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Trusses of the smallest total potential energy

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Abstract. The paper concerns the problem of minimization of the *total potential energy* of trusses subjected to static loads in the presence of prescribed displacements of selected supporting nodes. The positions of the internal (free) nodes are fixed and the supporting nodes are imposed, the member stiffnesses being design variables, while the truss volume represents the cost of the design. Due to the assumption of the stiffnesses being non-negative, the problem is reduced to a problem of optimization of structural topology. Upon eliminating all the design variables analytically the optimum design problem is eventually reduced to the two mutually dual problems expressed either in terms of member forces or in terms of displacements of free nodes. The problem setting concerning the case when the prescribed displacements of supports are the only loads applied (i.e. kinematic loads) assumes a particularly simple form. A specific numerical method of solving the stress-based auxiliary problem has been developed for the selected 2D and 3D optimal designs. The study is the first step towards topology optimization of trusses with distortions.

Key words: topology optimization; trusses; prescribed displacements.

1. INTRODUCTION

One of the method of rational designing engineering structures is introduction of distortions, or, in particular, prescribing displacements of supports. In case of skeletal structures under the term distortion one understands the presence of bars of initial lengths longer or shorter than the distance between the given nodes. By assembling the structure of such members one introduces an initial stress and deformations states. The topic of distortions in a continuum medium is indissolubly bonded with the theory of composites in which the Eshelby methods play a crucial role, see Mura [1]. Assessing sensitivity of response of a structure due to the presence of a local distortion in the form of a small inclusion or a small cavity is the subject of consideration of the series of papers on the topological derivative method, see e.g. Novotny and Sokołowski [2]. The topological derivative of the elastic energy stored in a linearly elastic body is determined by the Eshelby tensor, see Sec. 7.1 in Lewiński and Sokołowski [3]. The topological derivative concept applies also to discrete systems, e.g. to graphs whose all nodes are connected to a rigid support by springs, see Leugering and Sokołowski [4].
Distortions in skeletal structures are the tools of optimal design as well as the tools of optimal control of the structure during its exploitation; the relevant Virtual Distortion Method has been developed by Holnicki-Szulc [5]. New interesting examples of structures designed and constructed by applying distortions are the subject of the study by Bessini et al [6].

The present paper focuses on optimum design of trusses composed of linearly elastic members. The theory of response of trusses to the given static load and to prescribed displacements of supports is outlined in the manner that will be further directly applicable to the case of general distortions. Indeed, the prescribed displacements of supports can be viewed as boundary distortions.

Within the continuum media theory of equilibrium of nonlinearly elastic bodies the boundary value problems are usually formulated such that the displacements vanish on the support. The whole Chapter 6 of the book by Ciarlet [7], in which the implicit function theorem is used, concerns this special case. In particular, the Theorems 6.4-1 and 6.7-1 therein on the existence of solutions draw upon the assumption of homogeneity of the kinematic boundary conditions. The non-homogeneity of the kinematic conditions occurs only in Chapter 7 of this book in which the existence issue is discussed with using techniques of direct methods of calculus of variations.

The lectures on linear elasticity usually comprise the case of nonhomogeneous kinematic boundary conditions. The given field $\mathbf{U}$ of displacements on the supporting segment Γ_1 of the boundary of a domain Ω is assumed to be element of the space $H^{1/2}(\Gamma_1)$, see Eq.(3.10) in Duvaut and Lions [8]. Due to statical admissibility of the stress field $\boldsymbol{\sigma}$ the components of the vector field $\boldsymbol{\sigma}\mathbf{n}$ on the boundary ($\mathbf{n}$ being the unit outward normal to the domain) may be viewed as elements of the space $H^{-1/2}(\Gamma)$, Γ being the boundary. Thus, the product of the reactions $\boldsymbol{\sigma}\mathbf{n}$ and the displacement field $\mathbf{U}$ can be integrated over the support. Consequently, the Castigliano functional can be properly defined; its argument is a virtual stress field $\boldsymbol{\tau}$ within the given domain Ω; this functional reads

