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Robust controlled vector variational inequalities for multi-dimensional fractional control optimization problems

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This paper is devoted to study robust efficiency in terms of variational inequality for a class of multi-dimensional multi-objective first-order PDE-constrained fractional control optimization problems with data uncertainty (MMFP). We derive a robust controlled vector variational inequality (VI) together with its weak form and discuss equivalence between the solutions of (VI) and (MMFP) via imposing the suitable assumptions. Later on, we study a sufficient condition for the robust weak efficient solution of (MMFP) to be its robust efficient solution under the strict convexity assumption and give some applications to illustrate the established results.

Key words: fractional control optimization problem, convexity, robust efficient solution, uncertainty, variational inequality

1. Introduction

In order to model and address numerous optimization problems including data uncertainty, robust optimization is essential. In this method, one looks for a solution immune to all potential data uncertainties that might exist inside a particular uncertainty set. Therefore, robust optimization provides the optimum solution among the robustly viable options where the data are uncertain but restricted. Firstly, Ben-Tal and Nemirovski [4] presented the concept of an uncertain

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optimization problem and then solved it using a robust convex approach. Since then, robust optimization has gained much attention, see for instance, [3,7,15,16]. In 2015, Kim and Kim [12] investigated robust optimization problems with fractional objective function and gave optimality and duality results utilizing the parametric approach. Later on, Antczak [2] proposed the parametric robust necessary optimality conditions for an uncertain fractional programming problem and proved its sufficiency under the convexity assumption over involved functions. Recently, Jayswal and Baranwal [10] have modeled a multi-dimensional fractional control problem involving data uncertainty and established optimality and duality results using the parametric robust approach.

On the other hand, it has been observed that variational inequalities are very useful instruments for solving the optimization problems. In 1980, Giannessi [6] first introduced the concept of a vector variational inequality and proved the existence of its solutions. After that, this area gained vast attention, and many authors have efficiently investigated the association between variational inequalities and optimization problems (see [5, 13, 19] and their references). Recently, an intense study has been done to establish a specific alliance between the solution of the variational inequalities and various multi-dimensional optimization problems. Jayswal *et al.* [9] derived the relationships between the solutions of vector optimization problems and vector variational inequalities under the assumption of generalized convexity. In [18], Treanţă investigated the correlations between generalized vector variational inequalities and a class of robust multi-objective variational control problem. For more insights in this area one can go through [11, 14, 20] and references therein.

Motivated by the works mentioned earlier, this paper aims to investigate the multi-dimensional multi-objective fractional control optimization problems with data uncertainty (MMFP) and provide its efficiency via establishing connections with a new class of vector variational inequality which is introduced utilizing the robust approach. The management of this paper is as follows: Section 2 contains some basic concepts, construction of the problem (MMFP) and a non-fractional problem (NMMFP) associated with it and their robust counterparts (RMMFP) and (RNFP). Also, for the further solution, we have constructed the robust controlled vector variational inequality (VI) with its weak form (WVI). Section 3 presents the equivalence relations between robust (weak, proper) efficient solutions to the problem (MMFP) and solutions to the variational inequalities (VI) and (WVI). A characterization is also given for a weak efficient solution of (MMFP) becomes its efficient solution. Section 4 brings the paper to conclusions.

In this way, to the best of our knowledge, the presented associations between the considered problem and introduced variational inequality based on the two methods (the parametric and robust approach) is used for the first time in the literature for solving such a large class of multi-dimensional multi-objective fractional control optimization problems in which functionals involved data uncertainty.

2. Notations and preliminaries

The following notations and basic concepts are used throughout this paper.

- R^a, R^b and R^c are three Euclidean spaces of dimensions a, b and c, respectively.
- Let $\Omega = \Omega_{t_0,t_1} \subset R^b$ is a hyperparallelepiped, fixed by the diagonally opposite points $t_0 = (t_0^\alpha)$, $t_1 = (t_1^\alpha)$, $\alpha = \overline{1,b}$ and the point $t = (t^\alpha) \in \Omega$, $\alpha = \overline{1,b}$.
- $dt = dt^1 \wedge \cdots \wedge dt^b$ denote the volume element in $R^b \supset \Omega$.
- $u : \Omega \mapsto R^a$ be the piecewise smooth state function and its component $u(t) = (u^i(t)) \in R^a$, $i = \overline{1,a}$ and let U be the collection of such state functions. Also, $\frac{\partial u(t)}{\partial t^\alpha} = u_t$ denote the matrix of order $a \times b$ of partial derivative of $u(t)$ with respect to t^α, $\alpha = \overline{1,b}$.
- $v : \Omega \mapsto R^c$ be the continuous control functions and its component $v(t) = (v^j(t)) \in R^c$, $j = \overline{1,c}$ and let V be the collection of such control functions.
- T denotes the transpose of a vector.
- For any two points, $x = (x^l)$ and $y = (y^l)$, $l = \overline{1,p}$ in R^p, the following convention will be used
  $$x = y \Leftrightarrow x^l = y^l, \quad x \leqslant y \Leftrightarrow x^l \leqslant y^l, \quad x < y \Leftrightarrow x^l < y^l, \ l = \overline{1,p}.$$

Considering the above mathematical tools, we formulate the following multi-dimensional multi-objective first-order PDE constrained fractional control optimization problem with data uncertainty as:

$$\text{(MMFP)} \qquad \min_{(u(\cdot),v(\cdot))} \left\{ \frac{\int_\Omega \phi^l(t,u(t),v(t),m^l)\,\mathrm{d}t}{\int_\Omega \psi^l(t,u(t),v(t),n^l)\,\mathrm{d}t}, \ l = \overline{1,p}, \right\}$$

subject to

$$Y(t,u(t),v(t)) \leqslant 0, \tag{1}$$

$$u_t = H(t,u(t),v(t)), \tag{2}$$

$$u(t_0) = u_0, \ u(t_1) = u_1, \tag{3}$$

where $\phi^l : \Omega \times R^a \times R^c \times \mathcal{M}^l \mapsto R, \psi^l : \Omega \times R^a \times R^c \times \mathcal{N}^l \mapsto R \setminus \{0\}, l = \overline{1,p}$, $Y = (Y_\beta) : \Omega \times R^a \times R^c \mapsto R^q, \beta = \overline{1,q}, H = (H_\alpha^i) : \Omega \times R^a \times R^c \mapsto R^{ab}, i = \overline{1,a}$,

$\alpha = \overline{1,b}$, are continuously differentiable functionals, m^l and n^l are uncertain parameters for some convex compact subsets $\mathcal{M}^l$, $\mathcal{N}^l \subset R$, $l = \overline{1,p}$, respectively. The first-order PDE constraints H_α satisfy the complete integrability condition (closeness condition) $D_\gamma H_\alpha = D_\alpha H_\gamma$, $\gamma, \alpha = \overline{1,m}$, $\gamma \neq \alpha$, where D_α is the total derivative. Without loosing the generality of our considerations, we shall assume that $\phi(\cdot,p) \geqslant 0$ for every $p \in P$ and $\psi(\cdot,q) > 0$ for every $q \in Q$, throughout the article.

Then the robust counterpart of (MMFP) is given as:

$$\text{(RMMFP)} \quad \min_{(u(\cdot),v(\cdot))} \left\{ \frac{\int\limits_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(t,u(t),v(t),m^l)\,\mathrm{d}t}{\int\limits_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(t,u(t),v(t),n^l)\,\mathrm{d}t}, \quad l = \overline{1,p} \right\}$$

subject to constraints (1)–(3),

where $\phi^l, \psi^l, Y = (Y_\beta)$ and $H = (H^i_\alpha)$ are same as described for (MMFP).

The set of all feasible solutions to (RMMFP) (which is also the set of robust feasible solution to (MMFP)) is denoted by

$$\mathcal{D} = \{(u,v) \in U \times V \mid Y(t,u(t),v(t)) \leqslant 0,\ u_t = H(t,u(t),v(t)),\ u(t_0) = u_0, \\ u(t_1) = u_1,\ t \in \Omega\}.$$

Now, on the line of Jayswal [10], we construct a non-fractional problem for (MMFP) as:

$$\text{(NMMFP)} \quad \min_{(u(\cdot),v(\cdot))} \left\{ \int\limits_\Omega \phi^l(t,u(t),v(t),m^l)\,\mathrm{d}t \right. \\ \left. -P^l(t,u(t),v(t),m^l,n^l) \int\limits_\Omega \psi^l(t,u(t),v(t),n^l)\,\mathrm{d}t, \quad l = \overline{1,p} \right\},$$

subject to constraints (1)–(3),

where parameter $P := P^l(t,u(t),v(t),m^l,n^l) \in R_+$, $l = \overline{1,p}$.

The robust counterpart of (NMMFP) is given by

$$\text{(RNFP)} \quad \min_{(u(\cdot),v(\cdot))} \left\{ \int\limits_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(t,u(t),v(t),m^l)\,\mathrm{d}t \right. \\ \left. -P^l(t,u(t),v(t),m^l,n^l) \int\limits_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(t,u(t),v(t),n^l)\,\mathrm{d}t, \quad l = \overline{1,p} \right\},$$

subject to constraints (1)–(3).

We can clearly observe that $\mathcal{D}$ is the set of robust feasible solutions to (NMMFP) (feasible solutions to (RNFP)).

From now onwards, for simplicity of presentation, we use the following notions throughout the article given as $u = u(t)$, $v = v(t)$, $\overline{u} = \overline{u}(t)$, $\overline{v} = \overline{v}(t)$, $\pi = (t, u(t), v(t))$ and $f_u = \dfrac{\partial f}{\partial u}$ denotes the partial derivative of functional f with respect to u.

