctly defined. Let us show that this operator is a contraction. Let us fix $\varphi(\cdot) = \big(\varphi_1(\cdot),\ \varphi_2(\cdot)\big)$, $\psi(\cdot) = \big(\psi_1(\cdot),\ \psi_2(\cdot)\big) \in L^{\infty,e}\left(G,\ \mathbb{R}^{n_1+n_2}\right)$. For almost all $x \in G$, we derive

$$\|(K_1\varphi)(x) - (K_1\psi)(x)\| \leqslant \frac{a(X_1 - x_1)^{1-\alpha_1}}{\Gamma(\alpha_1)} \int_{x_1}^{X_1} (X_1 - \tau_1)^{\alpha_1 - 1}(\tau_1 - x_1)^{\alpha_1 - 1}$$

$$\times \Big[\|\varphi_1(\tau_1, x_2) - \psi_1(\tau_1, x_2)\| + \|\varphi_2(\tau_1, x_2) - \psi_2(\tau_1, x_2)\|\Big] \mathrm{d}\tau_1,$$

$$\|(K_2\varphi)(x) - (K_2\psi)(x)\| \leqslant \frac{a(X_2 - x_2)^{1-\alpha_2}}{\Gamma(\alpha_2)} \int_{x_2}^{X_2} (X_2 - \tau_2)^{\alpha_2 - 1}(\tau_2 - x_2)^{\alpha_2-1}$$
$$\times \left[\|\varphi_1(x_1,\tau_2) - \psi_1(x_1,\tau_2)\| - \|\varphi_2(x_1,\tau_2) - \psi_2(x_1,\tau_2)\|\right] \mathrm{d}\tau_2\,,$$

where $a = \max\{a_1,\ a_2,\ b_1,\ b_2\}$.

By the scheme from [34, Lemma 3.2] we can prove the inequality

$$\|(K_1\varphi)(x) - (K_1\psi)(x)\|$$
$$\leqslant \frac{a}{\Gamma(\alpha_1)} \int_{x_1}^{X_1} (\tau_1 - x_1)^{\alpha_1-1} \left(\|\varphi_1(\tau_1,\ x_2) - \psi_1(\tau_1,\ x_2)\|\right.$$
$$+ \|\varphi_2(\tau_1,x_2) - \psi_2(\tau_1,x_2)\|) + \Big(\mathbb{R}^{\alpha_1}_{X_1-}(\|\varphi_1(\cdot,x_2) - \psi_1(\cdot,x_2)\|$$
$$+ \|\varphi_2(\cdot,x_2) - \psi_2(\cdot,x_2)\|)\Big)(\tau_1,\ x_2)\,\mathrm{d}\tau_1,\ x \in G, a.e.$$

Using estimates (13) we have

$$\|(K_1\varphi)(x) - (K_1\psi)(x)\| \leqslant \frac{aM_{\alpha_1}}{\Gamma(\alpha_1)} \int_{x_1}^{X_1} (\tau_1 - x_1)^{\alpha_1-1}\big(\|\varphi_1(\cdot,x_2) - \psi_1(\cdot,x_2)\|_{[\tau_1,X_1]}$$
$$+\ \|\varphi_2(\cdot,x_2) - \psi_2(\cdot,x_2)\|_{[\tau_1,X_1]}\big)\,\mathrm{d}\tau_1$$
$$\leqslant \frac{aM_{\alpha_1}}{\Gamma(\alpha_1)}\big(\|\varphi_1(\cdot) - \psi_1(\cdot)\|_{\infty,e} + \|\varphi_2(\cdot) - \psi_2(\cdot)\|_{\infty,e}\big)$$
$$\times \int_{x_1}^{X_1} (\tau_1 - x_1)^{\alpha_1-1} e^{(X_1-\tau_1)k}\,\mathrm{d}\tau_1 e^{(X_2-x_2)k}$$
$$\leqslant \frac{aM_{\alpha_1}}{k^{\alpha_1}} e^{(X_1-x_1)k} e^{(X_2-x_2)k}\left(\|\varphi_1(\cdot) - \psi_1(\cdot)\|_{\infty,e}\right.$$
$$\left. +\ \|\varphi_2(\cdot) - \psi_2(\cdot)\|_{\infty,e}\right), \quad x \in G,\ a.e.,$$

where $M_{\alpha_1} = 1 + \dfrac{\sin\alpha_1\pi}{\alpha_1\pi}$.