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$$\wp(\boldsymbol{\tau}) = \frac{1}{2}\int_{\Omega} \boldsymbol{\tau}\cdot(\mathbf{A}\boldsymbol{\tau})d\Omega - \int_{\Gamma_1}(\boldsymbol{\tau}\mathbf{n})\cdot\mathbf{U}d\Gamma_1 \tag{1}$$

where $\mathbf{A}$ represents the tensor of flexibilities. Let Σ be the set of virtual stresses $\boldsymbol{\tau}$ which satisfy the equilibrium equations within the domain Ω and the natural boundary conditions $\boldsymbol{\tau}\mathbf{n} = \mathbf{T}$ on Γ_2, being the complementary segment of the boundary: $\Gamma_2 = \Gamma \setminus \Gamma_1$. According to Castigliano's theorem the field $\boldsymbol{\sigma}$, which is the stress field solving the boundary value problem of linear elasticity, can be constructed directly by the minimization process:

$$\wp(\boldsymbol{\sigma}) = \min_{\boldsymbol{\tau}\in\Sigma} \wp(\boldsymbol{\tau}) \tag{2}$$

The result (2) should be understood as follows: for the minimizer $\boldsymbol{\sigma}$, being statically admissible, one can find the displacement field $\mathbf{u}$ such that

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{A}\boldsymbol{\sigma} \ \text{ in } \ \Omega, \qquad \mathbf{u}=\mathbf{U} \ \text{ on } \ \Gamma_1 \tag{3}$$

where $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetric part of the gradient of $\mathbf{u}$. Moreover, under known assumptions the field $\mathbf{u}$ is unique. Then the triple $(\mathbf{u}, \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\sigma})$ solves the set of equations of linear elasticity and this solution is unique. The proof of the Castigliano theorem is delivered in Sec.3.5 of Duvaut and Lions[8] see also Nečas and Hlavaček [9].

For better understanding of the truss optimization problem let us recall now the *Isotropic Material Design* (IMD) problem of optimal distribution of the elastic moduli of isotropy: $k(x)$, $\mu(x)$ within a given design domain Ω, first proposed in Czarnecki [10]. The unit cost is assumed as equal to the trace of the Hooke tensor. In the case of isotropy the eigenvalues of the Hooke tensor are: $3k$, 2μ, 2μ , 2μ , 2μ , 2μ cf. Walpole [11]. Thus, the cost condition is assumed in the form

$$\int_{\Omega}(3k+10\mu)d\Omega \le \Lambda_0 \quad . \tag{4}$$

The fields $k(x)$, $\mu(x)$ are the design variables of the problem. We shall assume that k and μ are subject to the conditions: $k \ge 0$, $\mu \ge 0$, hence we admit the degenerated cases such that e.g. $k = 0$ and $\mu > 0$ or vice versa, while the case of $k = 0$ and $\mu = 0$ means that the material is absent. The body occupying the given domain Ω is subjected to the tractions of intensity $\mathbf{T}$ on the given segment Γ_2 of the boundary while the complementary segment Γ_1 is a support on which the displacement field $\mathbf{u}$ vanishes. In this problem with $\mathbf{u} = \mathbf{0}$ on Γ_1 maximization of the overall stiffness of the body means minimization of the compliance C defined by

$$C = \frac{1}{2}\int_{\Gamma_2} \mathbf{T}\cdot\mathbf{u}\, d\Gamma_2 \tag{5}$$

or, alternatively

$$C = \frac{1}{2}\min_{\boldsymbol{\tau}\in\Sigma}\int_{\Omega} \boldsymbol{\tau}\cdot(\mathbf{A}\boldsymbol{\tau})d\Omega \tag{6}$$

the integrand being

$$\boldsymbol{\tau}\cdot(\mathbf{A}\boldsymbol{\tau}) = \frac{1}{9k}(\mathrm{tr}\,\boldsymbol{\tau})^2 + \frac{1}{2\mu}\|\mathrm{dev}\,\boldsymbol{\tau}\|^2 \tag{7}$$

where $\mathrm{tr}\,\boldsymbol{\tau}$ and $\mathrm{dev}\,\boldsymbol{\tau}$ are the trace and deviator of the stress state $\boldsymbol{\tau}$; $\|\ \|$ represents the Euclidean norm. The problem of maximization of the stiffness of the body means minimization of C over all possible layouts of the bulk and shear moduli keeping the mentioned cost condition. The main feature of this approach is reducing the problem to a sequence of two minimization operations over independent variables. Indeed, the set Σ does not depend on the layout of the elastic moduli and the layout of the materials has nothing to do with the trial stress field. Thus, the order of the minimization operations can be changed and then minimization operation over the moduli can be performed analytically.