Definition 1. *A point $(\overline{u}, \overline{v}) \in \mathcal{D}$ is said to be a robust (weak) efficient solution to (MMFP), if it is a (weak) efficient solution to (RMMFP), that is, if there exists no other $(u, v) \in \mathcal{D}$, such that*

$$\frac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l)\,\mathrm{d}t}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l)\,\mathrm{d}t} \leqslant (<) \frac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\,\mathrm{d}t}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\,\mathrm{d}t}, \quad \forall\, l = \overline{1, p}.$$

Definition 2. *A point $(\overline{u}, \overline{v}) \in \mathcal{D}$ is said to be a robust (weak) efficient solution to (NMMFP), if it is (weak) efficient solution to (RNFP), that is, if there exists no other $(u, v) \in \mathcal{D}$, such that*

$$\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l)\,\mathrm{d}t - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l)\,\mathrm{d}t$$
$$\leqslant (<) \int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\,\mathrm{d}t - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\,\mathrm{d}t, \quad \forall\, l = \overline{1, p}.$$

In the following lemma, we establish a relationship between the robust (weak) efficient solutions to (MMFP) and (NMMFP), by following the analogous steps of Antczak [2] and Jayswal and Baranwal [10].

Lemma 1. *Let $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust weak efficient solution to (MMFP) then there exist $P^l(\overline{\pi}, m^l, n^l) \in R_+$, $l = \overline{1, p}$ such that $(\overline{u}, \overline{v})$ is a robust weak efficient solution to (NMMFP). Conversely, If $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust weak efficient solution to (NMMFP), with $P^l(\overline{\pi}, m^l, n^l) = \dfrac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\,\mathrm{d}t}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\,\mathrm{d}t}$, $l = \overline{1, p}$, then $(\overline{u}, \overline{v})$ is a robust weak efficient solution to (MMFP).*

Proof. Let $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust weak efficient solution to (MMFP), but not a robust weak efficient solution to (NMMFP), then there exist $(u, v) \in \mathcal{D}$, such that

$$\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\pi, m^l)\mathrm{d}\omega - P^l(\overline{\pi}, m^l, n^l)\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\pi, n^l)\mathrm{d}\omega$$
$$< \int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}\omega - P^l(\overline{\pi}, m^l, n^l)\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}\omega,\ \forall\, l = \overline{1,p}.$$

In particular, if we take $P^l(\overline{\pi}, m^l, n^l) = \dfrac{\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}\omega}{\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}\omega}$, then

$$\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\pi, m^l)\mathrm{d}\omega - \frac{\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}\omega}{\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}\omega}\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\pi, n^l)\mathrm{d}\omega$$
$$< \int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}\omega$$
$$- \frac{\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}\omega}{\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}\omega}\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}\omega,\ \forall\, l = \overline{1,p}.$$

Further, the above inequality yields

$$\frac{\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\pi, m^l)d\omega}{\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\pi, n^l)\mathrm{d}\omega} < \frac{\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}\omega}{\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}\omega},\ \forall\, l = \overline{1,p},$$

which contradicts $(\overline{u}, \overline{v})$ is a robust weak efficient solution to (MMFP).

Conversely, let $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust weak efficient solution to (NMMFP), and $P^l(\overline{\pi}, m^l, n^l) = \dfrac{\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\overline{\pi}, m^l)d\omega}{\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}\omega}$, $\forall\, l = \overline{1,p}$. Suppose, contrary to the result that $(\overline{u}, \overline{v})$ is not a robust weak efficient solution to (MMFP), then there exist $(u, v) \in \mathcal{D}$, such that

$$\frac{\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\pi, m^l)\mathrm{d}\omega}{\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\pi, n^l)\mathrm{d}\omega} < \frac{\int_\Omega \max_{m^l\in\mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}\omega}{\int_\Omega \min_{n^l\in\mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}\omega},\ \forall\, l = \overline{1,p}.$$

As $P^l(\overline{\pi}, m^l, n^l) = \dfrac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) \mathrm{d}\omega}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l) \mathrm{d}\omega}$, the above inequality becomes

$$\frac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) \mathrm{d}\omega}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) \mathrm{d}\omega} < P^l(\overline{\pi}, m^l, n^l),$$

or $\displaystyle\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) d\omega - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) d\omega < 0, \ \forall\, l = \overline{1, p}.$

Again, since $P^l(\overline{\pi}, m^l, n^l) = \frac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) d\omega}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l) \mathrm{d}\omega}$, we obtain

$$\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) d\omega - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l) d\omega = 0, \ \forall\, l = \overline{1, p}.$$

On combining the above two inequalities, we get

$$\begin{aligned} &\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) \mathrm{d}\omega - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) \mathrm{d}\omega \\ &\quad < \int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) \mathrm{d}\omega - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l) \mathrm{d}\omega, \ \forall\, l = \overline{1, p}, \end{aligned}$$

which contradicts that $(\overline{u}, \overline{v})$ is a robust weak efficient solution to (NMMFP). Hence the proof is complete. □

Lemma 2. *Let $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust efficient solution to (MMFP) then there exist $P^l(\overline{\pi}, m^l, n^l) \in R_+$, $l = \overline{1, p}$ such that $(\overline{u}, \overline{v})$ is a robust efficient solution to (NMMFP). Conversely, If $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust efficient solution to (NMMFP), with $P^l(\overline{\pi}, m^l, n^l) = \dfrac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) \mathrm{d}t}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l) \mathrm{d}t}$, $l = \overline{1, p}$, then $(\overline{u}, \overline{v})$ is a robust efficient solution to (MMFP).*

Proof. The proof is similar as given in [10] and Lemma 1, hence omitted. □

Definition 3. *A point* $(\overline{u}, \overline{v}) \in \mathcal{D}$ *is said to be a robust proper efficient solution to (NMMFP), if it is a robust efficient solution of (NMMFP) and there exists a scalar* $K > 0$*, such that for all* $l = \overline{1, p}$*,*

$$\begin{aligned}
&\left\{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\,\mathrm{d}t - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\,\mathrm{d}t\right\} \\
&\qquad - \left\{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l)\,\mathrm{d}t - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l)\,\mathrm{d}t\right\} \\
&\qquad \leqslant K \Bigg[\left\{\int_\Omega \max_{m^s \in \mathcal{M}^s} \phi^s(\pi, m^s)\,\mathrm{d}t - P^s(\overline{\pi}, m^s, n^s) \int_\Omega \min_{n^s \in \mathcal{N}^s} \psi^s(\pi, n^s)\,\mathrm{d}t\right\} \\
&\qquad - \left\{\int_\Omega \max_{m^s \in \mathcal{M}^s} \phi^s(\overline{\pi}, m^s)\,\mathrm{d}t - P^s(\overline{\pi}, m^s, n^s) \int_\Omega \min_{n^s \in \mathcal{N}^s} \psi^s(\overline{\pi}, n^s)\,\mathrm{d}t\right\}\Bigg]
\end{aligned}$$

is true for some $s = \overline{1, p}$ *such that*

$$\begin{aligned}
&\left\{\int_\Omega \max_{m^s \in \mathcal{M}^s} \phi^s(\pi, m^s)\,\mathrm{d}t - P^s(\overline{\pi}, m^s, n^s) \int_\Omega \min_{n^s \in \mathcal{N}^s} \psi^s(\pi, n^s)\,\mathrm{d}t\right\} \\
&\qquad > \left\{\int_\Omega \max_{m^s \in \mathcal{M}^s} \phi^s(\overline{\pi}, m^s)\,\mathrm{d}t - P^s(\overline{\pi}, m^s, n^s) \int_\Omega \min_{n^s \in \mathcal{N}^s} \psi^s(\overline{\pi}, n^s)\,\mathrm{d}t\right\},
\end{aligned}$$

whenever $(u, v) \in \mathcal{D}$ *and*

$$\begin{aligned}
&\left\{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l)\,\mathrm{d}t - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l)\,\mathrm{d}t\right\} \\
&\qquad < \left\{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\,\mathrm{d}t - P^l(\overline{\pi}, m^l, n^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\,\mathrm{d}t\right\}.
\end{aligned}$$

Next, following the footsteps of Jayswal and Baranwal [10], we extend the definitions of convex functional for the vector case and and give its generalization in order to establish the main results.

Definition 4. *A functional* $\int_\Omega \phi^l(\pi, \overline{m}^l)\mathrm{d}t$, $l = \overline{1, p}$ *is said to be convex at* $(\overline{u}, \overline{v})$, *if the following inequality*

$$\int_\Omega \phi^l(\pi, \overline{m}^l)\mathrm{d}t - \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l)\mathrm{d}t \geqslant \int_\Omega \left\{(u-\overline{u})\phi^l_u(\overline{\pi}, \overline{m}^l) + (v-\overline{v})\phi^l_v(\overline{\pi}, \overline{m}^l)\right\} \mathrm{d}t,$$

holds for all $(u, v) \in U \times V$.

Definition 5. *In the above definition, if we replace* $\geqslant$ *with* $>$ *and* $(u, v) \neq (\overline{u}, \overline{v})$, *then the functional is said to be strictly convex at* $(\overline{u}, \overline{v})$.

Definition 6. *A functional* $\int_\Omega \phi^l(\pi, \overline{m}^l)\mathrm{d}t$, $l = \overline{1, p}$ *is said to be pseudoconvex at* $(\overline{u}, \overline{v})$, *if the following inequality*

$$\int_\Omega \phi^l(\pi, \overline{m}^l)\mathrm{d}t - \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l)\mathrm{d}t < 0$$

$$\Rightarrow \int_\Omega \left\{(u-\overline{u})\phi^l_u(\overline{\pi}, \overline{m}^l) + (v-\overline{v})\phi^l_v(\overline{\pi}, \overline{m}^l)\right\}\mathrm{d}t < 0,$$

holds for all $(u, v) \in U \times V$, *equivalently*

$$\int_\Omega \left\{(u-\overline{u})\phi^l_u(\overline{\pi}, \overline{m}^l) + (v-\overline{v})\phi^l_v(\overline{\pi}, \overline{m}^l)\right\}\mathrm{d}t \geqslant 0$$

$$\Rightarrow \int_\Omega \phi^l(\pi, \overline{m}^l)\mathrm{d}t - \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l)\mathrm{d}t \geqslant 0.$$

Now via next example we provide the numerical proof of the above definition.

