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ALFRED TRZASKA*, KRYSTYNA SOBOWSKA**

# FLOWS WITH SUBSTANCE EXCHANGE AND CHANGE OF ITS VOLUME IN RADIAL GEOMETRY

# PRZEPŁYWY Z WYMIANĄ MASY I ZMIANĄ JEJ OBJĘTOŚCI W GEOMETRII RADIALNEJ

The subject of this paper is a certain model of the process of colmatage in which the swelling of colmatant settled in a porous medium proceeds. An axi-symmetric flow is considered. The paper is a continuation of the article (Trzaska, Sobowska 2000) in which a similar problem for the one-dimensional case was examined. It is assumed that the swelling occurs when colmatant contacts a certain substance.

In the accepted model it is assumed that suspension carrying solid particles of the colmatant is initially forced into the porous medium. The particles settle in the pores. This part of the process is conventionally called its first stage. The stage is described by a system of balance-transport (1) and kinetics (2) equations. The obtaining of a certain distribution of colmatant $P_0(r)$ settled in the porous medium (16) is the result of stage one of the process.

During stage two, substance which can cause the swelling of the colmatant settled earlier is forced into the medium. Part of this substance stays in the medium pores reacting on the colmatant and leading to the increase of its volume. This stage of the process is described by equations (17), (18), (24). The distribution of the colmatant in the medium pores $P(r,t)$ is obtained in the form (25).

In this paper computations are made which help to determine the distribution of pressure in the medium during stage two of the process. The equation of motion (26) was used. The flow at known discharge $q(t)$ was examined. The distribution of pressures $h(r,t)$ is expressed then by formula (29).

**Key words:** the flow with mass and momentum exchange, colmatage, filtration

Niniejsza publikacja jest kontynuacją pracy Trzaski i Sobowskiej (2000), w której podobny problem rozpatrywany był dla przypadku przepływu jednowymiarowego wzdłuż tworzących równoległych do osi przepływu. Tutaj rozważany jest model osiowo-symetryczny. Obydwa wymienione artykuły należą do serii prac poświęconych kolmatacji i będących efektem badań tego zjawiska prowadzonych przez ten zespół autorów. Przypomnijmy, że przez kolmatację rozumiemy zjawisko zachodzące w trakcie przepływu zawiesiny przez ośrodek porowaty i polegające na wymianie cząstek

* WYDZIAŁ GÓRNICZY, AKADEMIA GÓRNICZO-HUTNICZA, 30-059 KRAKÓW, AL. MICKIEWICZA 30.
** INSTYTUT MATEMATYKI, AKADEMIA GÓRNICZO-HUTNICZA, 30-059 KRAKÓW, AL. MICKIEWICZA 30.

stałych z ośrodka ciekłego do stałego. W efekcie dochodzi do zmian własności fizycznych ośrodka porowatego, zmienia się w szczególności jego porowatość i w konsekwencji współczynnik przepuszczalności. To z kolei powoduje zmianę ciśnienia oraz przy pewnych uwarunkowaniach wpływa na wielkość wydatku przepływu.

W dotychczasowych opracowaniach rozpatrywano procesy, w których własności fizyczne kolmatanta po jego osadzeniu się w porach ośrodka nie ulegają zmianie. W niniejszym artykule, podobnie jak w pracy Trzaski i Sobowskiej (2000) rozważany jest model procesu, w którym dochodzi do zmian objętości cząstek kolmatanta osadzonego w ośrodku porowatym. Przyjęto założenie, że zmiany te (pęcznienie, namnażanie) następują w wyniku kontaktu kolmatanta z pewną substancją.