The problem of minimization of the compliance given by (4), (5) over the non-negative bulk and shear moduli satisfying (4) reduces to the problem of the form

$$\min_{\boldsymbol{\tau}\in\Sigma}\int_{\Omega} F(\boldsymbol{\tau})d\Omega \tag{8}$$

where

$$F(\boldsymbol{\tau}) = \min\left\{\int_{\Omega}\left(\frac{1}{9k}(\mathrm{tr}\,\boldsymbol{\tau})^2 + \frac{1}{2\mu}\|\mathrm{dev}\,\boldsymbol{\tau}\|^2\right)d\Omega \,\middle|\, \text{over } k\ge 0, \mu \ge 0, \int_{\Omega}(3k+10\mu)d\Omega \le \Lambda_0\right\} \tag{9}$$

and the problem (9) can be solved by using the rule

$$\min\left\{\int_{\Omega}\left(\sum_{i=1}^{n}\frac{a_i(x)}{w_i(x)}\right)d\Omega \,\middle|\, \text{over: } w_i \ge 0,\ \int_{\Omega}\left(\sum_{i=1}^{n} w_i(x)\right)d\Omega \le \Lambda\right\} = \frac{1}{\Lambda}\left(\int_{\Omega}\left(\sum_{i=1}^{n}\sqrt{a_i(x)}\right)d\Omega\right)^2 \tag{10}$$

where $a_i(x) > 0$ are given functions in the domain Ω, Λ is a given positive constant while the functions $w_i(x)$, $i = 1,...,n$ are unknown, see Sec.1.3 in Lewiński [12]. The solution $w_i^*(x)$ to the problem (10) is given by

$$w_i^*(x) = \Lambda\frac{\sqrt{a_i(x)}}{\int_{\Omega}\sum_{j=1}^{n}\sqrt{a_j}\,d\Omega} \tag{11}$$

Here $w_1(x) = 3k(x)$, $w_2(x) = 10\mu(x)$, $n = 2$. Let us note: the explicit form of the function $F(\cdot)$ can be found by using the rule (10), which reduces the optimum design problem to the auxiliary problem (8) in which the design variables are absent and the only unknown is the stress field $\boldsymbol{\sigma}$ for which the functional attains its minimum.

Let us note that according to (11) the optimal bulk modulus k^* is proportional to $|\mathrm{tr}\,\boldsymbol{\sigma}|$ and the optimal μ^* is proportional to $\|\mathrm{dev}\,\boldsymbol{\sigma}\|$. The effective domain of the minimizer $\boldsymbol{\sigma}$ is the material domain, the remaining part becomes a void. The mathematical theory of the IMD method can be found in Bołbotowski and Lewiński [13].

Majority of optimum design problems concerns the case of **U** = **0**. Then the requirement of making the structure as stiff as possible reduces to the requirement of minimizing the compliance given by (4). It seems that almost no one paper on topology optimization up to 2011 discussed the case of kinematic loads (**T** = **0**, **U** ≠ **0**) or the case when both types of loads are simultaneously present (**T** ≠ **0**, **U** ≠ **0**). The method of tackling such optimum design problems has been put forward by Barbarosie and Lopes [14], Niu et al [15], Klarbring and Strömberg [16] and Klarbring [17]. These papers teach us that instead of minimizing the compliance one should minimize the functional

$$J = \frac{1}{2}\int_{\Gamma_2} \mathbf{T}\cdot\mathbf{u}\, d\Gamma_2 - \frac{1}{2}\int_{\Gamma_1} (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{U}\, d\Gamma_1 ; \qquad (12)$$