Example 1. Let $U = V = R$, $\mathcal{M}^l = [-1, 1]$, $l = \{1, 2\}$ and $t \in \Omega = [-1, 1] \times [-1, 1]$ fixed by the diagonally opposite points $t_0 := (t_0^1, t_0^2) = (-1, -1)$, $t_1 := (t_1^1, t_1^2) = (1, 1) \in R^2$, and the functional $\phi = (\phi^1, \phi^2) : \Omega \times R \times R \times \mathcal{M}^l \mapsto R^2$, defined as

$$\phi^1(\pi, m^1) = \left(\frac{u^2}{u^2 + v} + m^1\right),$$
$$\phi^2(\pi, m^2) = (u + v + 2m^2).$$

Now, consider $u = (t^1 + t^2),\ v = 0.2,\ \overline{m}^1 = \overline{m}^2 = 1$, then for $(\overline{u}, \overline{v}) = (0.6, 0.2)$, with $t^1 = t^2 = 0.3$, we have

$$\int_\Omega \phi^1(\pi, \overline{m}^1)\,\mathrm{d}t - \int_\Omega \phi^1(\overline{\pi}, \overline{m}^1)\,\mathrm{d}t, \int_\Omega \phi^2(\pi, \overline{m}^2)\,\mathrm{d}t - \int_\Omega \phi^2(\overline{\pi}, \overline{m}^2)\,\mathrm{d}t$$

$$= \int_\Omega \Big(\frac{(t^1+t^2)^2}{(t^1+t^2)^2+0.2} - 0.6428\Big)\mathrm{d}t, \int_\Omega (t^1 + t^2 - 0.6)\,\mathrm{d}t < (0,0).$$

$$\Rightarrow \int_\Omega \Big\{(u-\overline{u})\phi_u^1(\overline{\pi}, \overline{m}^1) + (v-\overline{v})\phi_v^1(\overline{\pi}, \overline{m}^1)\Big\}\mathrm{d}t,$$

$$\int_\Omega \Big\{(u-\overline{u})\phi_u^2(\overline{\pi}, \overline{m}^2) + (v-\overline{v})\phi_v^2(\overline{\pi}, \overline{m}^2)\Big\}\mathrm{d}t$$

$$= \left\{\int_\Omega 0.7653(t^1 + t^2 - 0.6)\,\mathrm{d}t, \int_\Omega (t^1 + t^2 - 0.6)\,\mathrm{d}t\right\} < (0,0),$$

which shows that the functional $\int_\Omega \phi(\pi, m)\,\mathrm{d}t$ is pseudoconvex. On the other hand, the inequality

$$\int_\Omega \phi^1(\pi, \overline{m}^1)\,\mathrm{d}t - \int_\Omega \phi^1(\overline{\pi}, \overline{m}^1)\,\mathrm{d}t - \int_\Omega \Big\{(u-\overline{u})\phi_u^1(\overline{\pi}, \overline{m}^1) + (v-\overline{v})\phi_v^1(\overline{\pi}, \overline{m}^1)\Big\}\mathrm{d}t,$$

$$\int_\Omega \phi^2(\pi, \overline{m}^2)\,\mathrm{d}t - \int_\Omega \phi^2(\overline{\pi}, \overline{m}^2)\,\mathrm{d}t - \int_\Omega \Big\{(u-\overline{u})\phi_u^2(\overline{\pi}, \overline{m}^2) + (v-\overline{v})\phi_v^2(\overline{\pi}, \overline{m}^2)\Big\}\mathrm{d}t$$

$$= \left\{\int_\Omega \left(\frac{(t^1+t^2)^2}{(t^1+t^2)^2+0.2} - 0.7653(t^1+t^2) - 0.1836\right)\mathrm{d}t, 0\right\} \ngeq (0,0),$$

shows the non-convex property of the considered functional.

Now, in order to establish main results, we introduce a new class of robust controlled vector variational inequality together with its weak form:
1. Find $(\overline{u}, \overline{v}) \in \mathcal{D}$ such that there exists no $(u, v) \in \mathcal{D}$ fulfilling

$$\text{(VI)} \quad \int_\Omega \Big[(u-\overline{u})\Big\{\max_{m^1 \in \mathcal{M}^1} \phi_u^1(\overline{\pi}, m^1) - P^1 \min_{n^1 \in \mathcal{N}^1} \psi_u^1(\overline{\pi}, n^1)\Big\}$$

$$+ (v-\overline{v})\Big\{\max_{m^1 \in \mathcal{M}^1} \phi_v^1(\overline{\pi}, m^1) - P^1 \max_{n^1 \in \mathcal{N}^1} \psi_v^1(\overline{\pi}, n^1)\Big\}\Big]\mathrm{d}t, \ldots,$$

$$\int_\Omega \Big[(u-\overline{u})\Big\{\max_{m^p\in\mathcal{M}^p}\phi_u^p(\overline{\pi},m^p)-P^p\min_{n^p\in\mathcal{N}^p}\psi_u^p(\overline{\pi},n^p)\Big\}$$
$$+(v-\overline{v})\Big\{\max_{m^p\in\mathcal{M}^p}\phi_v^p(\overline{\pi},m^p)-P^p\min_{n^p\in\mathcal{N}^p}\psi_v^p(\overline{\pi},n^p)\Big\}\Big]\mathrm{d}t \leqslant \mathbf{0}.$$

2. Find $(\overline{u},\overline{v})\in\mathcal{D}$ such that there exists no $(u,v)\in\mathcal{D}$ fulfilling

$$\text{(WVI)}\quad \int_\Omega \Big[(u-\overline{u})\Big\{\max_{m^1\in\mathcal{M}^1}\phi_u^1(\overline{\pi},m^1)-P^1\min_{n^1\in\mathcal{N}^1}\psi_u^1(\overline{\pi},n^1)\Big\}$$
$$+(v-\overline{v})\Big\{\max_{m^1\in\mathcal{M}^1}\phi_v^1(\overline{\pi},m^1)-P^1\max_{n^1\in\mathcal{N}^1}\psi_v^1(\overline{\pi},n^1)\Big\}\Big]\mathrm{d}t,\dots,$$
$$\int_\Omega \Big[(u-\overline{u})\Big\{\max_{m^p\in\mathcal{M}^p}\phi_u^p(\overline{\pi},m^p)-P^p\min_{n^p\in\mathcal{N}^p}\psi_u^p(\overline{\pi},n^p)\Big\}$$
$$+(v-\overline{v})\Big\{\max_{m^p\in\mathcal{M}^p}\phi_v^p(\overline{\pi},m^p)-P^p\min_{n^p\in\mathcal{N}^p}\psi_v^p(\overline{\pi},n^p)\Big\}\Big]\mathrm{d}t < \mathbf{0},$$

where $\mathbf{0}=(0,0,...,0)$ (p-tuple).

Following example shows that the introduced variational inequality (VI) attains solution.

Example 2. Let $a=c=1$, i.e. we are interested only in real-valued continuous control and affine piecewise smooth state functions, $\mathcal{M}^l=[1,4]$, $\mathcal{N}^l=[0.5,2.5]$, $l=\{1,2\}$, and $\Omega\subset R^2$ (i.e. $b=2$) is a square fixed by the diagonally opposite corners $t_0:=(t_0^1,t_0^2)=(0,0)$, $t_1:=(t_1^1,t_1^2)=(2,2)\in R^2$, $t\in\Omega=[0,2]\times[0,2]$. Consider the following multi-dimensional multi-objective first-order PDE constrained fractional control optimization problem with data uncertainty given by

$$\text{(FP1)}\quad \min_{(u(\cdot),v(\cdot))}\left\{\frac{\int_\Omega \phi^l(\pi,m^l)\,\mathrm{d}t}{\int_\Omega \psi^l(\pi,n^l)\,\mathrm{d}t}\right. := \left(\frac{\int_\Omega \phi^1(\pi,m^1)\,\mathrm{d}t}{\int_\Omega \psi^1(\pi,n^1)\,\mathrm{d}t},\ \frac{\int_\Omega \phi^2(\pi,m^2)\,\mathrm{d}t}{\int_\Omega \psi^2(\pi,n^2)\,\mathrm{d}t}\right)$$
$$=\left.\frac{\int_\Omega (m^1\exp\{u\}+v^2)\,\mathrm{d}t}{\int_\Omega (u^2+n^1u+v^2)\,\mathrm{d}t},\ \frac{\int_\Omega (2m^2u+v)\,\mathrm{d}t}{\int_\Omega (u+n^2v)\,\mathrm{d}t}\right\},$$

subject to

$$Y(\pi)=4u^2-9\leqslant 0, \tag{4}$$
$$\frac{\partial u}{\partial t^\alpha}=H_\alpha(\pi)=3-v,\ \alpha=\{1,2\}, \tag{5}$$
$$u(0,0)=0,\ u(2,2)=8. \tag{6}$$

Therefore, the parametric form is

$$\text{(NFP1)}\quad \min_{(u(\cdot),v(\cdot))} \left\{ \int_{\Omega} (m^1 \exp\{u\} + v^2)\,\mathrm{d}t - P^1 \int_{\Omega} (u^2 + n^1 u + v^2)\,\mathrm{d}t, \right.$$
$$\left. \int_{\Omega} (2m^2 u + v)\,\mathrm{d}t - P^2 \int_{\Omega} (u + n^2 v)\,\mathrm{d}t \right\}$$

subject to the constraints (4)–(6),

where $P := (P^l) = P^l(\pi, m^l, n^l) \in R^2_+,\ l = 1,2.$

Further the robust counterpart for (NFP1) is given by

$$\text{(RNFP1)}\quad \min_{(u(\cdot),v(\cdot))} \left\{ \int_{\Omega} \max_{m^1\in\mathcal{M}^1} (m^1 \exp\{u\} + v^2)\,\mathrm{d}t - P^1 \int_{\Omega} \min_{n^1\in\mathcal{N}^1} (u^2 + n^1 u + v^2)\,\mathrm{d}t, \right.$$
$$\left. \int_{\Omega} \max_{m^2\in\mathcal{M}^2} (2m^2 u + v)\,\mathrm{d}t - P^2 \int_{\Omega} \min_{n^2\in\mathcal{N}^2} (u + n^2 v)\,\mathrm{d}t \right\}$$

subject to the constraints (4)–(6).