W pracy badany jest model procesu składającego się z dwóch etapów. Pierwszy etap to zatłaczanie do ośrodka zawiesiny niosącej cząstki stałe. W trakcie przepływu zawiesiny dochodzi do osadzania się tych cząstek w przestrzeni porowej, zachodzi zjawisko kolmatacji. Ten etap opisano układem równań bilansu-transportu (1) i kinetyki (2). W wyniku rozwiązania układu (1), (2) z warunkami początkowo-brzegowymi (3), (5) uzyskano funkcję rozkładu kolmatanta w ośrodku porowatym w dowolnej chwili $t$ trwania procesu, a w szczególności w chwili $t_1$ zakończenia tego etapu (16). Uzyskane rozwiązanie stanowi warunek początkowy dla drugiego etapu procesu. Teraz do ośrodka zatłaczana jest substancja powodująca pęcznienie lub namnażanie kolmatanta osadzonego wcześniej. Do opisu tego etapu użyto układu równań (17), (18), (24), który rozpatrywano z warunkami początkowo-brzegowymi (19), (20), (23). Uzyskano funkcje koncentracji omawianej substancji w cieczy płynącej przez ośrodek oraz zatrzymanej w przestrzeni porowej i reagującej z kolmatantem. Następnie wyznaczono funkcję rozkładu pęczniejącego kolmatanta w ośrodku porowatym (25).

Kolejnym etapem pracy było wyznaczenie rozkładu ciśnienia panującego w ośrodku porowatym w trakcie drugiego etapu procesu. Wykorzystano równanie ruchu postaci (26) i uzyskano funkcję rozkładu ciśnienia dla przepływu przy zadanej, na przykład stałej wartości wydatku przepływu (29).

**Słowa kluczowe:** przepływy z wymiana masy i zmianą pędu, kolmatacja

## 1. Introduction

In this paper a certain model of the phenomenon of colmatage is presented. In this model the swelling of colmatant settled in a porous space during the flow of substance through an axi-symmetric medium occurs.

The paper is a continuation of the article (Trzaska, Sobowska 2000) in which a similar problem for the one-dimensional flow was examined. As in the study mentioned above (Trzaska, Sobowska 2000) we consider a process consisting of two stages. During stage one starting at the moment $t = 0$, substance carrying solid particles of the colmatant is forced through circular hole with a radius $R$ into homogeneous medium having initial porosity $\varepsilon_0$.

When the flow proceeds, particles of the colmatant settle in the medium pores, i.e. the phenomenon of colmatage takes place (Litwiniszyn, Bodziony 1961; Litwiniszyn 1966). In consequence, the changes of the medium porosity and the concentration of solid particles in the flowing suspension take place (Trzaska 1972, 1986, 1989). We assume that substance flowing into the medium has a constant volume concentration equal to n. Stage one of the process is over at the moment $t = t_1$. Now a liquid containing substance which brings about the swelling of the colmatant settled earlier is forced into the porous medium. Thus, further changes of porosity and the changes of the distribution of pressure in the flowing liquid take place (Trzaska 1986).

Our aim is to determine the distribution *P(x,t)* of the colmatant in the porous medium during both stages of the process. We will also determine the concentration *N(x,t)* of the colmatant dispersed in substance during stage one of the process, the concentration *C(x,t)* of substance in the liquid causing the swelling of the colmatant during stage two, and the quantity of this substance *U(x,t)* which reacting on the colmatant remains in the medium pores. The distribution of pressure *h(x,t)* in the medium during stage two of the process will be also determined.

## 2. Determination of the distibution of colmatant in the porous medium during stage one of the process

Functions *P(x,t)* and *N(x,t)* will be determined on the basis of the system of balance--transport equations:

$$\frac{\partial P(r,t)}{\partial t}+\frac{\partial[\varepsilon_0 - P(r,t)]N(r,t)}{\partial t}+\frac{q(t)}{r}\frac{\partial N(r,t)}{\partial t}=0 \tag{1}$$

where *q(t)* denotes a unitary discharge of flow during stage one, and the equations of kinetics

$$\frac{\partial P(r,t)}{\partial t}=\frac{\alpha q(t)}{r}N(r,t)[A-P(r,t)][P(r,t)+B] \tag{2}$$

where:

- $a$ — the coefficient of colmatage,
- $A,\ B$ — certain parameters dependent on the properties of the medium ($0 < A \le \varepsilon_0$, $B > 0$).