Similarly, we obtain the following inequality

$$\|(K_2\varphi)(x) - (K_2\psi)(x)\|$$
$$\leqslant \frac{aM_{\alpha_2}}{k^{\alpha_2}} e^{(X_1-x_1)k} e^{(X_2-x_2)k}\left(\|\varphi_1(\cdot) - \psi_1(\cdot)\|_{\infty,e} + \|\varphi_2(\cdot) - \psi_2(\cdot)\|_{\infty,e}\right),$$

where $M_{\alpha_2} = 1 + \dfrac{\sin\alpha_2\pi}{\alpha_2\pi}$.

We write the last two inequalities in the form

$$\|(K_1\varphi)(\cdot)-(K_1\psi)(\cdot)\|_{\infty,e} \leqslant \frac{aM_{\alpha_1}}{k^{\alpha_1}}\left(\|\varphi_1(\cdot)-\psi_1(\cdot)\|_{\infty,e}+\|\varphi_2(\cdot)-\psi_2(\cdot)\|_{\infty,e}\right),$$
$$\|(K_2\varphi)(\cdot)-(K_2\psi)(\cdot)\|_{\infty,e} \leqslant \frac{aM_{\alpha_2}}{k^{\alpha_2}}\left(\|\varphi_1(\cdot)-\psi_1(\cdot)\|_{\infty,e}+\|\varphi_2(\cdot)-\psi_2(\cdot)\|_{\infty,e}\right).$$

Summing these inequalities, we have

$$\|(K\varphi)(\cdot)-(K\psi)(\cdot)\|_{\infty,e} \leqslant \left(\frac{aM_{\alpha_1}}{k^{\alpha_1}}+\frac{aM_{\alpha_2}}{k^{\alpha_2}}\right)\|\varphi(\cdot)-\psi(\cdot)\|_{\infty,e}\ .$$

Thus due to the choice of the number k, the operator K is a contraction. By the Banach contraction principle, this operator has a unique fixed point, which is a unique solution $\psi(\cdot)=(\psi_1(\cdot),\,\psi_2(\cdot))\in L^{\infty,e}\left(G,\,\mathbb{R}^{n_1+n_2}\right)$ of the system (9), (10).

Now, for a fixed $(\theta_1,\ \theta_2)\in G$ we consider the following needle variation of the admissible control $u(\cdot)$:

$$\Delta_\varepsilon u(x)=\begin{cases} \upsilon-u(x), & x\in G_\varepsilon,\\ 0, & x\in G\backslash G_\varepsilon, \end{cases},$$

where $\upsilon\in V$ is an arbitrary fixed point, $\varepsilon>0$ is a sufficiently small parameter and $G_\varepsilon=(\theta_1,\,\theta_1+\varepsilon)\times(\theta_2,\,\theta_2+\varepsilon)\subset G$. The control $u_\varepsilon(\cdot)$ defined by the equality $u_\varepsilon(\cdot)=u(\cdot)+\Delta_\varepsilon u(\cdot)$ is an admissible control for all sufficiently small $\varepsilon>0$ and all $\upsilon\in V$ called a needle perturbation given by the control $u(\cdot)$. Obviously,

$$\Delta J(u)=J\left(u_\varepsilon\right)-J(u)=-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int\limits_{\theta_1}^{\theta_1+\varepsilon}\int\limits_{\theta_2}^{\theta_2+\varepsilon}(X_1-x_1)^{\alpha_1-1}(X_2-x_2)^{\alpha_2-1}$$
$$\times\left[H\left(x_1,x_2,\ \psi\left(x_1,x_2\right),\upsilon\right)-H\left(x_1,x_2,\ \psi\left(x_1,x_2\right),u\left(x_1,x_2\right)\right)\right]\mathrm{d}x_2\mathrm{d}x_1. \quad (14)$$

Then the following theorem is true. □

Theorem 2. *Let $u(\cdot)$ be a fixed admissible control, and $z(\cdot)=(z_1(\cdot),\ z_2(\cdot))$ and $\psi(\cdot)=\left(\psi^{(1)}(\cdot),\ \psi^{(2)}(\cdot)\right)$ the solutions of problems (1), (2) and (9), (10) corresponding to this control, respectively. Then for the optimality of the admissible control $u(\cdot)$ in problem (1)–(4) it is necessary and sufficient that for almost all $x\in G$ the maximum condition is satisfied:*

$$\max_{\upsilon\in V}H(x,\,\psi(x),\,\upsilon)=H(x,\,\psi(x),\,u(x)). \quad (15)$$