Let

$$\mathcal{D}_1 = \left\{ (u,v) \in U \times V :\ -\frac{3}{2} \leqslant u \leqslant \frac{3}{2},\ \frac{\partial u}{\partial t^1} = \frac{\partial u}{\partial t^2} = 3 - v,\ u(0,0) = 0,\ u(2,2) = 8 \right\}$$

be the collection of all robust feasible solutions to (FP1).

By direct computation, $(\bar{u}, \bar{v}) = (2t^1 + 2t^2, 1) \in \mathcal{D}_1$, at $t^1 = t^2 = 0$ is a solution to the associated variational inequality defined as

$$\int_{\Omega} \Big[(u - \bar{u}) \Big\{ \phi^1_u(\bar{\pi}, \bar{m}^1) - P^1 \psi^1_u(\bar{\pi}, \bar{n}^1) \Big\} + (v - \bar{v}) \Big\{ \phi^1_v(\bar{\pi}, \bar{m}^1) - P^1 \psi^1_v(\bar{\pi}, \bar{n}^1) \Big\} \Big]\,\mathrm{d}t,$$
$$\int_{\Omega} \Big[(u - \bar{u}) \Big\{ \phi^2_u(\bar{\pi}, \bar{m}^2) - P^2 \psi^2_u(\bar{\pi}, \bar{n}^2) \Big\} + (v - \bar{v}) \Big\{ \phi^2_v(\bar{\pi}, \bar{m}^2) - P^2 \psi^2_v(\bar{\pi}, \bar{n}^2) \Big\} \Big]\,\mathrm{d}t$$
$$= \int_{\Omega} 3(t^1 + t^2)\,\mathrm{d}t,\ \int_{\Omega} 12(t^1 + t^2)\,\mathrm{d}t \ \nleq (0,0),\ \forall (\bar{u}, \bar{v}) \neq (u,v) \in \mathcal{D}_1,$$

where $P^1 = 5,\ P^2 = 2$, uncertain parameters $\bar{m}^1 = \bar{m}^2 = 4,\ \bar{n}^1 = \bar{n}^2 = 0.5$.

3. Main results

In this section, we will derive some associations between the solutions of the multi-dimensional multi-objective control optimization problems with data uncertainty and the introduced robust controlled vector variational inequalities.

Theorem 1. *Let $\mathcal{D} \subset U \times V$ is a convex set, and let $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust proper efficient solution to (NMMFP) with* $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$ *and* $P^l(\overline{\pi}, m^l, n^l) = \dfrac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) \mathrm{d}t}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l) \mathrm{d}t}$, $l = \overline{1, p}$. *Then $(\overline{u}, \overline{v})$ is a solution to (VI).*

Proof. Let $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust proper efficient solution to (NMMFP). We proceed by the contradiction and assume that $(\overline{u}, \overline{v}) \in \mathcal{D}$ does not solves (VI), then there exist $(u, v) \in \mathcal{D}$, such that for all $l = \overline{1, p}$, we have

$$\int_\Omega \Big[(u - \overline{u})\Big\{ \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) - P^l \min_{n^l \in \mathcal{N}^l} \psi^l_u(\overline{\pi}, n^l)\Big\} + (v - \overline{v})\Big\{ \max_{m^l \in \mathcal{M}^l} \phi^l_v(\overline{\pi}, m^l) - P^l \min_{n^l \in \mathcal{N}^l} \psi^l_v(\overline{\pi}, n^l)\Big\}\Big] \mathrm{d}t < 0,$$

and for $s \neq l$

$$\int_\Omega \Big[(u - \overline{u})\Big\{ \max_{m^s \in \mathcal{M}^s} \phi^s_u(\overline{\pi}, m^s) - P^s \min_{n^s \in \mathcal{N}^s} \psi^s_u(\overline{\pi}, n^s)\Big\} + (v - \overline{v})\Big\{ \max_{m^s \in \mathcal{M}^s} \phi^s_v(\overline{\pi}, m^s) - P^s \min_{n^s \in \mathcal{N}^s} \psi^s_v(\overline{\pi}, n^s)\Big\}\Big] \mathrm{d}t \leqslant 0.$$

By taking $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$, $\forall (u, v) \in U \times V$ and $l = \overline{1, p}$, we obtain

$$\int_\Omega \Big[(u - \overline{u})\Big\{\phi^l_u(\overline{\pi}, \overline{m}^l) - P^l \psi^l_u(\overline{\pi}, \overline{n}^l)\Big\} + (v - \overline{v})\Big\{\phi^l_v(\overline{\pi}, \overline{m}^l) - P^l \psi^l_v(\overline{\pi}, \overline{n}^l)\Big\}\Big] \mathrm{d}t < 0, \tag{7}$$

and for $s \neq l$

$$\int_\Omega \Big[(u - \overline{u})\Big\{\phi^s_u(\overline{\pi}, \overline{m}^s) - P^s \psi^s_u(\overline{\pi}, \overline{n}^s)\Big\} + (v - \overline{v})\Big\{\phi^s_v(\overline{\pi}, \overline{m}^s) - P^s \psi^s_v(\overline{\pi}, \overline{n}^s)\Big\}\Big] \mathrm{d}t \leqslant 0. \tag{8}$$

Since $\mathcal{D} \subset U \times V$ is convex set, we can consider a sequence $\{\lambda_r\}$ of positive real numbers with $\lambda_r \to 0$ as $r \to \infty$, such that

$$(u^0, v^0) = (\overline{u}, \overline{v}) + \lambda_r((u - \overline{u}), (v - \overline{v})) \in \mathcal{D}.$$

Further, since the functional

$$\int_\Omega \phi^l(\pi, \overline{m}^l)\,\mathrm{d}t - P^l \int_\Omega \psi^l(\pi, \overline{n}^l)\,\mathrm{d}t, \quad l = \overline{1, p}$$

are continuously differentiable, using mean value theorem, we have

$$\begin{aligned}\int_\Omega \phi^l(\pi^0, \overline{m}^l)\,\mathrm{d}t - P^l \int_\Omega \psi^l(\pi^0, \overline{n}^l)\,\mathrm{d}t - \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l)\,\mathrm{d}t + P^l \int_\Omega \psi^l(\overline{\pi}, \overline{n}^l)\,\mathrm{d}t \\ = \int_\Omega \lambda_r \Big[(u - \overline{u})\Big\{\phi^l_u(\overline{\pi}, \overline{m}^l) - P^l \psi^l_u(\overline{\pi}, \overline{n}^l)\Big\} \\ + (v - \overline{v})\Big\{\phi^l_v(\overline{\pi}, \overline{m}^l) - P^l \psi^l_v(\overline{\pi}, \overline{n}^l)\Big\}\Big]\mathrm{d}t. \qquad (9)\end{aligned}$$

Dividing relation (9) by λ_r and taking the limit $r \to \infty$ both sides, we obtain

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{\lambda_r}\left[\int_\Omega \phi^l(\pi^0, \overline{m}^l)\,\mathrm{d}t - P^l \int_\Omega \psi^l(\pi^0, \overline{n}^l)\,\mathrm{d}t - \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l)\,\mathrm{d}t + P^l \int_\Omega \psi^l(\overline{\pi}, \overline{n}^l)\,\mathrm{d}t\right] \\ = \int_\Omega \Big[(u - \overline{u})\Big\{\phi^l_u(\overline{\pi}, \overline{m}^l) - P^l \psi^l_u(\overline{\pi}, \overline{n}^l)\Big\} + (v - \overline{v})\Big\{\phi^l_v(\overline{\pi}, \overline{m}^l) - P^l \psi^l_v(\overline{\pi}, \overline{n}^l)\Big\}\Big]\mathrm{d}t.\end{aligned}$$

In view of the inequality (7), it follows

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{\lambda_r}\Bigg[\int_\Omega \phi^l(\pi^0, \overline{m}^l)\,\mathrm{d}t - P^l \int_\Omega \psi^l(\pi^0, \overline{n}^l)\,\mathrm{d}t - \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l)\,\mathrm{d}t \\ + P^l \int_\Omega \psi^l(\overline{\pi}, \overline{n}^l)\,\mathrm{d}t\Bigg] < 0. \qquad (10)\end{aligned}$$

Since, proper efficiency of $(\overline{u},\overline{v}) \in \mathcal{D}$ implies the efficiency of $(\overline{u},\overline{v})$ in (NMMFP), thus we can consider the set

$$S = \left\{ s = \overline{1,p} \;\Big|\; \int_{\Omega} \phi^s(\overline{\pi},\overline{m}^s)\mathrm{d}t - P^s \int_{\Omega} \psi^s(\overline{\pi},\overline{n}^s)\mathrm{d}t - \int_{\Omega} \phi^s(\pi^0,\overline{m}^s)\mathrm{d}t \right.$$
$$\left. + P^s \int_{\Omega} \psi^s(\pi^0,\overline{n}^s)\mathrm{d}t \leqslant 0,\ \forall r \geqslant N \right\},$$

is non-empty.

