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Asymptotic behavior of a viscoelastic wave equation with a delay in internal fractional feedback

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We consider the viscoelastic wave equation with a time delay term in internal fractional feedback. By employing the energy method along with the Faedo-Galerkin procedure, we establish the global existence of solutions, subject to certain conditions. Additionally, we demonstrate how appropriate Lyapunov functionals can lead to general decay results of the energy.

Key words: global existence, general decay, relaxation function, delay fractional feedback, partial differential equations

1. Introduction

Mathematical modeling and analysis serve as indispensable tools in the realms of science and engineering, providing a systematic framework for understanding, predicting, and optimizing complex phenomena.This research is motivated by a diverse range of viscoelastic phenomena. Over the past two decades, fractional calculus has demonstrated its efficacy in control processing and various engineer-

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ing fields. The existing literature highlights the considerable attention given to linear wave equations due to their inherent structural properties [3,8,9,18,25,31].

In [25,26], the researchers investigated the blow-up and asymptotic behavior of a wave equation characterized by a time delay condition of fractional type. Focusing on the dynamics of the system, the study explored the conditions under which blow-up phenomena occurred and delved into the subsequent asymptotic patterns. By incorporating fractional-type time delay, the research introduced a nuanced dimension to the analysis, shedding light on the intricate interplay between temporal delays and wave dynamics. The findings contributed to a deeper understanding of the behavior of the system, offering insights into the emergence of blow-up and the long-term trends exhibited by the solution. This research bridged the gap between wave equations and fractional calculus, enriching the theoretical framework for understanding time-delayed wave phenomena with implications for various scientific and engineering applications.

We investigate the following problem:

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) + \displaystyle\int_0^t f(t-\sigma)\Delta w(x,\sigma)\,\mathrm{d}\sigma + \mu_1 w_t(x,t) & \\ \qquad + \mu_2 \partial_t^{\rho,\beta} w(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ w(x,t) = 0, & x \in \partial\Omega,\ t \geqslant 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \Omega, \\ w_t(x,t-\tau) = g_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geqslant 1)$ with a smooth boundary $\partial\Omega$, f is a function which will be specified later, $\tau > 0$ represents the delay and μ_1 and μ_2 are positive constants.The notation $\partial_t^{\rho,\beta}$ stands for the modified Caputo fractional derivative (see [7]) defined by:

$$\partial_t^{\rho,\beta} w(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t (t-\sigma)^{-\rho} e^{-\beta(t-\sigma)} w_\sigma(\sigma)\,\mathrm{d}\sigma, \quad 0 < \rho < 1, \quad \beta \geqslant 0$$

and u_0 , u_1, f_0 are given functions belonging to suitable spaces.

In recent years, the exploration of partial differential equations (PDEs) incorporating time delay effects has witnessed a notable surge in interest. This burgeoning area of research is underscored by a plethora of works, exemplified by references such as [2,3,28] and the comprehensive bibliography accompanying these studies. Significantly, the pivotal work of [6, 8] underscored the vulnerability of systems to instability with even minor delays in boundary control. This revelation has instigated a concerted effort in the scientific community to address stability concerns inherent in hyperbolic systems featuring input delay terms. The requisite introduction of additional control terms to mitigate such stability issues has been a subject of thorough investigation, extensively documented in works like [20] and the rich array of references encapsulated therein. Notably, the study in [20] scrutinized a system governed by the wave equation, specifically considering a linear boundary-damping term with a delay, thus contributing to the elucidation of dynamics in systems grappling with temporal delays in control mechanisms.

This study aims to investigate the existence and asymptotic stability of a viscoelastic wave equation embedded with a delay in internal fractional feedback. This nuanced exploration integrates the complexities of viscoelasticity and fractional calculus, addressing a dynamic system with a temporal delay in internal feedback mechanisms. The research aims to establish the existence of solutions for this intricate equation while unraveling the nuanced interplay between viscoelastic properties and fractional-order dynamics. Furthermore, the focus extends to the asymptotic stability analysis, probing the long-term behavior of solutions under the influence of delayed internal fractional feedback. By elucidating the stability characteristics, the study contributes to the broader understanding of the intricate dynamics inherent in viscoelastic wave equations, thereby fostering advancements in the theoretical underpinnings of these phenomena with potential implications for applications across various scientific and engineering domains. This study focused on investigating the following system:

$$\begin{cases} w_{tt} - \Delta w = 0, & x \in \Omega, \quad t > 0, \\ w(x,t) = 0, & x \in \Gamma_0, \quad t > 0, \\ \dfrac{\mathrm{d}w}{\mathrm{d}\nu}(x,t) = \mu_1 w_t(x,t) + \mu_2 w_t(x,t-\tau), & x \in \Gamma_0, \quad t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \Omega, \\ w(x,t-\tau) = g_0(x,t-\tau), & x \in \Omega, \quad t \in (0,\tau), \end{cases} \tag{2}$$

They found that the energy of the system is exponentially stable under the assumption

$$\mu_2 < \mu_1, \tag{3}$$

while a sequence of delays exists that can make the system unstable if (3) is not satisfied. The primary methodology employed in [20] involves an observability inequality coupled with a Carleman estimate. Similar findings were obtained when both the damping and delay were present within the domain. Notably, [30] achieved a parallel outcome to [20] in one spatial dimension by adopting a spectral analysis approach. As established by the aforementioned studies, the rate of decay of solutions is contingent upon the delay. In fact, as demonstrated in [25], the decay rate diminishes as the delay increases. In simpler terms, the decay process becomes slower as τ becomes larger.

The issue of time-varying delay in the wave equation has been recently investigated by Nicaise et al. [26] in the context of one spatial dimension. In their study, they established an exponential stability result subject to the following condition:

$$\mu_2 < \sqrt{1-d}\mu_1 , \tag{4}$$

where d is a constant such that

$$\tau'(t) \leqslant d < 1, \quad \forall t > 0. \tag{5}$$

The inclusion of the term viscoelastic in the study is justified from a physical perspective by the desire to capture the material properties that exhibit both viscous and elastic behavior. Viscoelastic materials possess characteristics of both viscosity, where they deform continuously under stress, and elasticity, where they can return to their original shape after deformation. This dual nature is prevalent in a variety of real-world materials, such as biological tissues, polymers, and certain geological substances. Understanding the viscoelastic properties is crucial in accurately modeling the behavior of these materials under dynamic conditions, especially in the context of wave equations where the response to stress and strain is dynamic. Therefore, the incorporation of the term viscoelastic in the equation reflects a commitment to representing the physical reality of materials that exhibit a combination of viscous and elastic traits, enhancing the relevance and applicability of the mathematical model to real-world scenarios.

Extensive research has been conducted on problems similar to (1) in bounded domains or in the entire N-dimensional space in the absence of the delay term, specifically when $\mu_2 = 0$. Over the past three decades, various authors have investigated the existence, blow-up, and asymptotic behavior of both smooth and weak solutions. For further information and references, please refer to [3, 6, 9] and the sources cited therein. In [15], the authors investigated the blow-up phenomenon in nonlinear viscoelastic wave equations, specifically focusing on solutions with positive initial energy. This research provided insights into the dynamics of viscoelastic systems, highlighting the conditions under which solutions exhibited blow-up behavior. The general decay behavior of solution energy in a viscoelastic equation featuring a nonlinear source was investigated in the literature [17]. The researchers explored the temporal evolution of energy within the system, offering insights into the dynamics and stability of viscoelastic equations with nonlinear influences. In [28], the authors focused on the asymptotic behavior of energy in materials characterized by partial viscoelasticity. Investigating the dynamics of energy within such materials provided valuable insights into their long-term behavior and contributed to the understanding of partially viscoelastic systems.

In [13], the authors focused on the stabilization of wave systems confronted with input delay in the boundary control. Employing a systematic approach, the authors delved into the intricacies of managing delayed control inputs in the context of wave systems. Through a comprehensive analysis, the study addressed the challenges posed by delays in boundary control and proposed effective strategies to achieve system stabilization. The work was grounded in established control theory principles, drawing on insights from previous research in wave systems with input delay. The methodologies presented contributed to the development of robust control strategies, offering valuable contributions to the broader field of control theory and its application to wave systems. The findings had implications for various domains, including engineering and physics, where the robust stabilization of dynamic systems was of paramount importance. In the following, we present a brief overview of key findings related to the viscoelastic wave equation. The viscoelastic wave equation is given by:

$$w_{tt} - \Delta w + \int_0^t f(t-\sigma)\Delta w(x,\sigma)\,\mathrm{d}\sigma + g(w_t) = h(u_t), \tag{6}$$

in $\Omega \times (0,\infty)$ has been extensively studied by Cavalcanti et al. [9]. This equation considers the case where $h = 0$ and $g(w_t) = a(x)w_t$, and is subject to initial conditions and Dirichlet-type boundary conditions. Specifically, Cavalcanti et al. focused on the problem formulated by

$$w_{tt} - \Delta w + \int_0^t f(t-\sigma)\Delta w(x,\sigma)\,\mathrm{d}\sigma + a(x)w_t = 0. \tag{7}$$

Here $a\colon \Omega \to \mathbb{R}^+$ is a function, which may be null in certain regions of the domain. Under the assumptions that $a(x) \geqslant a_0$ on a subset $\omega \subset \Omega$ and

$$-\zeta_1 f(t) \leqslant f'(t) \leqslant -\zeta_2 f(t), \quad \forall t \geqslant 0,$$

the authors established an exponential decay result with geometric restrictions on the subset ω. This finding was subsequently improved upon by Berrimi and Messaoudi [6], who obtained the same exponential decay result with weaker conditions on both a and f. Moreover, in [3], a more general abstract formulation of Eq. (6) was considered, leading to the derivation of a uniform stability result. Notably, the decay rates obtained in [3] align with those found in [6] for Eq. 6, highlighting the consistency of the findings across different studies.