here **u** and **σ** form the solution to the elasticity problem in which both the loads **T** and **U** are nonzero. In case of **T** ≠ **0**, **U** = **0** the functional J equals C and the task reduces to the minimum compliance problem. In the opposite case of **T** = **0**, **U** ≠ **0** minimization of J means maximization of the work of reactions **σn** on given displacements **U** of supports on the boundary Γ_1. This means that the designed structure is the stiffest, since its resistance due to the prescribed displacements of supports is the highest. However, the reason of choosing the functional (12) for the minimization process does not simply follows only from the discussion of the mentioned two extreme cases. The true reason is the equality: $J = \wp(\boldsymbol{\sigma})$ which defines a new meaning of the functional J and justifies calling it the *total potential energy*, as Klarbring [17] has suggested. The above arguments justify setting the minimization problem of the functional J given by (12) to formulate properly the optimum design problems concerning the structures simultaneously subjected to static and kinematic loads.

The subject of the present paper is optimum design of trusses: given are position of nodes, also those on which the structure is supported. The truss is subjected to nodal concentrated forces $P_1,...,P_s$ and - to the prescribed displacements of supports: $U_1,...,U_m$. The volume of the truss is bounded by a given value. The design variables: the axial stiffnesses of members EA_k, k = 1,…,e, are viewed as nonnegative, which means that the solutions, i.e. the optimal trusses, are admitted to be geometrically variable. The problem is thus posed as a problem of optimum structural topology in which the total potential energy J is minimized. We shall show that all the design variables can be analytically eliminated thus reducing the problem to the two mutually dual problems, the dual gap between them being zero. By solving the stressed-based problem one obtains explicit formulae for the optimal stiffnesses and the theorem on the constant stress distribution is an easy by-product of this part of the analysis. These conclusions correspond to the analogous properties of the optimal structures formed by the mentioned method of the *Isotropic Material Design*.

It occurs, however, that the growth of the minimized function in the stress-based problem concerning optimum design of trusses is very slow thus making difficulties in attaining the solution. For solving this problem the new numerical methods are proposed.

When the displacements **U** are prescribed we have no control of the values of reactions, hence the optimal designs are difficult to predict. Moreover, the kinematic load **U** acts on the boundary and causes reactions being self-equilibrated; consequently one may expect that the optimal designs will be composed of bars lying in a certain boundary zone only. This is not true in general, the Saint Venant principle does not hold in discrete systems - some reactions may transmit the stress along a line of bars to the other side of the support.

As mentioned, the kinematic loads **U** may be treated as boundary distortions. The formalism of internal distortions in trusses is similar. Thus, the present paper is an introduction to the problem of optimum design of trusses subjected to distortions of arbitrary nature.

A standard notation of linear algebra is applied. In particular, the scalar product of two vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$ is defined by $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + ... + a_nb_n$. The vectors will be viewed as columns, e.g. $\mathbf{a} = \left[a_1,...,a_n\right]^T$, hence $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^T\mathbf{b}$, where $()^T$ is the transposition operator. The identity matrix is represented by **I**. If **A** is a $m\times n$ matrix, then the image of the linear operator represented by **A** and the kernel of this operator are defined as below

$$Im(\mathbf{A}) = \left\{\mathbf{b}\in\mathbb{R}^m \,\middle|\, \exists\mathbf{v}\in\mathbb{R}^n, \mathbf{b} = \mathbf{A}\mathbf{v}\right\},$$
$$Ker(\mathbf{A}) = \left\{\mathbf{v}\in\mathbb{R}^n \,\middle|\, \mathbf{A}\mathbf{v} = \mathbf{0}\in\mathbb{R}^m\right\}, \qquad (13)$$

Above $(\mathbf{A}\mathbf{v})_k = \sum_{j=1}^{n} A_{kj}v_j$ is the k-the component of the vector **Av**. The diagonal matrix **A** of dimensions $n\times n$ will be denoted by $\mathrm{diag}\left\{A_{11},...,A_{nn}\right\}$.