Again from the differentiability of the functional

$$\int_{\Omega} \phi^s(\pi,\overline{m}^s)\mathrm{d}t - P^s \int_{\Omega} \psi^s(\pi,\overline{n}^s)\mathrm{d}t,$$

for $s \in S$, and using the mean value theorem, it follows

$$\int_{\Omega} \phi^s(\pi^0,\overline{m}^s)\mathrm{d}t - P^s \int_{\Omega} \psi^s(\pi^0,\overline{n}^s)\mathrm{d}t - \int_{\Omega} \phi^s(\overline{\pi},\overline{m}^s)\mathrm{d}t + P^s \int_{\Omega} \psi^s(\overline{\pi},\overline{n}^s)\mathrm{d}t$$
$$= \int_{\Omega} \lambda_r \Big[(u-\overline{u})\Big\{ \phi^s_u(\overline{\pi},\overline{m}^s) - P^s \psi^s_u(\overline{\pi},\overline{n}^s) \Big\}$$
$$+ \ (v-\overline{v})\Big\{ \phi^s_v(\overline{\pi},\overline{m}^s) - P^s \psi^s_v(\overline{\pi},\overline{n}^s) \Big\} \Big] \mathrm{d}t.$$

Dividing above relation by λ_r and taking the limit $r \to \infty$ both sides, we obtain

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \left[\int_{\Omega} \phi^s(\pi^0,\overline{m}^s)\mathrm{d}t - P^s \int_{\Omega} \psi^s(\pi^0,\overline{n}^s)\mathrm{d}t - \int_{\Omega} \phi^s(\overline{\pi},\overline{m}^s)\mathrm{d}t + P^s \int_{\Omega} \psi^s(\overline{\pi},\overline{n}^s)\mathrm{d}t \right]$$
$$= \int_{\Omega} \Big[(u-\overline{u})\Big\{ \phi^s_u(\overline{\pi},\overline{m}^s) - P^s \psi^s_u(\overline{\pi},\overline{n}^s) \Big\}$$
$$+ \ (v-\overline{v})\Big\{ \phi^s_v(\overline{\pi},\overline{m}^s) - P^s \psi^s_v(\overline{\pi},\overline{n}^s) \Big\} \Big] \mathrm{d}t. \tag{11}$$

By using the definition of the set S, we get

$$\int_{\Omega} \Big[(u-\overline{u})\Big\{ \phi^s_u(\overline{\pi},\overline{m}^s) - P^s \psi^s_u(\overline{\pi},\overline{n}^s) \Big\}$$
$$+ (v-\overline{v})\Big\{ \phi^s_v(\overline{\pi},\overline{m}^s) - P^s \psi^s_v(\overline{\pi},\overline{n}^s) \Big\} \Big] \mathrm{d}t \geqslant 0, \ \ \forall r \geqslant N. \tag{12}$$

On combining the inequalities (8) and (12), we obtain

$$\int_{\Omega}\Big[(u-\overline{u})\Big\{\phi^s_u(\overline{\pi},\overline{m}^s)-P^s\psi^s_u(\overline{\pi},\overline{n}^s)\Big\}$$
$$+(v-\overline{v})\Big\{\phi^s_v(\overline{\pi},\overline{m}^s)-P^s\psi^s_v(\overline{\pi},\overline{n}^s)\Big\}\Big]\mathrm{d}t=0,\quad \text{for } s\neq l,\ s\in S \text{ and } r\geqslant N.$$

In view of the inequality (11), it follows

$$\lim_{r\to\infty}\frac{1}{\lambda_r}\Bigg[\int_{\Omega}\phi^s(\pi^0,\overline{m}^s)\mathrm{d}t-P^s\int_{\Omega}\psi^s(\pi^0,\overline{n}^s)\mathrm{d}t-\int_{\Omega}\phi^s(\overline{\pi},\overline{m}^s)\mathrm{d}t$$
$$+P^s\int_{\Omega}\psi^s(\overline{\pi},\overline{n}^s)\mathrm{d}t\Bigg]=0. \tag{13}$$

Now, by combining the inequality (10) and the equation (13), we observe

$$\frac{\frac{1}{\lambda_r}\Bigg[\int_{\Omega}\phi^l(\overline{\pi},\overline{m}^l)\mathrm{d}t-P^l\int_{\Omega}\psi^l(\overline{\pi},\overline{n}^l)\mathrm{d}t-\int_{\Omega}\phi^l(\pi^0,\overline{m}^l)\mathrm{d}t+P^l\int_{\Omega}\psi^l(\pi^0,\overline{n}^l)\mathrm{d}t\Bigg]}{\frac{1}{\lambda_r}\Bigg[\int_{\Omega}\phi^s(\pi^0,\overline{m}^s)\mathrm{d}t-P^s\int_{\Omega}\psi^s(\pi^0,\overline{n}^s)\mathrm{d}t-\int_{\Omega}\phi^s(\overline{\pi},\overline{m}^s)\mathrm{d}t+P^s\int_{\Omega}\psi^s(\overline{\pi},\overline{n}^s)\mathrm{d}t\Bigg]}$$

tends to ∞ as $r\to\infty$, $\forall\ s\neq l,\ s\in S$, which contradicts $(\overline{u},\overline{v})$ is robust proper efficient solution to (NMMFP). This completes the proof. □

In the previous theorem, we have shown that the robust proper efficient solutions to (NMMFP) is a solution of defined robust controlled vector variational inequality (VI) as well. In the next theorem, we prove that the solution of robust controlled vector variational inequality (VI) is robust efficient solution of the problem (MMFP).

Theorem 2. *Let $(\overline{u},\overline{v})\in\mathcal{D}$ solves (VI) with $\max_{m^l\in\mathcal{M}^l}\phi^l(\pi,m^l)=\phi^l(\pi,\overline{m}^l)$, $\min_{n^l\in\mathcal{N}^l}\psi^l(\pi,n^l)=\psi^l(\pi,\overline{n}^l)$ and $P^l(\overline{\pi},m^l,n^l)=\dfrac{\int_{\Omega}\max_{m^l\in\mathcal{M}^l}\phi^l(\overline{\pi},m^l)\mathrm{d}t}{\int_{\Omega}\min_{n^l\in\mathcal{N}^l}\psi^l(\overline{\pi},n^l)\mathrm{d}t}$, $l=\overline{1,p}$. If, the functional $\int_{\Omega}\phi^l(\pi,\overline{m}^l)\mathrm{d}t-P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_{\Omega}\psi^l(\pi,\overline{n}^l)\mathrm{d}t$, $l=\overline{1,p}$ are convex at $(\overline{u},\overline{v})\in\mathcal{D}$, then $(\overline{u},\overline{v})$ is a robust efficient solution to (MMFP).*

Proof. Let $(\overline{u}, \overline{v}) \in \mathcal{D}$ solves (VI), therefore, there exists no other $(u, v) \in \mathcal{D}$ satisfying

$$\int_{\Omega} \Big[(u - \overline{u}) \Big\{ \max_{m^l \in \mathcal{M}^l} \phi_u^l(\overline{\pi}, m^l) - P^l \min_{n^l \in \mathcal{N}^l} \psi_u^l(\overline{\pi}, n^l \Big\}$$
$$+ (v - \overline{v}) \Big\{ \max_{m^l \in \mathcal{M}^l} \phi_v^l(\overline{\pi}, m^l) - P^l \max_{n^l \in \mathcal{N}^l} \psi_v^l(\overline{\pi}, n^l) \Big\} \Big] \mathrm{d}t \leqslant 0, \ \ \forall\, l = \overline{1, p}.$$

Since, $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$, $\forall\, l = \overline{1, p}$, one can say there exists no other $(u, v) \in \mathcal{D}$ such that

$$\int_{\Omega} \Big[(u - \overline{u}) \Big\{ \phi_u^l(\overline{\pi}, \overline{m}^l) - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \psi_u^l(\overline{\pi}, \overline{n}^l) \Big\}$$
$$+ (v - \overline{v}) \Big\{ \phi_v^l(\overline{\pi}, \overline{m}^l) - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \psi_v^l(\overline{\pi}, \overline{n}^l) \Big\} \Big] \mathrm{d}t \leqslant 0, \ \ \forall\, l = \overline{1, p}. \quad (14)$$

Now, we assume on the contrary that $(\overline{u}, \overline{v}) \in \mathcal{D}$ is not a robust efficient solution to (MMFP), then from Lemma 2 and Definition 2 there exist a point $(u, v) \in \mathcal{D}$ such that

$$\int_{\Omega} \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) \mathrm{d}t$$
$$\leqslant \int_{\Omega} \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l) \mathrm{d}t, \ \ \forall\, l = \overline{1, p}.$$

As $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$, $l = \overline{1, p}$, it follows

$$\int_{\Omega} \phi^l(\pi, \overline{m}^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \psi^l(\pi, \overline{n}^l) \mathrm{d}t$$
$$\leqslant \int_{\Omega} \phi^l(\overline{\pi}, \overline{m}^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \psi^l(\overline{\pi}, \overline{n}^l) \mathrm{d}t, \ \ \forall\, l = \overline{1, p}.$$

From the convexity of functional $\int_{\Omega} \phi^l(\pi, \overline{m}^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \psi^l(\pi, \overline{n}^l) \mathrm{d}t$, $l = \overline{1, p}$ at $(\overline{u}, \overline{v}) \in \mathcal{D}$, we get

$$\int_\Omega \phi^l(\pi,\overline{m}^l)\mathrm{d}t - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\pi,\overline{n}^l)\mathrm{d}t$$
$$-\int_\Omega \phi^l(\overline{\pi},\overline{m}^l)\mathrm{d}t + P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\overline{\pi},\overline{n}^l)\mathrm{d}t,$$
$$\geqslant \int_\Omega \Big[(u-\overline{u})\Big\{\phi_u^l(\overline{\pi},\overline{m}^l) - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi_u^l(\overline{\pi},\overline{n}^l)\Big\}$$
$$+ (v-\overline{v})\Big\{\phi_v^l(\overline{\pi},\overline{m}^l) - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi_v^l(\overline{\pi},\overline{n}^l)\Big\}\Big]\mathrm{d}t, \quad \forall\, l=\overline{1,p}.$$

On combining the above two inequalities, we obtain

$$\int_\Omega \Big[(u-\overline{u})\Big\{\phi_u^l(\overline{\pi},\overline{m}^l) - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi_u^l(\overline{\pi},\overline{n}^l)\Big\}$$
$$+ (v-\overline{v})\Big\{\phi_v^l(\overline{\pi},\overline{m}^l) - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi_v^l(\overline{\pi},\overline{n}^l)\Big\}\Big]\mathrm{d}t \leqslant 0, \quad \forall\, l=\overline{1,p},$$

which contradicts the inequality (14). This completes the proof. □

Now, via the robust weak controlled vector variational inequality (WVI), we obtain the robust weak efficient solutions to (MMFP).