Cavalcanti and Oquendo [11] employed the piecewise multipliers method to investigate a more general problem than the one addressed in [9]. Specifically, they examined the following problem:

$$w_{tt} - k_0 \Delta w + \int_{t_0} \operatorname{div}[a(x) f(t-\sigma) \Delta w(x,\sigma)]\, \mathrm{d}\sigma + b(x) g(w_t) + h(w) = 0. \quad (8)$$

Their study established stability results under certain conditions on the function g and when $a(x) + b(x) \geqslant \rho > 0$. If f decays exponentially and g is a linear function, an exponential stability result was proven. On the other hand, if f decays polynomially and g is a nonlinear function, a polynomial stability result was demonstrated. These findings highlight the influence of the decay behavior of f and the linearity / nonlinearity of g on the stability properties of the system.

The authors in [12] addressed the problem (7) with a constant coefficient $a(x) = a_0$ and demonstrated that the solution of the problem (7) exhibits exponential decay only if the relaxation kernel g also decays exponentially. This implies that the presence of the memory term can hinder exponential decay caused by the linear frictional damping term.

Cavalcanti et al. [8] conducted an investigation on the problem described by

$$|w_t|^{\rho} w_{tt} - \Delta w - \Delta w_{tt} + \int_0^t f(t-\sigma) \Delta w(x,\sigma)\, \mathrm{d}\sigma - \gamma \Delta w_t = 0, \quad \rho > 0, \quad (9)$$

in the domain $\Omega \times (0, \infty)$. They established a global existence result for $\gamma \geqslant 0$. Additionally, under the condition $\gamma > 0$ and assuming exponential decay of the function f, they obtained an exponential decay result for the solution.

Building upon the theoretical framework of potential well theory, Tatar and Messaoudi [18] undertook an extension of the results elucidated in [8] to a broader context. In this extended scenario, the investigation encompasses the introduction of an additional source term represented as $|w|^{p-2}w$ into the governing equation denoted as (9). This augmentation introduces a nonlinearity characterized by the power exponent p, enriching the mathematical model with a nonlinear term that influences the system's behavior. The incorporation of such a source term is significant as it reflects a departure from the linear dynamics considered in the initial formulation (9). Tatar and Messaoudi's extension contributes to the broader understanding of the impact of nonlinearity on the dynamics described by potential well theory, paving the way for more comprehensive analyses and applications in diverse scientific and mathematical contexts.

Messaoudi and Tatar [19] conducted a study on equation (9) with $\gamma = 0$ and demonstrated that the viscoelastic damping term is sufficiently strong to stabilize the system. Another noteworthy paper is [10], where the authors explored a similar problem to (6) but with a nonlinear feedback applied to the domain boundary Ω. They established uniform decay rates for the system's energy without imposing restrictive growth assumptions on the damping term. For additional insights, we refer to [1, 14], which provide results on asymptotic stability and global non-existence of the wave equation with memory-type boundary dissipation. In [29], a wave equation with acoustic and memory boundary conditions on a portion of the domain Ω boundary was investigated. The authors proved the existence and uniqueness of a global solution for this particular case.

Aounallah, Benaissa, and Zarai [2] recently investigated the system with a fractional time delay, given by

$$\begin{cases} w_{tt} - \Delta w + \mu_1 w_t(x,t) + \mu_2 \partial_t^{\rho,\beta} w_t(x,t-\tau) = w|w|^{p-1}, & x \in \Omega,\ t > 0, \\ w(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x \in \Omega, \\ w(x,t-\tau) = g_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \end{cases} \tag{10}$$

where they used semi-group theory to establish the existence of solutions for the problem and proved a decay rate estimate for the energy by introducing suitable Lyapunov functionals. Furthermore, they demonstrated that the solution blows up in finite time if the initial energy is negative, and under the condition

$$\beta^{\rho-1}\mu_2 < \mu_1 \tag{11}$$

the system is well-posed. It is worth mentioning that Benaissa and Gaouar [5] used the same method as in [2] to prove the well-posedness and exponential decay for the Lamè system with internal fractional delay and boundary damping of Neumann type.

In this study, we focus on problem (1). Our objective is twofold:

Firstly, by employing Faedo-Galerkin approximations along with energy estimates and imposing certain constraints on the parameters μ_1 and μ_2, we establish the well-posedness of the system.

Secondly, under the assumption that $\beta^{\rho-1}\mu_2 < \mu_1$, which relates to the relative weights of the delay term in the feedback and the term without delay, we demonstrate a general decay of the total energy in our problem. The proof methodology employed in this study leverages conceptual foundations established in prior works, particularly [2, 3], which have addressed wave equations incorporating delay. Our approach intricately incorporates estimates tailored to the specific nuances of the viscoelastic wave equation under consideration. The application of these specialized techniques enables the systematic construction of pertinent Lyapunov functionals, pivotal in establishing the desired outcomes of the analysis. Through this methodical utilization of established concepts and tailored estimates, we derive rigorous results integral to the understanding and characterization of the dynamics inherent in the viscoelastic wave equation with temporal delay.

The paper is structured as follows: in the subsequent section, we introduce the notation and provide a summary of useful lemmas for the convenience of the reader, without including their proofs. In Section 3, we establish the well-posedness of the solution. In Section 4, we establish a general decay of the energy defined by equation (40), provided that the weight of the delay term is smaller than the weight of the damping term.

2. Preliminaries

In this section, we provide the essential background information required to prove our main result. We introduce the following set of assumptions:

(A0) $f : \mathbb{R}+ \to \mathbb{R}+$ is a C^1 function satisfying

$$f(0) > 0 \quad \text{and} \quad 1 - \int_0^\infty f(\sigma)\,\mathrm{d}\sigma = l > 0.$$

(A1) There exists a positive, non-increasing differentiable function $\zeta(t)$ such that

$$f'(t) \leqslant -\zeta(t) f(t), \forall t \geqslant 0, \tag{12}$$

and

$$\int_0^\infty \zeta(t)\,\mathrm{d}t = \infty.$$

We suppose further that

$$\beta^{\rho-1}\mu_2 < \mu_1, \tag{13}$$

$$\tau\beta^{\rho-1}\mu_2 < \alpha < \tau\left[2\mu_1 - \beta^{\rho-1}\mu_2\right]. \tag{14}$$

We introduce the following notations:

$$(\varphi * \psi)(t) = \int_0^t \varphi(t-\tau)\psi(\tau)\,\mathrm{d}\tau,$$

$$(\varphi \diamond \psi)(t) = \int_0^t \varphi(t-\tau)|\psi(t) - \psi(\tau)|\,\mathrm{d}\tau,$$

$$(\varphi \circ \psi)(t) = \int_0^t \varphi(t-\tau) \int_\Omega |\psi(t) - \psi(\tau)|^2 \mathrm{d}x d\tau.$$

We present the following lemmas without providing their proofs.

Lemma 1. *[27] For any function $\varphi \in C^1(\mathbb{R})$ and any $\psi \in H^1(0,T)$, we have*

$$(\varphi * \psi)(t)\psi'(t) = -\frac{1}{2}\varphi(t)|\psi(t)|^2 + \frac{1}{2}(\varphi' \diamond \psi)(t) - \frac{1}{2}\frac{d}{dt}\left[(\varphi \diamond \psi)(t) - \left(\int_0^t \varphi(\sigma)\mathrm{d}\sigma\right)|\psi(t)|^2\right].$$

Lemma 2. *[27] For $w \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t f(t-\sigma)\left(w(t) - w(\sigma)\right)\mathrm{d}\sigma\right)^2 \mathrm{d}x \leqslant (1-l)B_{1,\Omega}^2 \left(f \circ \nabla w\right)(t), \qquad (15)$$

where $B_{1,\Omega}$ is the Poincaré constant and l is given in (A1).

Lemma 3. *[16] Let η be the function:*

$$\eta(y) = |y|^{\frac{(2\rho-1)}{2}}, \quad y \in \mathbb{R}, \quad 0 < \rho < 1.$$

Then the relationship between the "input" w and the "output" O of the system

$$\begin{cases} \partial_t v(y,t) + (y^2+\beta)v(y,t) - w(x,t)\eta(y) = 0, & y \in \mathbb{R}, t > 0, \beta \geqslant 0, \\ v(y,0) = 0, \\ O(t) = (\pi)^{-1}\sin(\rho\pi)\displaystyle\int_{-\infty}^{+\infty} v(y,t)\eta(y)dy, \end{cases} \qquad (16)$$

is given by

$$O = I^{1-\rho,\beta}w,$$

where

$$I^{\rho,\beta}w(t) = \frac{1}{\Gamma(\rho)}\int_0^t (t-\sigma)^{\rho-1}w(\sigma)e^{-\beta(t-\sigma)}\mathrm{d}\sigma.$$

Lemma 4. *[2] For all $\lambda \in D_\beta = \{\lambda \in \mathbb{C} : \Re e\lambda + \beta > 0\} \cup \{\lambda \in \mathbb{C} : \Im m\lambda \neq 0\}$,*

$$A_\lambda = \int_{-\infty}^{+\infty} \frac{\eta^2(y)}{\lambda+\beta+y^2}dy = \frac{\pi}{\sin(\rho\pi)}(\lambda+\beta)^{\rho-1}.$$

3. Well-posedness of the problem

This section is dedicated to presenting rigorous proof for the global existence and uniqueness of the solution to the problem (1). Our methodology entails the introduction of an additional unknown, facilitating the transformation of the original problem (1) into an equivalent form denoted as (19), outlined below. Through the systematic application of Faedo-Galerkin approximations and the judicious utilization of energy estimates, we will establish the existence and uniqueness of the solution to the transformed problem (19).