If $x\in V$ and the set V has a complex structure, then the minimization problem: $\min_{x\in V} f(x)$ will be written as: $\min\left\{ f(x) \mid x\in V\right\}$.

2. EQUATIONS OF STATICS OF TRUSSES SUBJECTED TO NODAL FORCES AND PRESCRIBED DISPLACEMENTS OF SUPPORTING NODES

Consider a truss of e bars; the supporting nodes are subject to given displacements $U_1,...,U_m$; the displacements $u_1,...,u_s$ of the free nodes of the truss determine the members' elongations $\Delta_1,...,\Delta_e$ by the equations

$$\Delta_k = \sum_{j=1}^{s} B_{kj}u_j + \sum_{l=1}^{m} \breve{B}_{kl}U_l \qquad (14)$$

where $\mathbf{B} = \left[B_{kj}\right]_{e\times s}$ and $\breve{\mathbf{B}} = \left[\breve{B}_{kl}\right]_{e\times m}$ are so-called geometric matrices composed of directional cosines. The prescribed displacements $U_1,...,U_m$ are accompanied by reactions $R_1,...,R_m$; the work of these reactions on the prescribed displacements is expressed by the scalar product **R**·**U**. Along the displacements $u_1,...,u_s$ the nodal forces $P_1,...,P_s$ are applied; their work on the unknown displacements is expressed by the scalar product $\mathbf{P}\cdot\mathbf{u}$. The static loads

$P_1,\ldots,P_s$ and the kinematic loads $U_1,\ldots,U_m$ cause axial member forces $N_1,\ldots,N_e$ and reactions $R_1,\ldots,R_m$. The equations of equilibrium of all nodes are expressed by one variational equation

$$\sum_{k=1}^{e} N_k \bar{\Delta}_k = \sum_{j=1}^{s} P_j \bar{u}_j + \sum_{l=1}^{m} R_l \bar{U}_l \qquad \forall\, \bar{\mathbf{u}} \in \mathbb{R}^s, \quad \forall\, \bar{\mathbf{U}} \in \mathbb{R}^m \tag{15}$$

where

$$\bar{\Delta}_k = \sum_{j=1}^{s} B_{kj} \bar{u}_j + \sum_{l=1}^{m} \breve{B}_{kl} \bar{U}_l \tag{16}$$

Substitution of (16) into (15) and making use of arbitrariness of $\bar{u}_j, \bar{U}_l$, $j = 1,\ldots,s$; $l = 1,\ldots,m$ leads to the nodal equilibrium equations

$$\sum_{k=1}^{e} B_{kj} N_k = P_j \quad , \quad j = 1,\ldots,s \tag{17}$$

$$\sum_{k=1}^{e} \breve{B}_{kl} N_k = R_l \quad , \quad l = 1,\ldots,m\,. \tag{18}$$

The constitutive equations linking the member forces $N_1,\ldots,N_e$ and the elongations $\Delta_1,\ldots,\Delta_e$ are assumed as linear, the distortions of members being not considered. Thus $\Delta_k = N_k l_k / (E A_k)$ where E represents Young's modulus and A_k is the area of the k-th bar, l_k being its length. The equations (14), (17), (18) and the constitutive equations form a solvable set of equations. In the matrix notation it reads

$$\begin{aligned} &\mathbf{\Delta} = \mathbf{B}\mathbf{u} + \breve{\mathbf{B}}\mathbf{U} \\ &\mathbf{B}^T\mathbf{N} = \mathbf{P}, \quad \breve{\mathbf{B}}^T\mathbf{N} = \mathbf{R} \\ &\mathbf{N} = \mathbf{E}\mathbf{\Delta} \end{aligned} \tag{19}$$

where the constitutive matrix has the diagonal form: $\mathbf{E} = diag\left\{ \frac{EA_1}{l_1},\ldots,\frac{EA_e}{l_e} \right\}$ and

$\mathbf{u} = [u_1,\ldots,u_s]^T$, $\mathbf{\Delta} = [\Delta_1,\ldots,\Delta_e]^T$, $\mathbf{P} = [P_1,\ldots,P_s]^T$,