Because of the assumption made earlier, the boundary condition will be accepted in the form:

$$N(R,t) = n \tag{3}$$

Let us denote:

$$Q(t)=\int_0^t q(t)dt \tag{4}$$

Let us notice that the wave front of the flowing liquid is described by the equation:

$$r^2 - R^2 = \frac{2}{\varepsilon_0}Q(t)$$

The boundary condition can be written in the form:

$$P(r,t) = 0, \qquad \text{when} \qquad r^2 - R^2 \geq \frac{2}{\varepsilon_0} Q(t) \tag{5}$$

The following is calculated from equation (2):

$$N(r,t) = \frac{r}{\alpha q(t)(A+B)} \frac{\partial \ln \dfrac{P(r,t)+B}{A-P(r,t)}}{\partial t} \tag{6}$$

Let us denote:

$$S(r,t) = \ln \frac{P(r,t)+B}{A-P(r,t)} - \ln \frac{B}{A} \tag{7}$$

Let us notice that on the strength of (5):

$$S(r,t) = 0, \qquad \text{when} \qquad r^2 - R^2 \geq \frac{2}{\varepsilon_0} Q(t) \tag{8}$$

and that:

$$P(r,t) = \frac{AB(e^{S(r,t)} - 1)}{A + Be^{S(r,t)}} \tag{9}$$

Relation (6) can have now the form:

$$N(r,t) = \frac{r}{\alpha q(t)(A+B)} \frac{\partial S(r,t)}{\partial t} \tag{10}$$

Formula (10) is taken into consideration in equation (1). We have:

$$\alpha(A+B)\frac{\partial P(r,t)}{\partial t} + \alpha(A+B)\frac{\partial[\varepsilon_0 - P(r,t)]N(r,t)}{\partial t} + \frac{1}{r}\frac{\partial S(r,t)}{\partial t} + \frac{\partial^2 S(r,t)}{\partial t \partial r} = 0$$

Hence after integrating and considering (5) and (7) we obtain the following equation:

$$\alpha(A+B)P(r,t) + [\varepsilon_0 - P(r,t)]\frac{r}{q(t)}\frac{\partial S(r,t)}{\partial t} + \frac{1}{r}S(r,t) + \frac{\partial S(r,t)}{\partial r} = 0 \tag{11}$$

Equation (9) being linearised we obtain:

$$P \approx \frac{ABS}{A+B}$$

The above formula is introduced into (11) and thus we have:

$$\left(\varepsilon_0 - \frac{ABS(r,t)}{A+B}\right)\frac{r}{q(t)}\frac{\partial S(r,t)}{\partial t} + \frac{\partial S(r,t)}{\partial r} + S(r,t)\left(\frac{1}{r} + \alpha AB\right) = 0 \tag{12}$$

The boundary condition for (12) is found taking relation (10) for $r = R$. We obtain:

$$n = \frac{R}{\alpha q(t)(A+B)} \frac{\partial S(R,t)}{\partial t}$$

Hence, after integrating with the condition (8) and taking into consideration (4):

$$S(R,t) = \frac{n\alpha Q(t)(A+B)}{R} \tag{13}$$

Equation (12) with the boundary condition (13) is solved using the Cauchy characteristic method. The system of the characteristics equations is as follows:

$$\frac{q(t)dt}{\left(\varepsilon_0 - \frac{ABS}{A+B}\right)r} = dr = -\frac{dS}{\left(\frac{1}{r} + \alpha AB\right)S}$$