To establish the existence and uniqueness of a solution to problem (1), we will follow the approach outlined in [23]. This approach involves introducing a new variable, which allows us to rewrite the problem in a suitable form.

Let

$$u(x,\xi,t) = w_t(x,t-\tau\xi), \quad x \in (0,1), \quad \xi \in (0,1), \quad t > 0. \tag{17}$$

Then, we have

$$\tau u_t(x,\xi,t) + u_\xi(x,\xi,t) = 0, \quad x \in (0,1), \quad \xi \in (0,1), \quad t > 0. \tag{18}$$

Therefore, problem (1) is equivalent to:

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) + \displaystyle\int_0^t f(t-\sigma)\Delta w(x,\sigma)\,\mathrm{d}\sigma + \mu_1 w_t(x,t) \\ \qquad + a_1 \displaystyle\int_{-\infty}^{+\infty} v(x,y,t)\eta(y)\,\mathrm{d}y = 0, \\ \partial_t v(x,y,t) + (y^2+\beta)v(x,y,t) - u(x,1,t)\eta(y) = 0, \\ \tau u_t(x,\xi,t) + u_\xi(x,\xi,t) = 0, \end{cases} \tag{19}$$

where $x \in \Omega$, $y \in (-\infty,+\infty)$, $\xi \in (0,1)$, $t > 0$ and $a_1 = (\pi)^{-1} sin(\rho\pi)\mu_2$. The above system is subject to the initial and boundary conditions

$$\begin{cases} w(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,y,0) = 0, & x \in \Omega,\ y \in (-\infty,+\infty), \\ u(x,0,t) = w_t(x,t), & x \in \Omega,\ t > 0, \\ u(x,\xi,0) = g_0(x,t-\tau), & x \in \Omega,\ \xi \in (0,1),\ t \in (0,\tau). \end{cases} \tag{20}$$

The existence result reads as follows:

Theorem 1. *Assume that $\beta^{\rho-1}\mu_2 \leqslant \mu_1$. Then given $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$, $g_0 \in L^2(\Omega \times (0,1))$ and $T > 0$, there exists a unique weak solution (w,v,u) of* *the problem (19)-(20) on* $(0,T)$ *such that*

$$w \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)),$$
$$w_t \in L^2(0,T; H_0^1(\Omega)) \cap L^2((0,T)\times\Omega).$$

Proof. The proof of Theorem 1 can be divided into two steps: first, we construct approximations, and second, we use specific energy estimates to pass to the limit.

Step 1: Approximate problem.

We construct approximations of the solution (w,v,u) by the Faedo-Galerkin method as follows. For every $n \geqslant 1$, let $Z_n = \operatorname{span}\{z_1, z_2, \ldots, z_n\}$ and $K_n = \operatorname{span}\{k_1, k_2, \ldots, k_n\}$ are the Hilbertian basis of the spaces $H_0^1(\Omega)$ and $L^2\,(\Omega \times (-\infty,+\infty))$ respectively.

Now, we define for $1 \leqslant j \leqslant n$, the sequence $\phi_j(x,\xi)$ as follows:

$$\phi_j(x,0) = w_j(x).$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\xi)$ over $L^2\,(\Omega\times(0,1))$ and denote with $U_n = \{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(w^n(x,t), v^n(x,y,t), u^n(x,\xi,t))$ $(n = 1,2,3,\ldots)$ in the form

$$\begin{aligned} w^n(x,t) &= \sum_{j=1}^{n} w_{nj}(t) z_j(x), \quad v^n(x,y,t) = \sum_{j=1}^{n} v_{nj}(t) k_j(x,y),\\ \text{and}\quad u^n(x,\xi,t) &= \sum_{j=1}^{n} u_{nj}(t)\phi_j(x,\xi), \end{aligned} \tag{21}$$

where (w_{nj}, v_{nj}, u_{nj}) $(n = 1,2,3,\ldots)$ are determined by the following ordinary differential equations:

$$\int_\Omega w_{tt}^n(x,t) z_j(x)\,\mathrm{d}x + \int_\Omega \nabla w^n(x,t)\nabla z_j(x)\,\mathrm{d}x - \int_0^t f(t-\sigma)\int_\Omega \nabla w^n(x,\sigma)\nabla z_j(x)\,\mathrm{d}x\,\mathrm{d}\sigma$$
$$+\mu_1 \int_\Omega w_t^n(x,t) z_j(x)\,\mathrm{d}x + a_1 \int_{-\infty}^{+\infty} \eta(y) \int_\Omega v^n(x,y,t) z_j(x)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{22}$$

$$\begin{aligned} w^n(0) &= w_{0n} = \sum_{j=1}^{n} (w_0, z_j) w_j \longrightarrow w_0 \quad \text{in } H_0^1(\Omega) \text{ as } n \longrightarrow +\infty,\\ w_t^n(0) &= w_{1n} = \sum_{j=1}^{n} (w_1, z_j) w_j \longrightarrow w_1 \quad \text{in } L^2(\Omega) \text{ as } n \longrightarrow +\infty, \end{aligned} \tag{23}$$

$$\int_{\Omega} v_t^n(x,y,t)k_j(x,y)\,\mathrm{d}x + \int_{\Omega}\left(y^2+\beta\right) v^n(x,y,t)k_j(x,y)\,\mathrm{d}x - \int_{\Omega}\eta(y)u^n(x,1,t)k_j(x,y)\,\mathrm{d}x = 0, \tag{24}$$

$$v^n(y,0) = v_{0n} = \sum_{j=1}^{n}(v_0,k_j)k_j \to v_0 = 0 \quad \text{in } L^2(\Omega\times(-\infty,+\infty)) \text{ as } n\to+\infty, \tag{25}$$

and

$$\int_{\Omega}\tau u_t^n(x,\xi,t)\phi_j(x,\xi)\,\mathrm{d}x + \int_{\Omega}u_{\xi}^n(x,\xi,t)\phi_j(x,\xi)\,\mathrm{d}x = 0, \tag{26}$$

$$u^n(\xi,0) = u_{0n} = \sum_{j=1}^{n}(g_0,\phi_j)\phi_j \to g_0 \quad \text{in } L^2(\Omega\times(0,1)) \text{ as } n\to+\infty. \tag{27}$$

In accordance with the standard theory of ordinary differential equations, the finite-dimensional problem (22)–(27) admits a solution $(q_{jn}(t),h_{jn}(t))_{j=1,n}$, defined on the interval $[0,t_n)$. The a priori estimates obtained indicate that t_n is equal to T.

The first estimate

Multiplying Eq. (22) by $w'_{n,j}(t)$, summing with respect to j and using Lemma 1, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(1-\int_0^t f(\sigma)\,\mathrm{d}\sigma\right)\|\nabla w^n(t)\|_2^2 + \|w_t^n(t)\|_2^2 + (f\circ\nabla w^n)(t)\right] + \mu_1\|w_t^n(t)\|_2^2 + \frac{1}{2}f(t)\|\nabla w^n(t)\|_2^2 - \frac{1}{2}(f'\circ\nabla w^n)(t) + a_1\int_{\Omega}w_t^n(x,t)\int_{-\infty}^{+\infty}v^n(x,y,t)\eta(y)\,\mathrm{d}y\,\mathrm{d}x = 0. \tag{28}$$

The last term on the left-hand side of (28) can be estimated as follows

$$
\begin{aligned}
&\left| \int_{\Omega} w_t^n(x,t) \int_{-\infty}^{+\infty} v^n(x,y,t)\eta(y)dy\,\mathrm{d}x \right| \\
&\leqslant \left(\int_{-\infty}^{+\infty} \frac{\eta^2(y)}{y^2+\beta} dy \right) \|w_t^n(x,t)\|_2^2 + \frac{1}{4} \int_{-\infty}^{+\infty} \int_{\Omega} \left(y^2+\beta\right) (v^n(x,y,t))^2 \mathrm{d}x\mathrm{d}y. \qquad (29)
\end{aligned}
$$

Inserting (29)in (28) and integrating over $(0,t)$, we get

$$
\begin{aligned}
&\frac{1}{2}\left[\left(1 - \int_0^t f(\sigma)\mathrm{d}\sigma \right) \|\nabla w^n(t)\|_2^2 + \|w_t^n(t)\|_2^2 + (f \circ \nabla w^n)(t) \right] \\
&- \frac{a_1}{4} \int_0^t \int_{-\infty}^{+\infty} \int_{\Omega} \left(y^2+\beta\right) (v^n(x,y,\sigma))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}\sigma \\
&+ \frac{1}{2} \int_0^t f(\sigma)\|\nabla w^n(\sigma)\|_2^2 \mathrm{d}\sigma - \frac{1}{2} \int_0^t (f' \circ \nabla w^n)(\sigma)\mathrm{d}\sigma \\
&+ \left(\mu_1 - \beta^{\rho-1}\mu_2 \right) \int_0^t \|w_\sigma^n(\sigma)\|_2^2 \mathrm{d}\sigma \leqslant \frac{1}{2}\|\nabla w_{0n}\|_2^2 + \frac{1}{2}\|w_{1n}\|_2^2. \qquad (30)
\end{aligned}
$$