Theorem 3. *Let* $(\overline{u},\overline{v}) \in \mathcal{D}$ *solves (WVI) and the functional*

$$\int_\Omega \phi^l(\pi,\overline{m}^l)\mathrm{d}t - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\pi,\overline{n}^l)\mathrm{d}t, \ \forall\, l=\overline{1,p}$$

are pseudoconvex at $(\overline{u},\overline{v}) \in \mathcal{D}$ *with* $\max_{m^l\in\mathcal{M}^l}\phi^l(\pi,m^l) = \phi^l(\pi,\overline{m}^l)$, $\min_{n^l\in\mathcal{N}^l}\psi^l(\pi,n^l) = \psi^l(\pi,\overline{n}^l)$ *and* $P^l(\overline{\pi},m^l,n^l) = \frac{\int_\Omega \max_{m^l\in\mathcal{M}^l}\phi^l(\overline{\pi},m^l)\mathrm{d}t}{\int_\Omega \min_{n^l\in\mathcal{N}^l}\psi^l(\overline{\pi},n^l)\mathrm{d}t}$, $l=\overline{1,p}$. *Then* $(\overline{u},\overline{v})$ *is a robust weak efficient solution to (MMFP).*

Proof. From the assumption, $(\overline{u},\overline{v}) \in \mathcal{D}$ solves (WVI), there exists no other $(u,v)\in\mathcal{D}$ satisfying

$$\int_\Omega \Bigg[(u-\overline{u})\left\{\max_{m^l\in\mathcal{M}^l}\phi_u^l(\overline{\pi},m^l) - P^l\min_{n^l\in\mathcal{N}^l}\psi_u^l(\overline{\pi},n^l\right\}$$
$$+(v-\overline{v})\left\{\max_{m^l\in\mathcal{M}^l}\phi_v^l(\overline{\pi},m^l) - P^l\max_{n^l\in\mathcal{N}^l}\psi_v^l(\overline{\pi},n^l)\right\}\Bigg]\mathrm{d}t < 0, \quad \forall\, l=\overline{1,p}.$$

Since, $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$, $\forall\, l = \overline{1, p}$, one can say there exists no other $(u, v) \in \mathcal{D}$ such that

$$\int_\Omega \Big[(u - \overline{u}) \Big\{ \phi^l_u(\overline{\pi}, \overline{m}^l) - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \psi^l_u(\overline{\pi}, \overline{n}^l) \Big\} + (v - \overline{v}) \Big\{ \phi^l_v(\overline{\pi}, \overline{m}^l) - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \psi^l_v(\overline{\pi}, \overline{n}^l) \Big\} \Big] \mathrm{d}t < 0, \quad \forall\, l = \overline{1, p}. \quad (15)$$

On the contrary, we assume that $(\overline{u}, \overline{v}) \in \mathcal{D}$ is not a robust weak efficient solution to (MMFP), then from Lemma 1 and Definition 2 there exist a point $(u, v) \in \mathcal{D}$ such that

$$\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) \mathrm{d}t < \int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l) \mathrm{d}t, \quad \forall\, l = \overline{1, p}.$$

As $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$, $\forall\, l = \overline{1, p}$, the above inequality becomes

$$\int_\Omega \phi^l(\pi, \overline{m}^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\pi, \overline{n}^l) \mathrm{d}t < \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\overline{\pi}, \overline{n}^l) \mathrm{d}t, \quad \forall\, l = \overline{1, p}.$$

The above inequality together with the pseudoconvexity of functional

$$\int_\Omega \phi^l(\pi, \overline{m}^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\pi, \overline{n}^l) \mathrm{d}t, \ l = \overline{1, p} \ \text{ at } \ (\overline{u}, \overline{v}) \in \mathcal{D}$$

gives

$$\int_\Omega \Big[(u - \overline{u}) \Big\{ \phi^l_u(\overline{\pi}, \overline{m}^l) - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \psi^l_u(\overline{\pi}, \overline{n}^l) \Big\} + (v - \overline{v}) \Big\{ \phi^l_v(\overline{\pi}, \overline{m}^l) - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \psi^l_v(\overline{\pi}, \overline{n}^l) \Big\} \Big] \mathrm{d}t < 0, \quad \forall\, l = \overline{1, p},$$

which contradicts the inequality (15). This completes the proof. □

Next, we prove the converse result of the above theorem.

Theorem 4. *Let $\mathcal{D} \subset U \times V$ is a convex set, $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust weak efficient solution to (MMFP) with $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$ and $P^l(\overline{\pi}, m^l, n^l) = \dfrac{\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\,\mathrm{d}t}{\int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\,\mathrm{d}t}$, $l = \overline{1, p}$. Then $(\overline{u}, \overline{v})$ is a solution to (WVI).*

Proof. Let $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust weak efficient solution to (MMFP), thus from Lemma 1 $(\overline{u}, \overline{v})$ is also a robust weak efficient solution to (NMMFP), then there exists no other $(u, v) \in \mathcal{D}$, such that

$$\int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l)\,\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l)\,\mathrm{d}t$$
$$< \int_\Omega \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\,\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\,\mathrm{d}t, \quad \forall\, l = \overline{1, p}.$$

Since, $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$, $l = \overline{1, p}$, it follows that no other $(u, v) \in \mathcal{D}$, satisfy

$$\int_\Omega \phi^l(\pi, \overline{m}^l)\,\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\pi, \overline{n}^l)\,\mathrm{d}t$$
$$< \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l)\,\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\overline{\pi}, \overline{n}^l)\,\mathrm{d}t, \quad \forall\, l = \overline{1, p}. \tag{16}$$

Since, $\mathcal{D}$ is convex, so for all $\lambda \in [0, 1]$, we consider

$$(u^0, v^0) = (\overline{u}, \overline{v}) + \lambda((u - \overline{u}), (v - \overline{v})) \in \mathcal{D}.$$

Therefore, by relation (16), we obtain there exists no other $(u^0, v^0) \in \mathcal{D}$ such that

$$\int_\Omega \phi^l(\pi^0, \overline{m}^l)\,\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\pi^0, \overline{n}^l)\,\mathrm{d}t$$
$$< \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l)\,\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\overline{\pi}, \overline{n}^l)\,\mathrm{d}t, \quad \forall\, l = \overline{1, p}. \tag{17}$$

Further, since the integral functionals

$$\int_\Omega \phi^l(\pi, \overline{m}^l)\,\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\pi, \overline{n}^l)\,\mathrm{d}t, \quad l = \overline{1, p},$$

is differentiable at $(u^0, v^0) \in \mathcal{D}$, following the same manner as in Theorem 1 and by considering relation (17), it results that there exists no other $(u, v) \in \mathcal{D}$ such that

$$\int_{\Omega} \Big[(u - \overline{u}) \Big\{ \phi_u^l(\overline{\pi}, \overline{m}^l) - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \psi_u^l(\overline{\pi}, \overline{n}^l) \Big\} + (v - \overline{v}) \Big\{ \phi_v^l(\overline{\pi}, \overline{m}^l) - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \psi_v^l(\overline{\pi}, \overline{n}^l) \Big\} \Big] \mathrm{d}t < 0, \ \forall\, l = \overline{1, p}.$$

Therefore, we conclude that $(\overline{u}, \overline{v})$ solves (WVI). □

The following theorem provides a characterization for a robust weak efficient solution to become a robust efficient solution of (MMFP).

Theorem 5. *Let $\mathcal{D} \subset U \times V$ is a convex set and the functional $\int_{\Omega} \phi^l(\pi, \overline{m}^l)\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \psi^l(\pi, \overline{n}^l)\mathrm{d}t$, $\forall\ l = \overline{1, p}$ are strictly convex at $(\overline{u}, \overline{v}) \in \mathcal{D}$ with $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$ and $P^l(\overline{\pi}, m^l, n^l) = \dfrac{\int_{\Omega} \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}t}{\int_{\Omega} \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}t}$, $l = \overline{1, p}$. If $(\overline{u}, \overline{v})$ is a robust weak efficient solution of (MMFP). Then, it is a robust efficient solution of (MMFP).*

Proof. Let $(\overline{u}, \overline{v}) \in \mathcal{D}$ is a robust weak efficient solution to (MMFP) then from Lemma 1 it is also a robust weak efficient solution to (NMMFP). On the contrary we assume that $(\overline{u}, \overline{v})$ is not a robust efficient solution to (MMFP), then from Lemma 2, there exist $(u^0, v^0) \in \mathcal{D}$, such that

$$\int_{\Omega} \max_{m^l \in \mathcal{M}^l} \phi^l(\pi^0, m^l)\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \min_{n^l \in \mathcal{N}^l} \psi^l(\pi^0, n^l)\mathrm{d}t \leqslant \int_{\Omega} \max_{m^l \in \mathcal{M}^l} \phi^l(\overline{\pi}, m^l)\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \min_{n^l \in \mathcal{N}^l} \psi^l(\overline{\pi}, n^l)\mathrm{d}t, \ \ \forall\, l = \overline{1, p}.$$

Since $\max_{m^l \in \mathcal{M}^l} \phi^l(\pi, m^l) = \phi^l(\pi, \overline{m}^l)$, $\min_{n^l \in \mathcal{N}^l} \psi^l(\pi, n^l) = \psi^l(\pi, \overline{n}^l)$, it follows

$$\int_{\Omega} \phi^l(\pi^0, \overline{m}^l)\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \psi^l(\pi^0, \overline{n}^l)\mathrm{d}t \leqslant \int_{\Omega} \phi^l(\overline{\pi}, \overline{m}^l)\mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_{\Omega} \psi^l(\overline{\pi}, \overline{n}^l)\mathrm{d}t, \ \ \forall\, l = \overline{1, p}.$$