Its solution has the form:

$$C_1 = rSe^{\alpha ABr} \tag{14}$$

$$C_2 = Q(t) - \frac{1}{2}\varepsilon_0 r^2 - \frac{rS}{(A+B)\alpha}$$

With respect to condition (13) the following relation occurs:

$$C_1(1-n) = \alpha n(A+B)e^{\alpha ABR}\left(C_2 + \frac{1}{2}\varepsilon_0 R^2\right)$$

We introduce formulae (14) into the above dependence. We obtain:

$$rS(1-n)e^{\alpha ABr} = \alpha n(A+B)e^{\alpha ABR}\left(Q(t) - \frac{1}{2}\varepsilon_0(r^2 - R^2) - \frac{rS}{(A+B)\alpha}\right)$$

Hence, the following is calculated:

$$S(r,t) = \frac{\alpha n(A+B)e^{\alpha ABR}\left[Q(t) - \frac{1}{2}\varepsilon_0(r^2 - R^2)\right]}{r[(1-n)e^{\alpha AB(r-R)} + n]} \tag{15}$$

Function $P(r,t)$ of the distribution of colmatant $0 \le t \le t_1$, i.e. during stage one of the process is found taking dependence (15) into account in (9). At the moment $t_1$ when the forcing in of the colmatant is over, this distribution has the form

$$P_0(r)=P(r,t_1)=\frac{AB\left\{\exp\frac{\alpha n(A+B)\left[Q(t_1)-\frac{1}{2}\varepsilon_0(r^2-R^2)\right]}{r[(1-n)e^{\alpha AB(r-R)}+n]}-1\right\}}{A+B\exp\frac{\alpha n(A+B)\left[Q(t_1)-\frac{1}{2}\varepsilon_0(r^2-R^2)\right]}{r[(1-n)e^{\alpha AB(r-R)}+n]}} \tag{16}$$

## 3. Distribution of colmatant in the porous medium during stage two of the process

At the moment $t = t_1$ substance causing the changes of the volume multiplication of the colmatant settled earlier is forced into the porous medium. We assume that part of the mentioned substance flowing through the medium passes to the medium pores and reacts on the colmatant causing its swelling. Let us denote by $C(r,t)$ function of position and time of the volume concentration of the substance in the liquid flowing through the medium, and by $U(r,t)$ its distribution in the medium pores at the moments $t \geq t_1$ of the lasting process. Both these functions are connected by the equation of balance-transport:

$$\frac{\partial[\varepsilon_0-P(r,t)]C(r,t)}{\partial t}+\frac{q_1(t)}{r}\frac{\partial C(r,t)}{\partial r}+\frac{\partial U(r,t)}{\partial t}=0 \tag{17}$$

and kinetics:

$$\frac{\partial U(r,t)}{\partial t}=\frac{1}{r}\alpha_1 q_1(t)C(r,t)P_0(r) \tag{18}$$

where $q_1(t)$ denotes the discharge of flow during stage two of the process, $\alpha_1$ is a certain constant and $P_0(r)$ is expressed by formula (16).

Let us denote $Q_1(t)=\int_{t_1}^{t} q_1(t)dt.$

The system of equations (17), (18) is solved with the initial-boundary conditions:

$$U(r,t)=0, \qquad \text{when} \qquad r^2-R^2\geq\frac{2}{\varepsilon_0}Q_1(t) \tag{19}$$

$$C(R,t)=C_0 \tag{20}$$

for $t \geq t_1$, $r^2-R^2\leq\frac{2}{\varepsilon_0}Q_1(t)$.