$\mathbf{U} = [U_1,\ldots,U_m]^T$, $\mathbf{R} = [R_1,\ldots,R_m]^T$, $\mathbf{N} = [N_1,\ldots,N_e]^T$

Let us introduce the stiffness matrix $\mathbf{K} = \mathbf{B}^T\mathbf{E}\mathbf{B}$ related to the unknowns $u_1,\ldots,u_s$. These unknowns are governed by the equation

$$\mathbf{K}\mathbf{u} = \mathbf{P} - \mathbf{B}^T\mathbf{E}\breve{\mathbf{B}}\mathbf{U} \tag{20}$$

The matrix $\mathbf{K}$ is of dimensions $s \times s$, is symmetric and $\det \mathbf{K} \geq 0$. In case of $\det \mathbf{K} > 0$ the solution of (20) is unique; having the displacements $u_1,\ldots,u_s$ one can compute the elongations $\Delta_1,\ldots,\Delta_e$ of bars by $(19)_1$ and then compute reactions by

$$\mathbf{R} = \breve{\mathbf{B}}^T\mathbf{E}\left(\mathbf{B}\mathbf{u} + \breve{\mathbf{B}}\mathbf{U}\right) \tag{21}$$

For future convenience let us introduce the new entities

$$\tilde{\mathbf{\Delta}} = \mathbf{\Delta} - \breve{\mathbf{B}}\mathbf{U}, \quad \tilde{\mathbf{\Delta}}^o = -\breve{\mathbf{B}}\mathbf{U} \tag{22}$$

The set of equations (19) assumes now the form

$$\begin{aligned} &\tilde{\mathbf{\Delta}} = \mathbf{B}\mathbf{u} \\ &\mathbf{B}^T\mathbf{N} = \mathbf{P} \\ &\mathbf{N} = \mathbf{E}\left(\tilde{\mathbf{\Delta}} - \tilde{\mathbf{\Delta}}^o\right) \end{aligned} \tag{23}$$

while reactions at supports are computed by $\mathbf{R} = \breve{\mathbf{B}}^T\mathbf{N}$. The equations (23) have now the form naturally appearing in the static problem of trusses with internal distortions. Let us introduce the Lagrange functional

$$L(\mathbf{v}) = \mathbf{P}\cdot\mathbf{v} - \frac{1}{2}\left(\mathbf{B}\mathbf{v} - \tilde{\mathbf{\Delta}}^o\right)\cdot\left[\mathbf{E}(\mathbf{B}\mathbf{v} - \tilde{\mathbf{\Delta}}^o)\right], \quad \mathbf{v} \in \mathbb{R}^s \tag{24}$$

and the Castigliano functional, the counterpart of the functional (1)

$$\Upsilon(\mathbf{n}) = \frac{1}{2}\mathbf{n}\cdot\left(\mathbf{E}^{-1}\mathbf{n}\right) + \mathbf{n}\cdot\tilde{\mathbf{\Delta}}^o, \qquad \mathbf{n} \in \mathbb{R}^e \tag{25}$$

where the component $\mathbf{n}\cdot\left(\mathbf{E}^{-1}\mathbf{n}\right)$ should be understood as below

$$\mathbf{n}\cdot\left(\mathbf{E}^{-1}\mathbf{n}\right) = \begin{cases} \mathbf{d}\cdot(\mathbf{E}\mathbf{d}) & \text{if } \mathbf{n} \in \mathrm{Im}(\mathbf{E}), \ \mathbf{n} = \mathbf{E}\mathbf{d}, \ \mathbf{d} \in \mathbb{R}^e \\ +\infty & \text{if } \mathbf{n} \notin \mathrm{Im}(\mathbf{E}) \end{cases} \tag{26}$$

The following theorems deliver useful reformulations of the problem (23)