Proof. Suppose that admissible control $u(\cdot)$ gives the minimum value of the functional (4). Then by (14), we have

$$-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int\limits_{\theta_1}^{\theta_1+\varepsilon}\int\limits_{\theta_2}^{\theta_2+\varepsilon}(X_1-x_1)^{\alpha_1-1}(X_2-x_2)^{\alpha_2-1}\Big(H\left(x_1,x_2,\,\psi(x_1,x_2),\,\upsilon\right)$$
$$-\;H\left(x_1,x_2,\,\psi(x_1,x_2),\,u(x_1,x_2)\right)\Big)\mathrm{d}x_2\mathrm{d}x_1 \geqslant 0. \tag{16}$$

Dividing both sides of (16) by ε^2 and passing to the limit as $\varepsilon \to 0$ we get

$$H\left(\theta_1,\theta_2,\,\psi\left(\theta_1,\theta_2\right),\,\upsilon\right)-H\left(\theta_1,\theta_2,\,\psi\left(\theta_1,\theta_2\right),\,u\left(\theta_1,\theta_2\right)\right)\leqslant 0. \tag{17}$$

Thus, for optimal control $u(\cdot)$, it is necessary to satisfy the condition (15). The equality (14) shows that this condition is sufficient for the optimality of the control $u(\cdot)$.

This completes the proof. □

Example

Let us consider the problem

$$\begin{aligned}
&D_1^{\frac{1}{2}}z_1^{(1)}=z_1^{(2)}-u, && z_1^{(1)}(0,\ x_2)=0, && (x_1,\,x_2)\in[0,\ 1]\times[0,\ 1],\\
&D_1^{\frac{1}{2}}z_1^{(2)}=u, && z_1^{(2)}(0,\ x_2)=0, && (x_1,\,x_2)\in[0,\ 1]\times[0,\ 1],\\
&D_1^{\alpha_2}z_2=u, && z_2(x_1,\ 0)=0, && V=[-1;\ 0]\cup\left[\frac{1}{2};\ 1\right],
\end{aligned}$$

$$J(u)=\int\limits_0^1\int\limits_0^1 u(x)\mathrm{d}x+2\int\limits_0^1 z_1^{(1)}(1,\ x_2)\mathrm{d}x_2\to\min.$$

For this problem, the Hamiltonian is

$$H=\psi_1^{(1)}\left(z_1^{(2)}-u\right)+\psi_1^{(2)}u+\psi_2 u-\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha_2)\sqrt{1-x_1}\,(1-x_2)^{1-\alpha_2}\,u,$$

and the solution of the adjoint problem has the form

$$\begin{aligned}
&\psi_1^{(1)}(x)=-2\Gamma\left(\alpha_2\right)\left(1-x_2\right)^{1-\alpha_2},\\
&\psi_1^{(2)}(x)=-2\sqrt{\pi}\Gamma\left(\alpha_2\right)\sqrt{1-x_1}(1-x_2)^{1-\alpha_2},\quad \psi_2(x)=0.
\end{aligned}$$

According to the principle of maximum

$$\max_{\upsilon\in V}\Big[\Gamma(\alpha_2)(1-x_2)^{1-\alpha_2}\left(2-3\sqrt{\pi}\sqrt{1-x_1}\right)\upsilon - 2\Gamma(\alpha_2)(1-x_2)^{1-\alpha_2} z_1^{(2)}(x)\Big]$$
$$=\Gamma(\alpha_2)(1-x_2)^{1-\alpha_2}\left(2-3\sqrt{\pi}\sqrt{1-x_1}\right)u(x)-2\Gamma(\alpha_2)(1-x_2)^{1-\alpha_2} z_1^{(2)}(x).$$

Hence it follows that the control that satisfies the Pontryagin's maximum principle has the form

$$u(x_1,x_2)=\begin{cases}-1, & (x_1,\ x_2)\in\left[0,\ 1-\frac{4}{9\pi}\right]\times[0,\ 1],\\ 1, & (x_1,\ x_2)\in\left[1-\frac{4}{9\pi},\ 1\right]\times[0,\ 1].\end{cases} \tag{18}$$