Multiplying Eq. (24) by $a_1 v_{jn}(t)$, summing with respect to j and integrating over $(0,t) \times (-\infty,+\infty)$,we get:

$$
\begin{aligned}
&\frac{a_1}{2}\|v^n\|^2_{L^2(\Omega\times(-\infty,+\infty))} + a_1 \int_0^t \int_{-\infty}^{+\infty} \int_{\Omega} \left(y^2+\beta\right) (v^n(x,y,\sigma))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}\sigma \\
&\qquad -a_1 \int_0^t \int_{\Omega} u^n(x,1,\sigma) \int_{-\infty}^{+\infty} v(x,y,\sigma)\eta(y)\mathrm{d}y\mathrm{d}x\mathrm{d}\sigma = 0. \qquad (31)
\end{aligned}
$$

The last term on the left-hand side of (31) can be estimated as follows

$$\begin{aligned}
&\left|\int_0^t\int_\Omega u^n(x,1,\sigma)\int_{-\infty}^{+\infty} v^n(x,y,\sigma)\eta(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\sigma\right| \\
&\quad\leqslant\left(\int_{-\infty}^{+\infty}\frac{|\eta(y)|^2}{y^2+\beta}\mathrm{d}y\right)\int_0^t\int_\Omega (u^n(x,1,\sigma))^2\mathrm{d}x\,\mathrm{d}\sigma \\
&\qquad+\frac{1}{4}\int_0^t\int_{-\infty}^{+\infty}\int_\Omega\left(y^2+\beta\right)(v^n(x,y,\sigma))^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\sigma. \qquad (32)
\end{aligned}$$

Consequently, equation (31) becomes

$$\begin{aligned}
&\frac{a_1}{2}\|v^n\|^2_{L^2(\Omega\times(-\infty,+\infty))}-\beta^{\rho-1}\mu_2\int_0^t\int_\Omega (u^n(x,1,\sigma))^2\mathrm{d}x\,\mathrm{d}\sigma \\
&\quad+\frac{3a_1}{4}\int_0^t\int_{-\infty}^{+\infty}\int_\Omega\left(y^2+\beta\right)(v^n(x,y,\sigma))^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\sigma\leqslant 0. \qquad (33)
\end{aligned}$$

Multiplying Eq. (26) by $\alpha u_{n,j}(t)$, summing with respect to j, and integrating over $(0,1)\times(0,t)$, we have:

$$\begin{aligned}
\frac{\alpha}{2}\|u^n\|^2_{L^2(\Omega\times(0,1))}&=-\frac{\alpha}{\tau}\int_0^t\int_0^1\int_\Omega u^n(x,\xi,\sigma)u^n_\xi(x,\xi,\sigma)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\sigma-\frac{\alpha}{2}\|u_{n0}\|^2_{L^2(\Omega\times(0,1))} \\
&=-\frac{\alpha}{2\tau}\int_0^t\int_\Omega\left[(u^n(x,1,\sigma))^2-(u^n(x,0,\sigma))^2\right]\mathrm{d}x\,\mathrm{d}\sigma \\
&\quad-\frac{\alpha}{2}\|u_{n0}\|^2_{L^2(\Omega\times(0,1))}. \qquad (34)
\end{aligned}$$

From (30), (33) and (34), we obtain

$$\begin{aligned}
&\mathcal{E}^n(t)+\left(\mu_1-\beta^{\rho-1}\mu_2-\frac{\alpha}{2\tau}\right)\int_0^t \|w_\sigma^n(\sigma)\|_2^2\mathrm{d}\sigma\\
&\quad+\left(\frac{\alpha}{2\tau}-\beta^{\rho-1}\mu_2\right)\int_0^t\int_\Omega (u^n(x,1,\sigma))^2\mathrm{d}x\mathrm{d}\sigma\\
&\quad+\frac{1}{2}\int_0^t f(\sigma)\|\nabla w^n(\sigma)\|_2^2\mathrm{d}\sigma-\frac{1}{2}\int_0^t (f'\circ\nabla w^n)(\sigma)\mathrm{d}\sigma\\
&\quad+\frac{a_1}{2}\int_0^t\int_{-\infty}^{+\infty}\int_\Omega \left(y^2+\beta\right)(v^n(x,y,\sigma))^2\mathrm{d}x\mathrm{d}y\mathrm{d}\sigma\leqslant\mathcal{E}^n(0),
\end{aligned}\tag{35}$$

where,

$$\begin{aligned}
\mathcal{E}^n(t)&=\frac{1}{2}\left[\left(1-\int_0^t f(\sigma)\mathrm{d}\sigma\right)\|\nabla w^n(t)\|_2^2+\|w_t^n(t)\|_2^2+(f\circ\nabla w^n)(t)\right]\\
&\quad+\frac{a_1}{2}\|v^n\|^2_{L^2(\Omega\times(-\infty,+\infty))}+\frac{\alpha}{2}\|u^n\|^2_{L^2(\Omega\times(0,1))}.
\end{aligned}\tag{36}$$

At this point, we suppose that $\beta^{\rho-1}\mu_2<\mu_1$ and us choose then α that satisfies inequality (13). Then, we can find two positive constants c_1 and c_2 such that:

$$\begin{aligned}
&\mathcal{E}^n(t)+c_1\int_0^t \|w_\sigma^n(\sigma)\|_2^2\mathrm{d}\sigma c_2\int_0^t\int_\Omega (u^n(x,1,\sigma))^2\mathrm{d}x\mathrm{d}\sigma\\
&\quad+\frac{1}{2}\int_0^t f(\sigma)\|\nabla w^n(\sigma)\|_2^2\mathrm{d}\sigma-\frac{1}{2}\int_0^t (f'\circ\nabla w^n)(\sigma)\mathrm{d}\sigma\\
&\quad+\frac{a_1}{2}\int_0^t\int_{-\infty}^{+\infty}\int_\Omega \left(y^2+\beta\right)(v^n(x,y,\sigma))^2\mathrm{d}x\mathrm{d}y\mathrm{d}\sigma\leqslant\mathcal{E}^n(0).
\end{aligned}\tag{37}$$

Now, in both cases and since the sequences $(w_{0n})_{n\in\mathbb{N}}$,$(w_{1n})_{n\in\mathbb{N}}$ and $(u_{0n})_{n\in\mathbb{N}}$ converge, we can find a positive constant C independent of n such that

$$\mathcal{E}^n(t)\leqslant C.\tag{38}$$

So, inequality (38) together with (36) give us, for all $n \in \mathbb{N}$, $t_n = T$; we deduce

$$\begin{aligned}
&(w^n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty\left(0,T;H_0^1(\Omega)\right),\\
&(w_t^n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right),\\
&(v^n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty\left(0,T;L^2\left((\Omega)\times(-\infty,+\infty)\right)\right),\\
&(u^n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty\left(0,T;L^2\left((\Omega)\times(0,1)\right)\right).
\end{aligned} \tag{39}$$

Consequently, we can deduce the following conclusions:

$$\begin{aligned}
w^n &\rightharpoonup \text{ weak* } w \text{ in } L^\infty\left(0,T;H_0^1(\Omega)\right),\\
w_t^n &\rightharpoonup \text{ weak* } w_t \text{ in } L^\infty\left(0,T;L^2(\Omega)\right)\\
v^n &\rightharpoonup \text{ weak* } v \text{ in } L^\infty\left(0,T;L^2\left((\Omega)\times(-\infty,+\infty)\right)\right)\\
u^n &\rightharpoonup \text{ weak* } u \text{ in } L^\infty\left(0,T;L^2\left((\Omega)\times(0,1)\right)\right).
\end{aligned}$$

From (39), we have $(w^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$. Then, $(w^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H_0^1(\Omega))$. Since $(w_t^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, then $(w_t^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, $(u^n)_{n\in\mathbb{N}}$ is bounded in $H^1(0,T;H^1(\Omega))$. Since the embedding $H^1(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$ is compact, using Aubin-Lions theorem [20], we can extract a subsequence $(w^{n'})_{n'\in\mathbb{N}}$ of $(w^n)_{n\in\mathbb{N}}$ such that

$$w^{n'} \to w \text{ strongly } L^2\left(0,T;L^2(\Omega)\right),$$

therefore,

$$w^{n'} \to \varphi \text{ strongly and a.e on}(0,T)\times(\Omega).$$

The proof now can be completed by arguing as in [20]. □

The Faedo-Galerkin method demonstrates notable efficacy within specific contextual domains, particularly in the treatment of non-linear problems. The rationale behind its application in the present scenario may be rooted in the congruence between the inherent attributes of the method and specific facets of the problem or the sought-after solution. An elucidation of the merits and discernments derived from the employment of the Faedo-Galerkin method would be advantageous. Such an exposition should delineate how this methodological approach significantly contributes to the comprehension and resolution of the linear problem under consideration. It is imperative to recognize that the selection of a particular methodological framework transcends conventional adherence and extends to the judicious identification of the most apt tool for the analytical task at hand.

4. Asymptotic behavior

In this section, our aim is to ascertain the convergence of the solution to the problem (19)-(20) towards the trivial steady state under the condition $\beta^{\rho-1}\mu_2 < \mu_1$. To attain this objective, we will employ the energy method in conjunction with the selection of an apt Lyapunov functional.