Theorem 2.1

The following statements are equivalent

a1) $\mathbf{u}$ is the solution to the system (23)

a2) $\mathbf{u}$ is the solution to the maximization problem

$$\max_{\mathbf{v}\in\mathbb{R}^s} L(\mathbf{v}) \tag{27}$$

Theorem 2.2

The following statements are equivalent

b1) $\mathbf{N}$ is the solution to the system (23) and then there exist $\mathbf{u}$, $\tilde{\mathbf{\Delta}}$ satisfying (23)

b2) $\mathbf{N}$ is the solution to the minimization problem

$$\min\left\{\Upsilon(\mathbf{n}) \,\middle|\, \mathbf{n} \in \mathbb{R}^e, \ \mathbf{B}^T\mathbf{n} = \mathbf{P}\right\} \tag{28}$$

The proofs of these theorems are given in the Appendices A and B, for the readers' convenience.

Let us note that if $\mathbf{E}$ is reversible, then

$$L(\mathbf{u}) = \Upsilon(\mathbf{N}) \tag{29}$$

Proof. Since $\mathbf{N}$ is statically admissible (statically compatible with the load $\mathbf{P}$), we have $(\mathbf{Bu})\cdot\mathbf{N}=\mathbf{P}\cdot\mathbf{u}$. Due to

$$\mathbf{Bu}=\mathbf{E}^{-1}\mathbf{N}+\tilde{\mathbf{\Delta}}^{o} \tag{30}$$

one gets

$$\left(\mathbf{E}^{-1}\mathbf{N}+\tilde{\mathbf{\Delta}}^{o}\right)\cdot\mathbf{N}=\mathbf{u}\cdot\mathbf{P} \tag{31}$$

hence

$$\frac{1}{2}\mathbf{N}\cdot\left(\mathbf{E}^{-1}\mathbf{N}\right)+\mathbf{N}\cdot\tilde{\mathbf{\Delta}}^{o}=\mathbf{P}\cdot\mathbf{u}-\frac{1}{2}\mathbf{N}\cdot\left(\mathbf{E}^{-1}\mathbf{N}\right) \tag{32}$$

By virtue of

$$\mathbf{N}\cdot\left(\mathbf{E}^{-1}\mathbf{N}\right)=\left(\tilde{\mathbf{\Delta}}-\tilde{\mathbf{\Delta}}^{o}\right)\cdot\left(\mathbf{E}\left(\tilde{\mathbf{\Delta}}-\tilde{\mathbf{\Delta}}^{o}\right)\right) \tag{33}$$

we find

$$\begin{aligned}&\frac{1}{2}\mathbf{N}\cdot\left(\mathbf{E}^{-1}\mathbf{N}\right)+\mathbf{N}\cdot\tilde{\mathbf{\Delta}}^{o}=\\&=\mathbf{P}\cdot\mathbf{u}-\frac{1}{2}\left(\mathbf{Bu}-\tilde{\mathbf{\Delta}}^{o}\right)\cdot\left[\mathbf{E}(\mathbf{Bu}-\tilde{\mathbf{\Delta}}^{o})\right]\end{aligned} \tag{34}$$

which is equivalent to (29). We conclude that the duality gap between the problems (27) and (28) vanishes.

The following notation will be used in the sequel

$$d_k(\mathbf{v})=\sum_{j=1}^{s}B_{kj}v_j \quad d_k^U(\mathbf{v})=d_k(\mathbf{v})+\sum_{l=1}^{m}\breve{B}_{kl}U_l \tag{35}$$

Hence (14) can be re-written by $\Delta_k=d_k^U(\mathbf{u})$.

3. THE TOTAL POTENTIAL ENERGY OF A TRUSS AND ITS LINK TO CASTIGLIANO'S THEOREM

The total potential energy of a truss is understood as in (12), or

$$J=\frac{1}{2}\mathbf{P}\cdot\mathbf{u}+\frac{1}{2}(-\mathbf{R}\cdot\mathbf{U}) \tag{36}$$

By assuming $\bar{\mathbf{u}}=\mathbf{u}$ and $\bar{\mathbf{U}}=\mathbf{U}$ in (15) one obtains the equality

$$\mathbf{N}\cdot\mathbf{\Delta}=\mathbf{P}\cdot\mathbf{u}+\mathbf{R}\cdot\mathbf{U} \tag{37}$$