Then, by Theorem 2, the function $u(\cdot)$ defined by equality (18) is an optimal control, and the corresponding optimal trajectory has the form

$$z_1^{(1)}(x)=\begin{cases}-x_1+\dfrac{\sqrt{x_1}}{\Gamma\left(\frac{3}{2}\right)}, & (x_1,\ x_2)\in\left[0,\ 1-\frac{4}{9\pi}\right]\times[0,\ 1],\\ \dfrac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\sqrt{x_1}-2\sqrt{x_1-1+\frac{4}{9\pi}}\right)+2\left(x_1-1+\frac{4}{9\pi}\right)-\dfrac{3\sqrt{\pi}}{2\Gamma\left(\frac{3}{2}\right)}x_1, & \\ \qquad (x_1,\ x_2)\in\left[1-\frac{4}{9\pi},\ 1\right]\times[0,\ 1],\end{cases}$$

$$z_1^{(2)}(x)=\begin{cases}-\dfrac{\sqrt{x_1}}{\Gamma\left(\frac{3}{2}\right)}, & (x_1,\ x_2)\in\left[0,\ 1-\frac{4}{9\pi}\right]\times[0,\ 1],\\ \dfrac{1}{\Gamma\left(\frac{3}{2}\right)}\left[2\sqrt{x_1-1+\frac{4}{9\pi}}-\sqrt{x_1}\right], & (x_1,\ x_2)\in\left[1-\frac{4}{9\pi},\ 1\right]\times[0,\ 1],\end{cases}$$

$$z_2(x)=\begin{cases}-\dfrac{x_2^{\alpha_2}}{\Gamma(\alpha_2+1)}, & (x_1,\ x_2)\in\left[0,\ 1-\frac{4}{9\pi}\right]\times[0,\ 1],\\ \dfrac{x_2^{\alpha_2}}{\Gamma(\alpha_2+1)}, & (x_1,\ x_2)\in\left[1-\frac{4}{9\pi},\ 1\right]\times[0,\ 1],\end{cases}$$

and the minimum value of the functional

$$J(u)=\frac{4}{\sqrt{\pi}}-\frac{8}{3\pi}-7.$$

6. Conclusion

In this paper we consider the optimal control problem associated with the fractional Roesser model described by Caputo partial derivatives. Using the Banach contraction principle, we prove the existence and uniqueness of a solution to the corresponding boundary problem for a fixed admissible control. The formulated problem of optimal control is studied using a new version of the incremental method, in which the concept of an adjoint integral equation is essential. In turn, the adjoint equation is the sum of partial operators of fractional integration with weight. Using commutation of fractional integrals with power functions, one can go directly from weighted fractional integrals to nonweighted fractional integrals. We prove the existence and uniqueness of a solution to the adjoint problem. Then a necessary and sufficient optimality condition is derived in the form of the Pontryagin's maximum principle.The approach presented here can be applied to the derivation of various necessary optimality conditions for an optimal control problem in which the system is controlled by a nonlinear fractional Roesser model. It is commendable that the described technique can be useful from the point of view of calculating the gradient of the functional. In this case, the functional gradient formula can be applied to find an approximate solution to the problem posed using the gradient method.

References

[1] S. Abbas, M. Benchohra and A.N. Vityuk: On fractional order derivatives and Darboux problem for implicit differential equations. *Fractional Calculus and Applied Analysis*, **15**(2), (2012), 168–182. DOI: 10.2478/s13540-012-0012-5

[2] O.P. Agrawal: A general formulation and solution scheme for fractional optimal control problems. *Nonlinear Dynamics*, **38**(1), (2004), 323–337. DOI: 10.1007/s11071-004-3764-6

[3] I. Basdouri, S. Kasmi and J. Lerbet: Practical Mittag-Leffler stability of quasi-one-sided Lipschitz fractional order systems. *Archives of Control Sciences*, **33**(1), (2023), 55–70. DOI: DOI: 10.24425/acs.2023.145113

[4] K. Benyettou and D. Bouagada: Admissibility tests for multidimensional singular fractional continuous-time models. *Archives of Control Sciences*, **32**(3), (2022), 607–625. DOI: 10.24425/acs.2022.142851

[5] V. Capasso, S. Anita and V. Arnautu: *An Introduction to Optimal Control Problems in Life Sciences and Economics*. Birkhäuser, Boston 2010.

[6] M. Bergounioux and L. Bourdin: Pontryagin maximum principle for general Caputo fractional optimal control problems with Bolza cost and terminal constaints. *ESAIM: Control, Optimisation and Calculus of Variations*, **26** (2020), 1–38. DOI: 10.1051/cocv/2019021

[7] A.H. Bhrawy and M.A. Zaky: Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation. *Nonlinear Dynamics*, **80**(1-2), (2015), 101–116. DOI: 10.1007/s11071-014-1854-7

[8] K. Diethelm: *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition using Differential Operators of Caputo Type*. Lecture Notes in Matematics, 2004, Spinger-Verlag, Berlin 2010.