To define the energy functional for problem (19)–(20), we introduce a positive constant α that satisfies the inequality (14). The energy functional is then defined as follows:

$$\mathcal{E}(t) = \frac{1}{2}\left[\left(1 - \int_0^t f(\sigma)\mathrm{d}\sigma\right)\|\nabla w(t)\|_2^2 + \|w_t(t)\|_2^2 + (f \circ \nabla w)(t)\right] + \frac{a_1}{2}\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega\times(0,1))}. \tag{40}$$

Our objective now is to establish that the energy $\mathcal{E}(t)$ is a monotonically decreasing function along the trajectories. Specifically, we have the following result:

Lemma 5. *Suppose that (A0) and (A1) hold and let (u, φ, z) be a solution of the problem (19)- (20). Then, the energy functional defined by (40) is a nonincreasing function, that is there exists a positive constant C such that*

$$\mathcal{E}'(t) \leqslant -C\left(\|w_t(t)\|_2^2 + \int_\Omega (u(x,1,t))^2\mathrm{d}x\right) - \frac{1}{2}f(t)\|\nabla w(t)\|_2^2 + \frac{1}{2}(f' \circ \nabla w)(t) - \frac{a_1}{2}\int_{-\infty}^{+\infty}\int_\Omega \left(y^2+\beta\right)|v(x,y,t)|^2\mathrm{d}x\mathrm{d}y \leqslant 0, \quad \forall t \geqslant 0. \tag{41}$$

Proof. Multiplying the first equation in (19) by w_t, integrating over Ω and using integration by parts, we get 1, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left[\|\nabla w(t)\|_2^2+\|w_t(t)\|_2^2\right]+\mu_1\|w_t(t)\|_2^2$$
$$+a_1\int_\Omega w_t(x,t)\int_{-\infty}^{+\infty}v^n(x,y,t)\eta(y)\,\mathrm{d}y\,\mathrm{d}x$$
$$=\int_0^t f(t-\sigma)\int_\Omega \nabla w(t)\nabla w(\sigma)\,\mathrm{d}x\,\mathrm{d}\sigma. \tag{42}$$

Now, using Lemma 1, the first term in the right-hand side of (42) can be rewritten as follows

$$\int_0^t f(t-\sigma)\int_\Omega \nabla w(t)\nabla w(\sigma)\,\mathrm{d}x\,\mathrm{d}\sigma+\frac{1}{2}f(t)\|\nabla w(t)\|_2^2$$
$$=\frac{1}{2}\frac{d}{dt}\left[\int_0^t f(\sigma)\,\mathrm{d}\sigma\|\nabla w(t)\|_2^2-(f\circ\nabla w)(t)\right]-\frac{1}{2}(f'\circ\nabla w)(t). \tag{43}$$

Consequently, equality (42) becomes

$$\frac{1}{2}\frac{d}{dt}\left[\left(1-\int_0^t f(\sigma)\,\mathrm{d}\sigma\right)\|\nabla w(t)\|_2^2+\|w_t(t)\|_2^2+(f\circ\nabla w)(t)\right]$$
$$=-\mu_1\|w_t(t)\|_2^2-a_1\int_\Omega w_t(x,t)\int_{-\infty}^{+\infty}v^n(x,y,t)\eta(y)\,\mathrm{d}y\,\mathrm{d}x$$
$$-\frac{1}{2}f(t)\|\nabla w(t)\|_2^2+\frac{1}{2}(f'\circ\nabla w)(t). \tag{44}$$

We multiply the second equation in (19) by $a_1 v$ and integrate the result over $\Omega\times(\infty,+\infty)$, to obtain:

$$\frac{a_1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))}=-a_1\int_{-\infty}^{+\infty}\int_\Omega\left(y^2+\beta\right)(y^n(x,y,t))^2\,\mathrm{d}x\,\mathrm{d}y$$
$$+a_1\int_\Omega u(x,1,t)\int_{-\infty}^{+\infty}v(x,y,t)\eta(y)\,\mathrm{d}y\,\mathrm{d}x. \tag{45}$$

We multiply the second equation in (19) by αu and integrate the result over $\Omega \times (0,1)$, to obtain:

$$\frac{\alpha}{2}\frac{d}{dt}\int_0^1\int_\Omega (u(x,\xi,t))^2 \mathrm{d}x d\xi = -\frac{\alpha}{2\tau}\int_0^1\int_\Omega \frac{\partial}{\partial \xi}\Big(u(x,\xi,t)\Big)^2 \mathrm{d}x\mathrm{d}\xi$$
$$= -\frac{\alpha}{2\tau}\int_\Omega \Big[(u(x,1,t))^2 - (u(x,0,t))^2\Big]\mathrm{d}x. \quad (46)$$

From (44), (45), (46), using the equation (17) and Young inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) = &-\left[\mu_1 - \beta^{\rho-1}\mu_2 - \frac{\alpha}{2}\right]\|w_t(t)\|_2^2 \\ &- \left[\frac{\alpha}{2\tau} - \beta^{\rho-1}\mu_2\right]\int_\Omega (u(x,1,t))^2 \mathrm{d}x \\ &- \frac{1}{2}f(t)\|\nabla w(t)\|_2^2 + \frac{1}{2}(f' \circ \nabla w)(t) \\ &- \frac{a_1}{2}\int_{-\infty}^{+\infty}\int_\Omega \Big(y^2+\beta\Big)\,|v(x,y,t)|^2 \mathrm{d}x\mathrm{d}y. \end{aligned} \quad (47)$$

Then, using (14) our conclusion holds. □

Our stability result reads as follows:

Theorem 2. *Let w be the solution of (1). Assume that $\beta^{\rho-1}\mu_2 < \mu_1$ and f satisfies $(A0)$ and $(A1)$. Then, there exist two positive constants K and λ such that the energy of problem (1) satisfies*

$$\mathcal{E}(t) \leqslant K e^{-\lambda \int_0^t \zeta(s)ds}, \quad \forall t \geqslant 0. \quad (48)$$

The proof of Theorem 2 will be carried out by utilizing several Lemmas. We will introduce a functional $\mathcal{L}(t)$, which is equivalent to the energy $\mathcal{E}(t)$, and satisfies the following condition:

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\gamma\mathcal{L}(t), \quad \forall t \geqslant 0,$$

where γ is a positive constant. In order to construct such functional, let us first define the following

$$I_1(t) = \int_0^t w w_t \mathrm{d}x. \quad (49)$$

Then, we have the following estimate.

Lemma 6. *Let (w, v, u) be the solution of (19)-(20), then for any $\alpha_1 > 0$, we have*

$$\frac{d}{dt} I_1(t) \leqslant \left(1 + \frac{\mu_1}{4\alpha_1}\right) \|w_t\|_2^2 - \left(\frac{l}{2} - \alpha_1 B_{1,\Omega}^2 \left(\mu_1 + \beta^{\rho-1}\mu_2\right)\right) \|\nabla w\|_2^2$$
$$+ \frac{a_1}{4\alpha_1} \int_\Omega \int_{-\infty}^{+\infty} \left(y^2 + \beta\right) |v(x, y, t)|^2 dy dx + \frac{1-l}{2l} (f \circ \nabla w)(t). \tag{50}$$

Proof. Using the first equation in (19), a direct computation leads to the following identity

$$I_1'(t) = \|w_t\|_2^2 - \|\nabla w\|_2^2 + \int_\Omega \nabla w(t) \int_0^t f(t-\sigma) \nabla w(\sigma) ds dx \tag{51}$$
$$- \mu_1 \int_\Omega w_t w dx - a_1 \int_\Omega w \int_{-\infty}^{+\infty} v(x, y, t) \eta(y) dy dx. \tag{52}$$

Now, the third term in the right-hand side of (51) can be estimated as follows:

$$\int_\Omega \nabla w(t) \int_0^t f(t-\sigma) \nabla w(\sigma) d\sigma dx \leqslant \frac{1}{2} \|\nabla w(t)\|_2^2$$
$$+ \frac{1}{2} \int_\Omega \left(\int_0^t f(t-\sigma) \nabla w(\sigma) d\sigma \right)^2 dx$$
$$\leqslant \frac{1}{2} \|\nabla w(t)\|_2^2 + \frac{1}{2} \int_\Omega \left(\int_0^t f(t-\sigma) |\nabla w(\sigma) - \nabla w(t)| + |\nabla w(t)| d\sigma \right)^2 dx.$$

Using the estimate (15) in Lemma 2, Young's inequality and the fact that $\int_0^t f(\sigma) d\sigma \leqslant \int_0^\infty f(\sigma) d\sigma = 1 - l$, We get for any $\nu > 0$, (see relation (20) in [17]).

$$\begin{aligned}
&\int_\Omega \left(\int_0^t f(t-\sigma)|\nabla w(\sigma) - \nabla w(t)| + |\nabla w(t)| \mathrm{d}\sigma \right)^2 \mathrm{d}x \\
&\quad \leqslant (1+\nu) \int_\Omega \left(\int_0^t f(t-\sigma)|\nabla w(t)| \mathrm{d}\sigma \right)^2 \mathrm{d}s \\
&\qquad + \left(1 + \frac{1}{\nu}\right) \int_\Omega \left(\int_0^t f(t-\sigma)|\nabla w(\sigma) - \nabla w(t)| \mathrm{d}\sigma \right)^2 \mathrm{d}x \\
&\quad \leqslant \left(1 + \frac{1}{\nu}\right)(1-l)(f \circ \nabla w)(t) + (1+\nu)(1-l)^2 \|\nabla w(t)\|_2^2 . \qquad (53)
\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}
\int_\Omega \nabla w(t) \int_0^t f(t-\sigma)\nabla w(\sigma) \mathrm{d}\sigma \mathrm{d}x \leqslant{} & \frac{1}{2}\left(1 + \frac{1}{\nu}\right)(1-l)(f \circ \nabla w)(t) \\
& + \frac{1}{2}\left(1 + (1+\nu)(1-l)^2\right) \|\nabla w(t)\|_2^2 . \qquad (54)
\end{aligned}$$