Function *U(r,t)* is eliminated from the system (17), (18) and the following equation is obtained:

$$\frac{r}{q_1(t)}\frac{\partial[\varepsilon_0 - P(r,t)]C(r,t)}{\partial t}+\frac{\partial C(r,t)}{\partial r}+\alpha_1 C(r,t)P_0(r)=0$$

Its solution satisfying condition (20) is function:

$$C(r,t)=C_0 e^{-\alpha_1\int_R^r P_0(r)dr} \tag{21}$$

We place formula (21) into equation (18) and obtain:

$$\frac{\partial U(r,t)}{\partial t}=\frac{\alpha_1}{r}q_1(t)P_0(r)C_0 e^{-\alpha_1\int_R^r P_0(r)dr}$$

We integrate it with respect to *t* taking condition (19) into account. In this way we get the other function:

$$U(r,t)=\frac{\alpha_1}{r}\left[Q_1(t)-\frac{\varepsilon_0}{2}(r^2-R^2)\right]P_0(r)C_0 e^{-\alpha_1\int_R^r P_0(r)dr} \tag{22}$$

when $t \geq t_1$, $r^2 - R^2 \leq \frac{2}{\varepsilon_0}Q_1(t)$.

Now we proceed to determine the volume concentration of the colmatant settled in the porous medium during stage one of the process and swelling during stage two in result of its reaction on the substance forced into. Let us notice that at the moment $t_1$ which is the end of stage two the relation occurs:

$$P(r,t_1) = P_0(r), \qquad \text{when} \qquad r^2 - R^2 \geq \frac{2}{\varepsilon_0}Q_1(t) \tag{23}$$

where $P_0(r)$ is expressed by formula (16).

We assume that swelling (multiplication) proceeds according to the principles of the third kinetics. Thus, we have:

$$\frac{\partial P(r,t)}{\partial t}=\alpha_2 U(r,t)P(r,t)[\varepsilon_0 - P(r,t)] \quad \text{for} \quad t \geq t_1, r^2 - R^2 \leq \frac{2}{\varepsilon_0}Q_1(t) \tag{24}$$

Function *U(r,t)* occuring in the above equation is expressed by formula (22), and $\alpha_2$ is a certain constant. Equation (24) is solved with the initial condition (23). After some transformations and when formula (22) is taken into consideration, it has the form:

$$\frac{\partial}{\partial t}\ln\frac{P(r,t)}{\varepsilon_0 - P(r,t)}=\frac{\alpha_1\alpha_2}{r}C_0\varepsilon_0 P_0(r)Q_1(t)e^{-\alpha_1\int_R^r P_0(r)dr}$$

Integrating the above relation with the condition (23) we obtain:

$$\ln\frac{P(r,t)}{\varepsilon_0 - P(r,t)} - \ln\frac{P_0(r)}{\varepsilon_0 - P_0(r)} W(r)\int_{t_2}^{t} Q_1(t)dt$$

where:

$$W(r) = \frac{\alpha_1\alpha_2}{r} C_0\varepsilon_0 P_0(r) e^{-\alpha_1\int_R^r P_0(r)dr}, \quad t_2 = Q^{-1}\left[\frac{\varepsilon_0}{2}(r^2 - R^2)\right]$$

Let us notice that $t_2$ is the time necessary for the wave front to reach point $r$. Hence, after transformation we obtain:

$$P(r,t) = \frac{\varepsilon_0 P_0(r)}{[\varepsilon_0 - P_0(r)]e^{-W(r)\int_{t_2}^{t} Q_1(t)dt} + P_0(r)} \quad \text{for} \quad t \geq t_1, r^2 - R^2 \leq \frac{2}{\varepsilon_0} Q_1(t) \tag{25}$$

Formula (25) determines the distribution of colmatant in the porous medium at an arbitrary moment $t \leq t_1$ of the duration of stage two of the process (Fig.1).

Let us consider what form formula (25) takes in the case when the discharge of flow is constant $q_1(t) = q_1 = \text{const}$. Then, function $Q_1(t)$ is expressed by the formula:

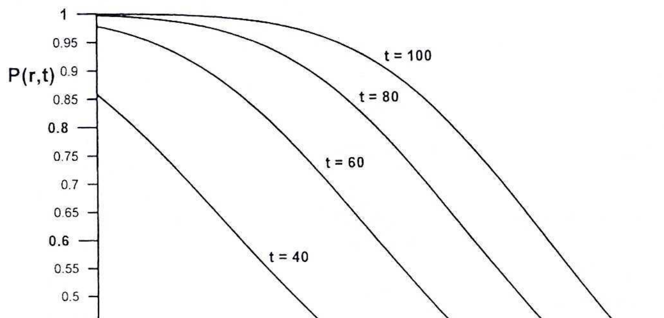


Fig. 1.