From strict convexity of the functional

$$\int_\Omega \phi^l(\pi,\overline{m}^l)\mathrm{d}t - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\pi,\overline{n}^l)\mathrm{d}t,\ l=\overline{1,p},$$

at $(\overline{u},\overline{v}) \in \mathcal{D}$, we get

$$\begin{aligned}&\int_\Omega \phi^l(\pi^0,\overline{m}^l)\mathrm{d}t - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\pi^0,\overline{n}^l)\mathrm{d}t\\ &\qquad - \int_\Omega \phi^l(\overline{\pi},\overline{m}^l)\mathrm{d}t + P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\overline{\pi},\overline{n}^l)\mathrm{d}t,\\ &> \int_\Omega \Big[(u^0-\overline{u})\Big\{\phi_u^l(\overline{\pi},\overline{m}^l) - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi_u^l(\overline{\pi},\overline{n}^l)\Big\}\\ &\qquad + (v^0-\overline{v})\Big\{\phi_v^l(\overline{\pi},\overline{m}^l) - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi_v^l(\overline{\pi},\overline{n}^l)\Big\}\Big]\mathrm{d}t,\\ &\qquad \forall\, l=\overline{1,p} \text{ and } (u^0,v^0)\neq(\overline{u},\overline{v})\in\mathcal{D}.\end{aligned}$$

On combining the above two inequalities, we obtain

$$\begin{aligned}&\int_\Omega \Big[(u^0-\overline{u})\Big\{\phi_u^l(\overline{\pi},\overline{m}^l) - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi_u^l(\overline{\pi},\overline{n}^l)\Big\}\\ &\quad + (v^0-\overline{v})\Big\{\phi_v^l(\overline{\pi},\overline{m}^l) - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi_v^l(\overline{\pi},\overline{n}^l)\Big\}\Big]\mathrm{d}t < 0,\quad \forall\, l=\overline{1,p},\end{aligned}$$

which shows that $(\overline{u},\overline{v})$ is not a solution to (WVI) and hence from the Theorem 4 $(\overline{u},\overline{v})$ is not a robust weak efficient solution to (NMMFP) and we arrive on contradiction. This completes the proof. □

Here we present an illustrative application which utilize the established results of the paper.

Example 3. A manufacturing company produces some goods and wants to minimize the production cost of two products as well as maximize their profits, respectively. The total production costs are $\phi(\pi,m) := (\phi^1(\pi,m^1),\phi^2(\pi,m^2)) = \Big((u\exp(4+u)+m^1u+v^2),(u^2+v^2+m^2u)\Big)$ and total profits are $\psi(\pi,m) := (\psi^1(\pi,m^1),\psi^2(\pi,m^2)) = \Big((u\log(5-u)-n^1v),(2u-n^2v)\Big)$, where $u=u(t)\in R$ is the level of output, $v=v(t)\in R$ is the control function, $t=(t^1,t^2)\in R^2$ is the time variable, and $m^1,m^2\in[-3,3]$, $n^1,n^2\in[-1,1]$ are data uncertainties. The nature of control over the level of output is defined by the dynamical system

$$u_t = H(\pi) = 3 - v.$$

The production costs should be minimized and profits should be maximized subject to constraint

$$Y(\pi) = 4u^2 - 9 \leqslant 0.$$

The endpoint conditions are $u(0,0) = 0,\ u(1,1) = 4$. The problem is to find an efficient pair of output and control functions of time which minimizes the production costs and maximizes the profits simultaneously, in the face of data uncertainties.

The above industrial test problem can be mathematically formulated as a multi-dimensional multi-objective first-order PDE constrained fractional control optimization problem with data uncertainty as:

$$\text{(FP2)}\quad \min_{(u(\cdot),v(\cdot))} \left\{ \frac{\int_\Omega \phi(\pi,m)\,dt^1 dt^2}{\int_\Omega \psi(\pi,n)\,dt^1 dt^2} := \left(\frac{\int_\Omega \phi^1(\pi,m^1)\,dt^1 dt^2}{\int_\Omega \psi^1(\pi,n^1)\,dt^1 dt^2}, \frac{\int_\Omega \phi^2(\pi,m^2)\,dt^1 dt^2}{\int_\Omega \psi^2(\pi,n^2)\,dt^1 dt^2} \right) = \frac{\int_\Omega (u\exp(4+u) + m^1 u + v^2)\,dt^1 dt^2}{\int_\Omega (u\log(5-u) - n^1 v)\,dt^1 dt^2}, \frac{\int_\Omega (u^2 + v^2 + m^2 u)\,dt^1 dt^2}{\int_\Omega (2u - n^2 v)\,dt^1 dt^2} \right\},$$

subject to

$$Y(\pi) = 4u^2 - 9 \leqslant 0, \tag{18}$$

$$\frac{\partial u}{\partial t^\alpha} = H_\alpha(\pi) = 3 - v,\ \alpha = \{1,2\}, \tag{19}$$

$$u(0,0) = 0,\ u(1,1) = 4. \tag{20}$$

where $U = V = R$, $M^1 = M^2 = [-3,3]$, $N^1 = N^2 = [-1,1]$, $\Omega \subset R^2$ is a square fixed by the diagonally opposite points $t_0 := (t_0^1, t_0^2) = (0,0)$, $t_1 := (t_1^1, t_1^2) = (1,1) \in R^2$ and $t \in \Omega$.

Thus, the parametric form of (FP2) is

$$\text{(NFP2)}\quad \min_{(u(\cdot),v(\cdot))} \left\{ \int_\Omega (u\exp(4+u) + m^1 u + v^2)\,dt^1 dt^2 - P^1 \int_\Omega (u\log(5-u) - n^1 v)\,dt^1 dt^2, \int_\Omega (u^2 + v^2 + m^2 u)\,dt^1 dt^2 - P^2 \int_\Omega (2u - n^2 v)\,dt^1 dt^2 \right\}$$

subject to the constraints (18)–(20),

where $P := (P^l) = P^l(\pi, m^l, n^l) \in R_+^2,\ l = 1,2$.

The robust counterpart for (NFP2) is given by

$$\text{(RNFP2)}\quad \min_{(u(\cdot),v(\cdot))}\left\{\int_\Omega \max_{m^1\in\mathcal{M}^1}(u\exp(4+u)+m^1u+v^2)\,\mathrm{d}t^1\,\mathrm{d}t^2 - P^1\int_\Omega \min_{n^1\in\mathcal{N}^1}(u\log(5-u)-n^1v)\,\mathrm{d}t^1\,\mathrm{d}t^2,\right.$$
$$\left.\int_\Omega \max_{m^2\in\mathcal{M}^2}(u^2+v^2+m^2u)\,\mathrm{d}t^1\,\mathrm{d}t^2 - P^2\int_\Omega \min_{n^2\in\mathcal{N}^2}(2u-n^2v)\,\mathrm{d}t^1\,\mathrm{d}t^2\right\}$$

subject to the constraints (18)–(20).

We denote the set of robust feasible solutions as

$$\mathcal{D}_2=\left\{(u,v)\in U\times V:\ -\frac{3}{2}\leqslant u\leqslant\frac{3}{2},\ \frac{\partial u}{\partial t^1}=\frac{\partial u}{\partial t^2}=3-v,\ u(0,0)=0,\ u(1,1)=4\right\}.$$

Since, the following inequality

$$\int_\Omega \phi^l(\pi,\overline{m}^l)\,\mathrm{d}t - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\pi,\overline{n}^l)\,\mathrm{d}t - \int_\Omega \phi^l(\overline{\pi},\overline{m}^l)\,\mathrm{d}t$$
$$+P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\overline{\pi},\overline{n}^l)\,\mathrm{d}t,$$
$$-\int_\Omega\Big[(u-\overline{u})\Big\{\phi^l_u(\overline{\pi},\overline{m}^l)-P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi^l_u(\overline{\pi},\overline{n}^l)\Big\}$$
$$+(v-\overline{v})\Big\{\phi^l_v(\overline{\pi},\overline{m}^l)-P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\psi^l_v(\overline{\pi},\overline{n}^l)\Big\}\Big]\,\mathrm{d}t$$
$$=\Bigg(\int_\Omega (u\exp(4+u)+\overline{m}^1u+v^2)\,\mathrm{d}t^1\,\mathrm{d}t^2 - P^1\int_\Omega (u\log(5-u)-\overline{n}^1v)\,\mathrm{d}t^1\,\mathrm{d}t^2$$
$$-\int_\Omega (\overline{u}\exp(4+\overline{u})+\overline{m}^1\overline{u}+\overline{v}^2)\,\mathrm{d}t^1\,\mathrm{d}t^2 + P^1\int_\Omega (\overline{u}\log(5-\overline{u})-\overline{n}^1\overline{v})\,\mathrm{d}t^1\,\mathrm{d}t^2$$
$$-\int_\Omega\Big[(u-\overline{u})\Big\{\exp(4+\overline{u})+\overline{u}\exp(4+\overline{u})+\overline{m}^1-P^1\Big(\log(5-\overline{u})-\frac{\overline{u}}{(4-\overline{u})}\Big)\Big\}$$
$$+(v-\overline{v})\Big\{2\overline{v}+P^1\overline{n}^1\Big\}\Big]\,\mathrm{d}t^1\,\mathrm{d}t^2,$$

$$\int_\Omega (u^2+v^2+\overline{m}^2 u)\,\mathrm{d}t^1\mathrm{d}t^2 - P^2\int_\Omega (2u-\overline{n}^2 v)\,\mathrm{d}t^1\mathrm{d}t^2$$

$$-\int_\Omega (\overline{u}^2+\overline{v}^2+\overline{m}^2\overline{u})\,\mathrm{d}t^1\mathrm{d}t^2 + P^2\int_\Omega (2\overline{u}-\overline{n}^2\overline{v})\,\mathrm{d}t^1\mathrm{d}t^2$$

$$-\int_\Omega \left[(u-\overline{u})\left\{2\overline{u}+\overline{m}^2-2P^2\right\}+(v-\overline{v})\left\{2\overline{v}+P^2\overline{n}^2\right\}\right]\mathrm{d}t^1\mathrm{d}t^2\Bigg)$$