Next, Young inequality and Poincaré's inequality imply that, for any $\delta_1 > 0$

$$\int_\Omega w_t w \,\mathrm{d}x \leqslant \delta_1 B_{1,\Omega}^2 \int_\Omega |\nabla w(t)|^2 \mathrm{d}x + \frac{1}{4\alpha_1}\|w_t(t)\|_2^2, \qquad (55)$$

and

$$\begin{aligned}
\int_\Omega w \int_{-\infty}^{+\infty} v(x,y,t)\eta(y) \mathrm{d}y \mathrm{d}x \leqslant{} & \alpha_1 B_{1,\Omega}^2 \left(\int_{-\infty}^{+\infty} \frac{\eta^2(y)}{y^2+\beta} \mathrm{d}y \right) \|\nabla w(t)\|_2^2 \\
& + \frac{1}{4\alpha_1} \int_\Omega \int_{-\infty}^{+\infty} (y^2+\beta)|v(x,y,t)|^2 \mathrm{d}y \mathrm{d}x. \qquad (56)
\end{aligned}$$

By inserting the estimates (54), (55) and (56) into (51) and choosing $\nu = l/(1-l)$, then (50) holds. □

For the second lemma, we introduce the functional

$$I_2(t) := \frac{a_1}{2}\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \frac{a_1}{2}\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)\left[\int_0^t y(x,y,\sigma)\,\mathrm{d}\sigma - \frac{\tau\eta(y)}{y^2+\beta}\int_0^1 g_0(x,t-\tau)\,\mathrm{d}\xi + \frac{\eta(y)w_0}{y^2+\beta}\right]^2 \mathrm{d}y\,\mathrm{d}x. \tag{57}$$

It satisfies an estimate stated in the

Lemma 7. *Let (w,v,u) be a solution of (19)and (20), then for any $\alpha_1>0$, we have*

$$I_2'(t) \leqslant \left(-a_1+\frac{3a_1}{4\alpha_1}\right)\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)v^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x + \beta^{\rho-1}\mu_2\alpha_1\int_\Omega|u(x,1,t)|^2\mathrm{d}x - a_1\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \beta^{\rho-1}\mu_2\alpha_1 B^2_{1,\Omega}\|\nabla w\|_2^2 + \beta^{\rho-1}\mu_2\tau^2\alpha_1\int_\Omega\int_0^1|u(x,\xi,t)|^2\mathrm{d}\xi\,\mathrm{d}x. \tag{58}$$

Proof. By taking the time derivative of (57), we get

$$I_2'(t) = -a_1\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)v^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x + a_1\int_\Omega u(x,1,t)\int_{-\infty}^{+\infty}\eta(y)v(x,y,t)\,\mathrm{d}y\,\mathrm{d}x + a_1\int_\Omega\int_{-\infty}^{+\infty}\left\{(y^2+\beta)v(x,y,t)\left[\int_0^t v(x,y,\sigma)\,\mathrm{d}\sigma - \frac{\tau\eta(y)}{y^2+\beta}\int_0^1 g_0(x,t-\tau)\,\mathrm{d}\xi + \frac{\eta(y)w_0}{y^2+\beta}\right]\right\}\mathrm{d}y\,\mathrm{d}x.$$

Using the second equation in (19), we have

$$\int_0^t v(x,y,\sigma)\,\mathrm{d}\sigma = -\frac{v(x,y,t)}{y^2+\beta} + \frac{\eta(y)}{y^2+\beta}\int_0^t u(x,1,\sigma)\,\mathrm{d}\sigma. \tag{59}$$

Using the third equation in (19), the last term in the left-hand side of (59) can be handled as

$$\int_0^t u(x,1,\sigma)\,\mathrm{d}\sigma = -\tau\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi + \tau\int_0^1 g_0(x,t-\tau)\,\mathrm{d}\xi + w(x,t) - w_0\,. \tag{60}$$

Consequently, using Lemma 4, we arrive at

$$\begin{aligned} I_2'(t) = &- a_1\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)v^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x - a_1\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))} \\ &+ a_1\int_\Omega u(x,1,t)\int_{-\infty}^{+\infty}\eta(y)v(x,y,t)\,\mathrm{d}y\,\mathrm{d}x \\ &- a_1\tau\int_\Omega\int_{-\infty}^{+\infty}\eta(y)v(x,y,t)\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}x \\ &+ a_1\int_\Omega\int_{-\infty}^{+\infty}\eta(y)v(x,y,t)w(x,t)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{61}$$

Using Young and Poicarè inequalities, we obtain (61).

$$\begin{aligned} a_1\int_\Omega u(x,1,t)\int_{-\infty}^{+\infty}\eta(y)v(x,y,t)\,\mathrm{d}y\,\mathrm{d}x \leqslant{}& \alpha_1\beta^{\rho-1}\mu_2\int_\Omega |u(x,1,t)|^2\,\mathrm{d}x \\ &+ \frac{a_1}{4\alpha_1}\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)v^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{62}$$

$$a_1\tau\int_\Omega\int_{-\infty}^{+\infty}\eta(y)v(x,y,t)\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}x \leqslant \tau^2\alpha_1\beta^{\rho-1}\mu_2\int_\Omega\int_0^1|u(x,\xi,t)|^2\mathrm{d}\xi\,\mathrm{d}x$$
$$+\frac{a_1}{4\alpha_1}\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)v^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x. \tag{63}$$

$$a_1\int_\Omega\int_{-\infty}^{+\infty}\eta(y)v(x,y,t)w(x,t)\,\mathrm{d}y\,\mathrm{d}x \leqslant \alpha_1 B_{1,\Omega}^2\beta^{\rho-1}\mu_2\|\nabla u(t)\|_2^2$$
$$+\frac{a_1}{4\alpha_1}\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)v^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x. \tag{64}$$

By inserting the estimates (62), (63)and (64)into (61) , we have

$$\begin{aligned}I_2'(t) \leqslant{}& \left(-a_1+\frac{3a_1}{4\alpha_1}\right)\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)v^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x\\
&+\beta^{\rho-1}\mu_2\alpha_1\int_\Omega|u(x,1,t)|^2\mathrm{d}x - a_1\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))}\\
&+\beta^{\rho-1}\mu_2\alpha_1 B_{1,\Omega}^2\|\nabla u\|_2^2+\beta^{\rho-1}\mu_2\tau^2\alpha_1\int_\Omega\int_0^1|u(x,\xi,t)|^2\mathrm{d}\xi\,\mathrm{d}x.\end{aligned} \tag{65}$$

For the third lemma, we introduce the functional

$$I_3(t) := \int_\Omega\int_0^1 e^{-2\tau\xi}u^2(x,\xi,t)\,\mathrm{d}\xi\,\mathrm{d}x. \tag{66}$$

It satisfies an estimate stated in the below Lemma:

Lemma 8. *Let (w,v,u) be a solution of (19)-(20). Then we have*

$$\frac{dI_3(t)}{dt} \leqslant -\xi I_3(t)-\frac{\gamma_1}{2\tau}\int_\Omega|u(x,1,t)|^2\mathrm{d}x+\frac{1}{2\tau}\int_\Omega|u(x,0,t)|^2\mathrm{d}x. \tag{67}$$

Proof. Differentiating (66), we obtain

$$\frac{dI_3}{dt}(t) = -\frac{1}{\tau}\int_\Omega\int_0^1 e^{-2\tau\xi} u u_\xi(x,\xi,t)\,\mathrm{d}\xi\,\mathrm{d}x$$
$$= -\int_\Omega\int_0^1 \xi e^{-2\tau\xi}|u(x,\xi,t)|^2\mathrm{d}\xi\,\mathrm{d}x - \frac{1}{2\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\xi}\left(e^{-2\tau\xi}|u(x,\xi,t)|^2\right)\mathrm{d}\xi\,\mathrm{d}x.$$

Then, there exists a positive constant α_2 such that (67) holds.

For the last lemma, we introduce the functional

$$I_4(t) := -\int_\Omega w_t \int_0^t f(t-\sigma)\left(w(t)-w(\sigma)\right)\mathrm{d}\sigma\,\mathrm{d}x. \tag{68}$$

Next, we prove the following Lemma for further analysis:

Lemma 9. *Let (w, v, u) be the solution of (19)-(20), then, we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_4(t) \leqslant{} & \left(\alpha_3(1+\mu_1) - \int_0^t f(s)\mathrm{d}s\right)\|w_t\|_2^2 + \left(\alpha_2 + 2\alpha_2(1-l)^2\right)\|\nabla u\|_2^2 \\ & + \alpha_4 a_1 \int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)|v(x,y,t)|^2\mathrm{d}y\,\mathrm{d}x - \frac{B_{1,\Omega}^2 f(0)}{4\alpha_3}(f'\circ\nabla w)(t) \\ & + \left(\frac{\beta^{\rho-1}\mu_2(1-l)B_{1,\Omega}^2}{4\alpha_4} + \frac{B_{1,\Omega}^2\mu_1}{4\alpha_3} + \frac{1-l}{2\alpha_2} + 2\alpha_2(1-l)\right)(f\circ\nabla w)(t),\end{aligned} \tag{69}$$

where α_i, $(i = 2, 3, 4)$ are arbitrary positive constants.