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$$Q_1(t)=\int_{t_1}^{t} q_1(t)dt=q_1(t-t_1)$$

Hence

$$Q_1^{-1}(r)=\frac{r}{q_1}+t_1$$

and

$$Q_1^{-1}=\left[\frac{\varepsilon_0}{2}(r^2-R^2)\right]=\frac{\varepsilon_0}{2q_1}(r^2-R^2)+t_1$$

Thus, we have:

$$\int_{t_2}^{t} Q_1(t)dt=\int_{\frac{\varepsilon_0}{2q_1}(r^2-R^2)+t_1}^{t} q_1(t-t_1)dt=\frac{1}{2}q_1(t-t_1)^2-\frac{\varepsilon_0^2}{8q_1}(r^2-R^2)^2$$

So, we can write formula (25) in the form:

$$P(r,t)=\frac{\varepsilon_0 P_0(r)}{[\varepsilon_0-P_0(r)]e^{-W(r)\left[\frac{1}{2}q_1(t-t_1)^2-\frac{\varepsilon_0^2}{8q_1}(r^2-R^2)^2\right]}}$$

for $t > t_1,\ r^2-R^2 \le \frac{2}{\varepsilon_0}q_1(t-t_1)$

## 4. Distribution of pressure in the porous medium and the discharge of flow during stage two of the process

The distribution of pressure in the porous medium during stage two is determined basing on the equation of motion:

$$\frac{\partial h(r,t)}{\partial r}=-\frac{\alpha_0 q_1(t)}{r[\varepsilon(r,t)]^3} \qquad \text{for} \qquad t\ge t_1, r^2-R^2\le\frac{2}{\varepsilon_0}Q_1(t) \tag{26}$$

where $\varepsilon(r,t)$ denotes the porosity of the medium, $\alpha_0$ — a certain constant.

Let us notice that $\varepsilon(r,t)=\varepsilon_0-P(r,t)$.

Taking into account the above relation in which $P(r,t)$ is expressed by formula (25), the following dependence is obtained in equation (26):

$$\frac{\partial h(r,t)}{\partial r} = -\frac{\alpha_0 q_1(t)}{r\varepsilon_0^3}\left[1+\frac{P_0(r)}{\varepsilon_0 - P_0(r)} e^{W(r)\int_{t_2}^{t} Q_1(t)dt}\right]^3 \tag{27}$$

We assume that the deposit pressure has the value $h_1$. Thus at the wave front $r = \sqrt{R^2 + \frac{2}{\varepsilon_0} Q_1(t)}$ the relation occurs:

$$h\left(\sqrt{R^2 + \frac{2}{\varepsilon_0} Q_1(t)}, t\right) = h_1 \tag{28}$$

Equation (27) is integrated with the condition (28). We obtain the following formula:

$$h(r,t) = h_1 + \frac{\alpha_0 q_1(t)}{\varepsilon_0^3} \int_r^{\sqrt{R^2+\frac{2}{\varepsilon_0}Q_1(t)}} \frac{1}{r}\left[1+\frac{P_0(r)}{\varepsilon_0 - P_0(r)} e^{W(r)\int_{t_2}^{t} Q_1(t)dt}\right] dr \tag{29}$$

for $t \geq t_1, \quad R \leq r \leq \sqrt{R^2 + \frac{2}{\varepsilon_0} Q_1(t)}$

The above formula allows one to determine the pressure in the pores medium in the case when the discharge of flow, i.e. function $q_1(t)$ is known. It can be applied, for example, when the flow is accomplished at the constant discharge $q_1(t)$ = const.

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