[9] S. Dipierro, E. Palatucci and G. Valdinoci: Dislocation dynamics in crystals: A macroscopic theory in a fractional Laplace setting. *Communications in Mathematical Physics*, **333**(2), (2015), 1061–1105. DOI: 10.1007/s00220-014-2118-6

[10] L. Fang, Ti-Jun Xiao and H. Xu: On nonlinear neutral fractional integrodifferential inclusions with infinite delay. *Journal of Applied Mathematics*, **2012** (2012), 1–19. DOI: 10.1155/2012/916543

[11] R. Gabasov and F.M. Kirillova: High-order necessary conditions for optimality. *SIAM Journal on Control*, **10**(1), (1972), 127–168. DOI: 10.1137/0310012

[12] W.M. Haddad and A. L'Afflitto: Finite-time partial stability theory and fractional Lyapunov differential inequalities. *2015 American Control Conference (ACC)*, Chicago, IL, USA, (2025), 5347–5352. DOI: 10.1109/ACC.2015.7172175

[13] D. Idczak: Bang-bang principle for linear and non-linear Goursat-Darboux problem. *International Journal of Control*, **76**(11), (2003), 1089–1094. DOI: 10.1080/002071703100 0123283

[14] D. Idczak, R. Kamocki, M. Majewski and S. Walczak: Existence of optimal solutions to Lagrange problems for Roesser type systems of the first and fractional orders. *Applied Mathematics and Computation*, **66**(C) (2015), 809–819. DOI: 10.1016/j.amc.2015.05.142

[15] J. Liang, Y. Mu and X. Ti-Jun: Nonlocal Cauchy problems for fractional differential equations involving ψ-Hilfer multivariable operators. *Fractional Calculus and Applied Analysis*, **23**(4), (2020), 1090–1124. DOI: 10.1515/fca-2020-0056

[16] R. Kamocki: Pontryagin maximum principle for fractional ordinary optimal control problems. *Mathematical Methods in the Applied Sciences*, **37**(11), (2014), 1668–1686. DOI: 10.1002/mma.2928

[17] R. Kamocki and M. Majewski: Fractional linear control systems with Caputo derivative and their optimization. *Optimal Control Applications and Methods*, **36**(6), (2014), 953-967. DOI: 10.1002/oca.2150

[18] R. Kamocki: Necessary and sufficient optimality conditions for fractional nonhomogeneous Roesser model. *Optimal Control Applications and Methods*, **37**(4), (2015), 1–16. DOI: 10.1002/oca.2180

[19] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo: *Theory and Applications of Fractional Differential Equations*. Volume 204 of North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam 2006.

[20] J. Klamka: Local controllability of fractional discrete-time semilinear systems. *Acta Mechanica et Automatica*, **5**(2), (2012), 55–58. DOI: 10.2316/P.2012.769-028

[21] P. Kulczycki, J. Korbicz and J. Kacprzyk: *Fractional Dynamical Systems: Methods, Algorithms and Applications*. Springer, 2022.

[22] I. Lasiecka and R. Triggiani: Domains of Fractional Powers of Matrix-valued Operators: A General Approach. In: Arendt, W., Chill, R., Tomilov, Y. (eds.) Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics. *Operator Theory: Advances and Applications*, **250** Birkhäuser, Cham. DOI: 10.1007/978-3-319-18494-4-20

[23] M.S. Mahmoud and Yuanqing Xia: *Applied Control Systems Design*. Springer, 2012.

[24] E.N. Mahmudov: *Approximation and Optimization of Discrete and Differential Inclusions*. Elsevier, Boston, USA 2011.

[25] E.N. Mahmudov and Sh.Sh. Yusubov: Nonlocal boundary value problems for hyperbolic equations with a Caputo fractional derivative. *Journal of Computational and Applied Mathematics*, **398**(2), (2021), 1–15. DOI: 10.1016/j.cam.2021.113709

[26] E.N. Mahmudov: Optimization of higher-order differential inclusions with special boundary value conditions. *Journal of Optimization Theory and Applications*, **192**(3), (2022), 36–55. DOI: 10.1007/s10957-021-01936-6

[27] E.N. Mahmudov and D.I. Mastaliyeva: Optimal control of higher-order retarded differential inclusions. *Journal of Industrial and Management Optimization*, **19**(9), (2023), 6544–6557. DOI: 10.3934/jimo.2022226

[28] R. Metzler and J. Klafter: The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Physics Reports*, **339**(1), (2000), 1–77. DOI: 10.1016/S0370-1573(00)00070-3

[29] M.A. Nazra, L.Y. Zulakmal and R. Revina: On LQ optimization problem subject to fractional order irregular singular systems. *Archives of Control Sciences*, **30**(4), (2020), 745–756. DOI: 110.24425/acs.2020.135850

[30] V. Obukhovskii and J.C. Yao: Some existence results for fractional functional differential equations. *Fixed Point Theory*, **11**(1), (2010), 85–96.