Proof. By derivative of ψ, we obtain

$$\begin{aligned}I_4'(t) ={} & -\int_\Omega w_{tt}(t)\int_0^t f(t-\sigma)\left(w(t)-w(\sigma)\right)\mathrm{d}\sigma\,\mathrm{d}x \\ & - \int_\Omega w_t(t)\int_0^t f'(t-\sigma)\left(w(t)-w(\sigma)\right)\mathrm{d}s\,\mathrm{d}x \\ & - \left(\int_0^t f(\sigma)\mathrm{d}\sigma\right)\|w_t(t)\|_2^2.\end{aligned} \tag{70}$$

Using problem (19) and using integration by parts over Ω, we get

$$
\begin{aligned}
I_4'(t) = & \int_\Omega \nabla w(t) \int_0^t f(t-\sigma)\left(\nabla w(t) - \nabla w(\sigma)\right) \mathrm{d}\sigma \mathrm{d}x \\
& - \int_\Omega \left\{ \int_0^t f(t-\sigma) \nabla w(\sigma) \mathrm{d}s \right\} \left\{ \int_0^t f(t-\sigma)\left(\nabla w(t) - \nabla w(\sigma)\right) \mathrm{d}\sigma \right\} \mathrm{d}x \\
& - \int_\Omega w_t(t) \int_0^t f'(t-\sigma)\left(w(t) - w(\sigma)\right) \mathrm{d}\sigma \mathrm{d}x - \left(\int_0^t f(\sigma) \mathrm{d}\sigma \right) \|w_t(t)\|_2^2 \\
& + a_1 \int_\Omega \left\{ \int_{-\infty}^{+\infty} \mu(y) v(x,y,t) \mathrm{d}y \right\} \left\{ \int_0^t f(t-\sigma)\left(w(t) - w(\sigma)\right) \mathrm{d}\sigma \right\} \mathrm{d}x \\
& + \mu_1 \int_\Omega w_t(t) \int_0^t f(t-\sigma)\left(w(t) - w(\sigma)\right) \mathrm{d}\sigma \mathrm{d}x.
\end{aligned} \tag{71}
$$

Similarly as in (69), we estimate the right-hand side terms of (71) as follows: First, using Young inequality and (15), we obtain for any $\alpha_2 > 0$,

$$
\begin{aligned}
& \left| \int_\Omega \nabla w(t) \int_0^t f(t-s)\left(\nabla w(t) - \nabla w(s)\right) \mathrm{d}s \mathrm{d}x \right| \\
& \qquad \leqslant \alpha_2 \|\nabla w(t)\|_2^2 + \frac{(1-l)}{4\alpha_2} \left(f \circ \nabla w\right)(t).
\end{aligned} \tag{72}
$$

Also, the second term can be estimated as follows (see [14])

$$
\begin{aligned}
& \left| \int_\Omega \left\{ \int_0^t f(t-s) \nabla w(s) \mathrm{d}s \right\} \left\{ \int_0^t f(t-s)\left(\nabla w(t) - \nabla w(s)\right) \mathrm{d}s \right\} \mathrm{d}x \right| \\
& \qquad \leqslant 2\alpha_2 (1-l)^2 \|\nabla w(t)\|_2^2 + (2\alpha_2 + \frac{1}{4\alpha_2})(1-l)\left(f \circ \nabla w\right)(t).
\end{aligned} \tag{73}
$$

Concerning the third term, we have for $\alpha_3 > 0$,

$$\left|\int_\Omega w_t(t)\int_0^t f'(t-s)\left(w(t)-w(s)\right)\mathrm{d}s\mathrm{d}x\right| \leqslant \alpha_3\|w_t(t)\|_2^2 - \frac{B_{1,\Omega}^2 f(0)}{4\alpha_3}\left(f'\circ\nabla w\right)(t). \tag{74}$$

The fifth term can be estimated as follows:

$$\begin{aligned}
&\int_\Omega\left\{\int_{-\infty}^{+\infty}\mu(y)v(x,y,t)\mathrm{d}y\right\}\left\{\int_0^t f(t-s)\left(w(t)-w(s)\right)\mathrm{d}s\right\}\mathrm{d}x\\
&\leqslant \frac{1}{4\alpha_4}\int_\Omega\left\{\int_{-\infty}^{+\infty}\frac{\mu^2(y)}{y^2+\beta}\mathrm{d}y\right\}\left\{\int_0^t f(t-s)\left(w(t)-w(s)\right)\mathrm{d}s\right\}^2\mathrm{d}x\\
&\quad+\alpha_4\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)|v(x,y,t)|^2\mathrm{d}y\mathrm{d}x\\
&\leqslant \frac{A_0B_{1,\Omega}^2(1-l)}{4\alpha_4}\left(f\circ\nabla w\right)(t)\\
&\quad+\alpha_4\int_\Omega\int_{-\infty}^{+\infty}(y^2+\beta)|v(x,y,t)|^2\mathrm{d}y\mathrm{d}x,\quad \alpha_4>0.
\end{aligned}\tag{75}$$

For the sixth term, we have

$$\left|\int_\Omega w_t(t)\int_0^t f'(t-s)\left(\nabla w(t)-\nabla w(s)\right)\mathrm{d}s\mathrm{d}x\right| \leqslant \alpha_3\|w_t(t)\|_2^2 + \frac{B_{1,\Omega}^2}{4\alpha_3}\left(f\circ\nabla w\right)(t). \tag{76}$$

Inserting the above estimates (72)–(76) into (71), the assertion of the Lemma 9 is established. □

Proof. of Theorem 2, we define the following Lyapunov function $\mathcal{L}$ as:

$$\mathcal{L}(t) = d_1E(t) + d_2I_1(t) + d_2I_2(t) + d_3I_3(t) + I_4(t), \tag{77}$$

where d_i, $(i = 1, 2, 3)$ are positive real numbers which will be chosen later.

Since the function f is positive and continuous and $f(0) > 0$, then for all $t_0 > 0$, we get

$$\int_0^t f(s)\mathrm{d}s \geqslant \int_{t_0}^t f(s)\mathrm{d}s = f_0 > 0, \quad \forall t \geqslant t_0 . \tag{78}$$

Now, using (50), (58), (67) and (69), we get, for all $t \geqslant t_0$,

$$\begin{aligned}
\mathcal{L}'(t) \leqslant & \left(-Cd_1 + d_2\left(1 + \frac{\mu}{4\alpha_1}\right) + (\alpha_3(1+\mu) - f_0) + \frac{d_3}{2\tau}\right)\|w_t\|_2^2 \\
& + \left(\alpha_2\left(1 + 2(1-l)^2\right) - d_2\left(\frac{l}{2} - \alpha_1 B_{1,\Omega}^2(\mu_1 + 2\beta^{\rho-1}\mu_2)\right)\right)\|\nabla w\|_2^2 \\
& + a_1\left(\frac{-d_1}{2} + \alpha_4\right)\int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta)|v(x,y,t)|^2 \mathrm{d}y\mathrm{d}x - d_2 a_1 \|v\|^2_{L^2(\Omega\times(-\infty,+\infty))} \\
& + \left(\frac{d_1}{2} - \frac{B_{1,\Omega}^2 f(0)}{4\alpha_3}\right)(f' \circ \nabla w)(t) \\
& + \left(-Cd_1 + d_2\beta^{\rho-1}\mu_2\alpha_1 - \frac{d_3\gamma_1}{2\tau}\right)\int_\Omega u^2(x,1,t)\mathrm{d}x \\
& + \left(\frac{(1-l)d_2}{2l} + \left(\frac{B_{1,\Omega}^2\mu_1}{4\alpha_3} + \frac{\beta^{\rho-1}\mu_2(1-l)B_{1,\Omega}^2}{4\alpha_4}\right.\right. \qquad (79) \\
& \left.\left. + \frac{1-l}{2\alpha_2} + 2\alpha_2(1-l)\right)\right)(f \circ \nabla w)(t) \\
& + \left(-d_3\xi e^{-2\tau} + d_2\beta^{\rho-1}\mu_2\tau^2\alpha_1\right)\int_\Omega \int_0^1 u^2(x,\xi,t)\mathrm{d}\xi\mathrm{d}x. \qquad (80)
\end{aligned}$$

At this point, we choose α_1 small that

$$\alpha_1 B_{1,\Omega}^2(\mu_1 + 2\beta^{\rho-1}\mu_2) \leqslant \frac{l}{4} .$$

Now, we pick d_2 large that

$$d_2\left(\frac{l}{2} - \alpha_1 B_{1,\Omega}^2(\mu_1 + 2\beta^{\rho-1}\mu_2)\right) \geqslant 2\alpha_2\left(1 + 2(1-l)^2\right.$$

After that, we pick d_3 large that

$$d_3 \geqslant \max\left\{\frac{\tau d_2\beta^{\rho-1}\mu_2\alpha_1}{\gamma_1}, \frac{d_2\beta^{\rho-1}\mu_2\tau^2\alpha_1}{2\xi e^{-2\tau}}\right\}.$$

Then, we select α_3 large enough that

$$\alpha_3(1+\mu) > 2f_0 .$$

Finally, we choose d_1 large enough such that

$$\frac{d_1}{2} \geqslant \max\left\{\frac{d_2\left(1+\frac{\mu}{4\alpha_1}\right)+(\alpha_3(1+\mu)-f_0)+\frac{d_3}{2\tau}}{C}, \frac{d_2\beta^{\rho-1}\mu_2\alpha_1-\frac{d_3\gamma_1}{2\tau}}{C}, .\right.$$
$$\left. 2\alpha_4, \frac{B^2_{1,\Omega}f(0)}{8\alpha_3}\right\}.$$