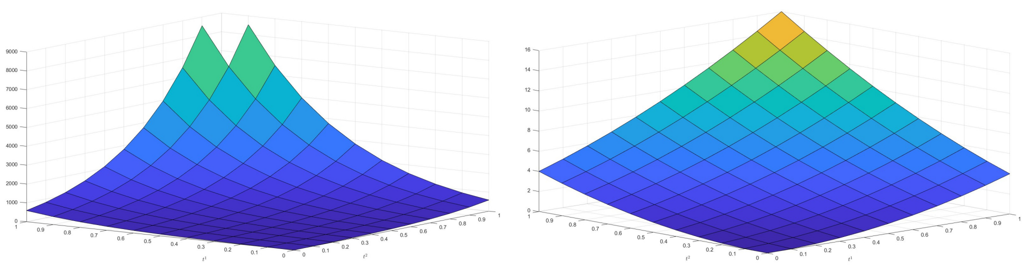
$$=\Bigg(\int_\Omega 2(t^1+t^2)(\exp(4+2t^1+2t^2)-\log(5-2t^1-2t^2)$$

$$-\exp(4)-3+\log(5))+6t^1+6t^2\,\mathrm{d}t^1\mathrm{d}t^2,$$

$$\int_\Omega 4(t^1+t^2)^2\,\mathrm{d}t^1\mathrm{d}t^2\Bigg)=(e_1,e_2)>(0,0),\quad \text{(see Fig. 1(a) and 1(b)).}$$

holds, where $\overline{m}^l=3$, $\overline{n}^l=-1$ and $P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)=\dfrac{\int_\Omega \max_{m^l\in\mathcal{M}^l}\phi^l(\overline{\pi},m^l)\,\mathrm{d}t}{\int_\Omega \min_{n^l\in\mathcal{N}^l}\psi^l(\overline{\pi},n^l)\,\mathrm{d}t}=1$, $l=1,2$, therefore the integral functionals

$$\int_\Omega \phi^l(\pi,\overline{m}^l)\,\mathrm{d}t - P^l(\overline{\pi},\overline{m}^l,\overline{n}^l)\int_\Omega \psi^l(\pi,\overline{n}^l)\,\mathrm{d}t,\quad \forall\, l=\overline{1,p},\ l=1,2$$



(a) (b)

Figure 1: The figures (a) and (b) show that the expressions e_1 and e_2 are taking positive values, respectively, in the domain Ω

are convex at $(\overline{u}, \overline{v}) = (2t^1 + 2t^2, 1)$, at $t^1 = t^2 = 0$. Also, $(\overline{u}, \overline{v})$ solves the robust controlled vector variational inequality:

$$
\text{(VI2)} \quad \int_\Omega \Big[(u - \overline{u}) \Big\{ \max_{m^1 \in \mathcal{M}^1} \phi^1_u(\overline{\pi}, m^1) - P^1 \min_{n^1 \in \mathcal{N}^1} \psi^1_u(\overline{\pi}, n^1 \Big\}
+ (v - \overline{v}) \Big\{ \max_{m^1 \in \mathcal{M}^1} \phi^1_v(\overline{\pi}, m^1) - P^1 \max_{n^1 \in \mathcal{N}^1} \psi^1_v(\overline{\pi}, n^1) \Big\} \Big] \mathrm{d}t,
$$

$$
\int_\Omega \Big[(u - \overline{u}) \Big\{ \max_{m^2 \in \mathcal{M}^2} \phi^2_u(\overline{\pi}, m^2) - P^2 \min_{n^2 \in \mathcal{N}^2} \psi^2_u(\overline{\pi}, n^2) \Big\}
+ (v - \overline{v}) \Big\{ \max_{m^2 \in \mathcal{M}^2} \phi^2_v(\overline{\pi}, \overline{m}^2) - P^2 \min_{n^2 \in \mathcal{N}^2} \psi^2_v(\overline{\pi}, n^2) \Big\} \Big] \mathrm{d}t
$$

$$
= \int_\Omega \Big[(u - \overline{u}) \Big\{ \exp(4 + \overline{u}) + \overline{u} \exp(4 + \overline{u}) + \overline{m}^1 - P^1 \Big(\log(5 - \overline{u}) - \frac{\overline{u}}{(4 - \overline{u})} \Big) \Big\}
+ (v - \overline{v}) \Big\{ 2\overline{v} + P^1 \overline{n}^1 \Big\} \Big] \mathrm{d}t^1 \mathrm{d}t^2,
$$

$$
\int_\Omega \Big[(u - \overline{u}) \Big\{ 2\overline{u} + \overline{m}^2 - 2P^2 \Big\} + (v - \overline{v}) \Big\{ 2\overline{v} + P^2 \overline{n}^2 \Big\} \Big] \mathrm{d}t^1 \mathrm{d}t^2
$$

$$
= \Bigg(\int_\Omega 2(\exp(4) + 3 - \log(5))(t^1 + t^2) \mathrm{d}t^1 \mathrm{d}t^2, \ \int_\Omega 2(t^1 + t^2) \mathrm{d}t^1 \mathrm{d}t^2 \Bigg) > (0, 0),
$$

$\forall (t^1, t^2) \in \Omega$.

Now, it's remaining to show that the point $(\overline{u}, \overline{v}) = (2t^1 + 2t^2, 1), \ t^1 = t^2 = 0$ is a robust efficient solution to the problem (FP2), we observe

$$
\int_\Omega \phi^l(\pi, \overline{m}^l) \mathrm{d}t - P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\pi, \overline{n}^l) \mathrm{d}t - \int_\Omega \phi^l(\overline{\pi}, \overline{m}^l) \mathrm{d}t
+ P^l(\overline{\pi}, \overline{m}^l, \overline{n}^l) \int_\Omega \psi^l(\overline{\pi}, \overline{n}^l) \mathrm{d}t,
$$

$$
= \Bigg(\int_\Omega (u \exp(4 + u) + \overline{m}^1 u + v^2) \mathrm{d}t^1 \mathrm{d}t^2 - P^1 \int_\Omega (u \log(5 - u) - \overline{n}^1 v) \mathrm{d}t^1 \mathrm{d}t^2
- \int_\Omega (\overline{u} \exp(4 + \overline{u}) + \overline{m}^1 \overline{u} + \overline{v}^2) \mathrm{d}t^1 \mathrm{d}t^2 + P^1 \int_\Omega (\overline{u} \log(5 - \overline{u}) - \overline{n}^1 \overline{v}) \mathrm{d}t^1 \mathrm{d}t^2,
$$

$$\int_{\Omega}(u^2+v^2+\overline{m}^2u)\,\mathrm{d}t^1\,\mathrm{d}t^2 - P^2\int_{\Omega}(2u-\overline{n}^2v)\,\mathrm{d}t^1\,\mathrm{d}t^2$$

$$\left.-\int_{\Omega}(\overline{u}^2+\overline{v}^2+\overline{m}^2\overline{u})\,\mathrm{d}t^1\,\mathrm{d}t^2 + P^2\int_{\Omega}(2\overline{u}-\overline{n}^2\overline{v})\,\mathrm{d}t^1\,\mathrm{d}t^2\right)$$

$$=\left(\int_{\Omega}2(t^1+t^2)(\exp(4+2t^1+2t^2)-\log(5-2t^1-2t^2))+6t^1+6t^2\,\mathrm{d}t^1\,\mathrm{d}t^2,\right.$$

$$\left.\int_{\Omega}4(t^1+t^2)^2+2(t^1+t^2)\,\mathrm{d}t^1\,\mathrm{d}t^2\right) > (0,0),\ \forall (t^1,t^2)\in\Omega.$$

holds. Thus, we conclude $(\overline{u},\overline{v}) = (2t^1+2t^2,1),\ t^1=t^2=0$ is a robust efficient solution to the problem (NFP2) with $\overline{m}^1=\overline{m}^2=3,\ \overline{n}^1=\overline{n}^2=-1$ and $P^l = \dfrac{\int_{\Omega}\max_{m^l\in\mathcal{M}^l}\phi^l(\overline{\pi},m^l)\,\mathrm{d}t}{\int_{\Omega}\min_{n^l\in\mathcal{N}^l}\psi^l(\overline{\pi},n^l)\,\mathrm{d}t} = 1,\ l=1,2$ and thus from Lemma 2, it is a robust efficient solution to (FP2). Hence, an efficient pair of output and control function, which minimizes the production costs and maximizes the profits simultaneously in the face of data uncertainties is obtained at $(\overline{u},\overline{v}) = (0,1)$.

Remark 1. *It should be noted that no results based on variational inequality have been found in the literature to show the effectiveness of the two approaches used for solving a multi-dimensional multi-objective fractional control optimization problem with first-order PDE constraints as even for solving such that one considered in the above example. This follows from the fact that the involved objective functional are ratio of two functionals with data uncertainty. For this reason, it is not possible to apply the results existing in the literature which have been established either for the class of multi-dimensional multi-objective optimization problems (not having the ratio of two functionals as objectives) or the class of optimization problems that do not involve data uncertainty in them (see, for example, [9, 11, 20]).*

4. Conclusions

In this paper, we have introduces a new robust controlled vector variational inequality with its weak form to study the efficiency of an uncertain multi-dimensional multi-objective first-order PDE-constrained fractional control optimization problem. To solve such uncertain control problems, we use the approach which is based on the robust optimization method and the parametric method. The associated non-fractional multi-dimensional multi-objective control problem with first-order PDE constraints has been constructed since the parametric method was applied to examine the multi-dimensional multi-objective first-order PDE-constrained fractional control problem. The robust method has formulated a robust counterpart for this uncertain control problem. The equivalence between robust efficient solutions of the original uncertain multi-dimensional multi-objective fractional control problem and its associated non-fractional control problem derived in the parametric approach has been established. Further, we have introduced a new robust controlled vector variational inequalities with its weak form. We have established several equivalence results between the variational inequalities and the problem mentioned above. In particular, under convexity and pseudoconvexity assumptions, the equivalence has been established between robust efficient solutions (weak and proper) in the considered uncertain PDE-constrained multi-objective multi-dimensional fractional control problem and the robust controlled vector variational inequalities constructed for the parametric non-fractional control problem. The presence of uncertainty in function and the use of the robust and parametric approach to construct the variational inequality in this context is an element of novelty. Also, some applications are given to better illustrate the motivation of the suggested results of the paper.

Still, there are a few intriguing study areas left. If the method presented in this paper can be applied to additional kinds of nonconvex multi-dimensional multi-objective first-order PDE-constrained fractional control optimization problems, it would be interesting to explore. In later publications, we will look into these questions.

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