[31] I. Podlubny: *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. **198** of Mathematics in Science and Engineering, Academic Press, San Diego, Ca 1999.

[32] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mishenko: *Mathematical Theory of Optimal Processes*, Nauka, Moscow 1983; Gordon and Breach, New York 1986.

[33] A.B. Salati, M. Shamsi and D.F.M. Torres: Direct transcription methods based on fractional integral approximation formulas for solving nonlinear fractional optimal control problems. *Communications in Nonlinear Science and Numerical Simulation*, **67** (2019), 334–350. DOI: 10.48550/arXiv.1805.06537

[34] S.G. Samko, A.A. Kilbas and O.I. Marichev: *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishers, Yverdon, Switzerland 1993.

[35] B. Sikora and N. Matlok: On controllability of fractional positive continuous-time linear systems with delay. *Archives of Control Sciences*, **31**(1), (2021), 29–51. DOI: 110.24425/acs.2021.136879

[36] H.T. Tuan, A. Czornik, J.J. Nieto and M. Niezabitowski: Global attractivity for some classes of Riemann-Liouville fractional differential systems. *Journal of Integral Equations and Applications*, **31**(2), (2017), 265–282. DOI: 10.1216/JIE-2019-31-2-265

[37] C. Tunç, O. Tunç and J.C. Yao: On the new qualitative results in integro-differential equations with Caputo fractional derivative and multiple kernels and delays. *Journal of Nonlinear and Convex Analysis* **23**(11), (2022), 2577–2591.

[38] Sh.Sh. Yusubov: Necessary conditions of optimality for systems with impulse control. *Computational Mathematics and Mathematical Physics*, **45** (2005), 222–226.

[39] Sh.Sh. Yusubov: Necessary optimality conditions for singular controls. *Computational Mathematics and Mathematical Physics*, **47** (2007), 1446–1451.

[40] Sh.Sh. Yusubov: Necessary optimality conditions for singular control in one system with distributed parameters. *Journal of Computer and Systems Sciences International*, **47**(1), (2008), 8–13. DOI: 10.1134/S1064230708010024

[41] Sh.Sh. Yusubov: On an optimality of the singular with respect to components controls in the Goursat-Dourboux systems. *Problemy Upravleniya*, **5** (2014), 2–6, (in Russian).

[42] Sh.Sh. Yusubov: Necessary conditions of optimality for the quasi-singular relative to the component controls in the Goursat-Darboux systems. *TWMS Journal of Pure and Applied Mathematics*, **6**(2), (2015), 246–253.

[43] Sh.Sh. Yusubov: Boundary value problems for hyperbolic equations with a Caputo fractional derivative. *Advanced Mathematical Models and Applications*, **5** (2020), 192–204.

[44] Sh.Sh. Yusubov and E.N. Mahmudov: Optimality conditions of singular controls for systems with Caputo fractional derivatives. *Journal of Industrial and Management Optimization*, **19**(1), (2022), 246–264. DOI: 10.3934/jimo.2021182

[45] Sh.Sh. Yusubov and E.N. Mahmudov: Necessary and sufficient optimality conditions for fractional Fornasini-Marchesini model. *Journal of Industrial and Management Optimization*, **19**(10), (2023), 7221–7244. DOI: 10.3934/jimo.2022260

[46] Sh.Sh. Yusubov and E.N. Mahmudov: Some necessary optimality conditions for systems with fractional Caputo derivatives. *Journal of Industrial and Management Optimization*, **19**(12), (2023), 8831–8850. DOI: 10.3934/jimo.2023063

[47] Sh.Sh. Yusubov and E.N. Mahmudov: Necessary optimality conditions for quasi-singular controls for systems with Caputo fractional derivatives. *Archives of Control Sciences*, **33**(3), (2023), 463–496. DOI: 10.24425/acs.2023.146955