Consequently, from the above, we deduce that there existe tow positive constants m and c_2 such that (79) becomes

$$\mathcal{L}'(t) \leqslant -m\mathcal{E}(t) + c_2\left(f\circ\nabla w\right)(t), \quad \text{for all } t \geqslant t_0. \tag{81}$$

Multiplying (81), by $\zeta(t)$, we arrive at

$$\zeta(t)\mathcal{L}'(t) \leqslant -m\zeta(t)\mathcal{E}(t) + c_2\zeta(t)\left(f\circ\nabla w\right)(t), \quad \text{for all } t \geqslant t_0 .$$

Recalling (A1) and using (5), we get

$$\begin{aligned}\zeta(t)\mathcal{L}'(t) &\leqslant -m\zeta(t)\mathcal{E}(t) - c_2\left(f'\circ\nabla w\right)(t)\\ &\leqslant -m\zeta(t)\mathcal{E}(t) - 2c_2\mathcal{E}'(t), \quad \text{for all } t \geqslant t_0.\end{aligned}$$

That is

$$(\zeta(t)\mathcal{L}(t) + 2c_2\mathcal{E}(t))' - \zeta'(t)\mathcal{L}(t) \leqslant -m\zeta(t)\mathcal{E}(t), \quad \text{for all } t \geqslant t_0 .$$

Using the fact that $\zeta'(t) \leqslant 0$ and letting

$$G(t) = \zeta(t)\mathcal{L}(t) + 2c_2\mathcal{E}(t) \sim \mathcal{E}(t),$$

we have

$$G'(t) \leqslant -m\zeta(t)\mathcal{E}(t) \leqslant -w\zeta(t)G(t), \quad \forall t \geqslant t_0. \tag{82}$$

A simple integration of (82) over (t_0, t) leads to

$$G(t) \leqslant G(t_0)e^{-w\int_{t_0}^{t}\zeta(s)\,\mathrm{d}s}, \quad t \geqslant t_0.$$

Using G(t) and $\mathcal{E}(t)$ are equivalent, we obtein

$$\mathcal{E}(t) \leqslant k e^{-w\int_{t_0}^{t}\zeta(s)\,\mathrm{d}s}, \quad t \geqslant t_0. \tag{83}$$

The final step in proving Theorem 2 is to establish the equivalence between $\mathcal{L}(t)$ and $\mathcal{E}(t)$. To accomplish this, we present the following lemma.

Lemma 10. *For any (w, v, u) solution of problem (19), the following inequality holds:*

$$\epsilon_1\mathcal{E}(t) \leqslant \mathcal{L}(t) \leqslant \epsilon_2\mathcal{E}(t), \tag{84}$$

where ϵ_1 and ϵ_2 are positive constants.

Proof. We consider the functional

$$H(t) = d_2 I_1(t) + d_2 I_2(t) + d_2 I_3(t) + I_4(t) \tag{85}$$

and show that

$$|H(t)| \leqslant C\mathcal{E}(t), \quad C > 0. \tag{86}$$

Using Young inequality, Poincaré's inequality and Lemma 2, we obtain

$$\begin{aligned}|I_4(t)| &\leqslant \frac{1}{2}\|w_t(t)\|^2 + \frac{1}{2}\int_\Omega\left(\int_0^t f(t-\sigma)(w(t)-w(\sigma))\mathrm{d}\sigma\right)^2\mathrm{d}x\\ &\leqslant \frac{1}{2}\|w_t(t)\|^2 + \frac{1}{2}(1-l)B_{1,\Omega}^2\,(f\circ\nabla w)\,(t).\end{aligned} \tag{87}$$

Similarly, we have

$$\begin{aligned}|d_2 I_1 + d_3 I_3| \leqslant{}& \frac{d_2}{2}\|w_t(t)\|^2 + \frac{d_2 B_{1,\Omega}^2}{2}\|\nabla w(t)\|_2^2\\ &+ d_3\widetilde{c}\int_\Omega\int_0^1 |u(x,\xi,t)|^2\mathrm{d}\xi\,\mathrm{d}x.\end{aligned} \tag{88}$$

Finally, from (59) and (60), we get

$$\begin{aligned}\int_0^t (y^2+\beta)v(x,y,\sigma)\,\mathrm{d}\sigma ={}& w(x,t)\eta(y) - v(x,y,t) - w_0(x)\eta(y)\\ &- \eta(y)\tau\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi + \eta(y)\tau\int_0^1 g_0(x,t-\tau)\,\mathrm{d}\xi.\end{aligned} \tag{89}$$

Thus

$$
\begin{aligned}
&\left(y^2+\beta\right)\left(\int_0^t v(x,y,\sigma)\,\mathrm{d}\sigma-\frac{\eta(y)\tau}{y^2+\beta}\int_0^1 g_0(x,t-\tau)\,\mathrm{d}\xi+\frac{w_0(x)\eta(y)}{y^2+\beta}\right)^2\\
&\quad\leqslant\frac{|v(x,y,t)|^2}{y^2+\beta}+A_0|w(x,t)|^2+A_0\tau^2\left(\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi\right)^2\\
&\quad+2\frac{|v(x,y,t)w(x,t)\eta(y)|}{y^2+\beta}+2A_0\tau\left|w(x,t)\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi\right|\\
&\quad+2\frac{\left|\eta(y)\tau v(x,y,t)\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi\right|}{y^2+\beta}.
\end{aligned}
\tag{90}
$$

Integrating over $\Omega\times(-\infty,+\infty)$, we obtain the following expression:

$$
\begin{aligned}
\frac{2}{a_1}|I_2(t)|&\leqslant\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))}+A_0\|w(x,t)\|_2^2\\
&+\int_\Omega\int_{-\infty}^{+\infty}\frac{|v(x,y,t)|^2}{y^2+\beta}\,\mathrm{d}y\,\mathrm{d}x+A_0\tau^2\int_\Omega\left(\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi\right)^2\mathrm{d}x\\
&+2\left|\int_\Omega\int_{-\infty}^{+\infty}\frac{v(x,y,t)w(x,t)\eta(y)}{y^2+\beta}\,\mathrm{d}y\,\mathrm{d}x\right|\\
&+2A_0\tau\left|\int_\Omega w(x,t)\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi\,\mathrm{d}x\right|\\
&+2\tau\left|\int_\Omega\int_{-\infty}^{+\infty}\frac{\eta(y)v(x,y,t)\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi}{y^2+\beta}\,\mathrm{d}y\,\mathrm{d}x\right|.
\end{aligned}
\tag{91}
$$

Now, we will estimate the right hand side of (91). First using Holder's inequality, we have

$$\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi \leqslant \left(\int_0^1 |u(x,\xi,t)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}. \tag{92}$$

Applying Young's inequality and using the fact $\frac{1}{y^2+\beta} \leqslant \frac{1}{\beta}$, we get:

$$\left|\int_\Omega \int_{-\infty}^{+\infty} \frac{v(x,y,t)w(x,t)\mu(y)}{y^2+\beta}\,\mathrm{d}y\,\mathrm{d}x\right| \leqslant \frac{1}{2\beta}\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \frac{A_0}{2}\|w\|_2^2, \tag{93}$$

$$\tau\left|\int_\Omega w(x,t)\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi\,\mathrm{d}x\right| \leqslant \frac{\tau^2}{2}\int_\Omega\int_0^1 |u(x,\xi,t)|^2\,\mathrm{d}\xi\,\mathrm{d}x + \frac{1}{2}\|w\|_2^2, \tag{94}$$

and

$$\begin{aligned}\tau\left|\int_\Omega\int_{-\infty}^{+\infty} \frac{\eta(y)v(x,y,t)\int_0^1 u(x,\xi,t)\,\mathrm{d}\xi}{y^2+\beta}\,\mathrm{d}y\,\mathrm{d}x\right| &\leqslant \frac{\tau^2 A_0}{2}\int_\Omega\int_0^1 |u(x,\xi,t)|^2\,\mathrm{d}\xi\,\mathrm{d}x \\ &\quad + \frac{1}{2\beta}\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))}.\end{aligned} \tag{95}$$

By inserting the estimates (92), (93), (94) and (95) into (91), we get

$$\begin{aligned}|I_2(t)| \leqslant{}& \frac{a_1}{2}\left(1+\frac{3}{\beta}\right)\|v\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \frac{3\beta^{\rho-1}B^2_{1,\Omega}}{2}\|\nabla w(t)\|_2^2 \\ &+ \frac{3\beta^{\rho-1}}{2}\int_\Omega\int_0^1 |u(x,\xi,t)|^2\,\mathrm{d}\xi\,\mathrm{d}x.\end{aligned} \tag{96}$$

By utilizing the inequality $1-\int_0^t f(\sigma)\,\mathrm{d}\sigma \geqslant l$, along with equations (40), (87), (88), and (96), we can derive equation (86) with the existence of a positive constant C. It becomes evident that, based on (77), (86), and by selecting a sufficiently large value for d_1, our desired result is proven. □

5. Conclusion

The exploration of the asymptotic stability of a viscoelastic wave equation with a delay is a commendable endeavor that sheds light on the intricate dynamics inherent in such systems. The consideration of viscoelasticity and temporal delays adds layers of complexity to the analysis, making it a challenging yet intellectually stimulating subject. In this work, we examined the viscoelastic wave equation, incorporating a time delay term in internal fractional feedback. The investigation utilized the energy method in conjunction with the Faedo-Galerkin procedure to rigorously establish the global existence of solutions, contingent upon specific conditions. Furthermore, the study demonstrated the efficacy of employing suitable Lyapunov functionals, elucidating their role in yielding comprehensive decay results for the energy within the system.

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