By virtue of $(19)_3$ we have

$$\mathbf{N}\cdot\left(\mathbf{E}^{-1}\mathbf{N}\right)=\mathbf{P}\cdot\mathbf{u}+\mathbf{R}\cdot\mathbf{U} \tag{38}$$

and hence

$$2J=\mathbf{N}\cdot\left(\mathbf{E}^{-1}\mathbf{N}\right)-2\mathbf{R}\cdot\mathbf{U} \tag{39}$$

while by using $(19)_2$ we get

$$2J=\mathbf{N}\cdot\left(\mathbf{E}^{-1}\mathbf{N}\right)-2(\breve{\mathbf{B}}^{T}\mathbf{N})\cdot\mathbf{U} \tag{40}$$

hence, see (25)

$$J=\Upsilon(\mathbf{N}) \tag{41}$$

We conclude that the function subjected to minimization in Castigliano's theorem (28) represents the total potential energy of a truss.

4. TRUSSES OF MINIMAL POTENTIAL ENERGY

4.1. Formulation of the optimization problem

In case of $\mathbf{P}\neq\mathbf{0}$ and $\mathbf{U}=\mathbf{0}$ the total potential energy equals $J=\mathbf{P}\cdot\mathbf{u}/2$ and represents the compliance of the truss. In case of a one unit force applied at a node the value $2J$ is equal to the projection of the displacement of this node on the direction of the force. Minimization of J means minimization of this displacement. In case of many loads applied the compliance is the weighted sum of the displacements with weights proportional to the magnitudes of the forces. Minimization of J means maximization of stiffness.

In case of $\mathbf{P}=\mathbf{0}$, $\mathbf{U}\neq\mathbf{0}$ the total potential energy equals $J=-\mathbf{R}\cdot\mathbf{U}/2$. Minimization of J means maximization of the work of reactions done on the prescribed displacements of nodes of the supports. Thus, $\mathbf{R}\cdot\mathbf{U}$ measures the resistance of the truss subjected to the imposed displacements of supports and the value $-\mathbf{R}\cdot\mathbf{U}$ is the compliance due to prescribed displacements. The bigger the resistance the better the design of the truss.

In case of $\mathbf{P}\neq\mathbf{0}$ and $\mathbf{U}\neq\mathbf{0}$ the total potential energy J is a linear combination of both the compliances with the same weights ½. Thus, minimization of J is a certain compromise between compliances due to these two various loading conditions.

Assume that the positions of nodes are given, hence also the bars' lengths are prescribed. We consider the designs in which the cost is given as EV, V being the volume of the material of all bars; we require that this cost does not exceed a limit cost Λ_o or, we require that

$$\sum_{k=1}^{e}EA_k l_k\le\Lambda_o \tag{42}$$

We note the units: [Λ_o] = Nm.

Consider the optimum design problem

$$J_{opt}=\min\left\{J\,\middle|\,EA_1\ge 0,\ldots,EA_e\ge 0 \text{ such that } \sum_{k=1}^{e}EA_k l_k\le\Lambda_o\right\} \tag{43}$$

where J is viewed as a function of: the design variables $EA_1,\ldots,EA_e$ and the behavioural variables according to Secs.2,3. According to (41) and (28) the problem (43) can be re-written in the explicit form

$$J_{opt}=\min_{\substack{EA_1\ge0,\ldots,EA_e\ge0\\ \sum_{k=1}^{e}EA_k l_k\le\Lambda_o}}\ \min_{\substack{\mathbf{n}\in\mathbb{R}^e\\ \mathbf{B}^T\mathbf{n}=\mathbf{P}}}\left\{\frac{1}{2}\mathbf{n}\cdot\left(\mathbf{E}^{-1}\mathbf{n}\right)-\left(\breve{\mathbf{B}}^{T}\mathbf{n}\right)\cdot\mathbf{U}\right\} \tag{44}$$

where the first component in the curly brackets should be understood as in (26). The matrix $\mathbf{B}$ does not involve the design variables $EA_1,\ldots,EA_e$, hence problem (44) can be re-written as below, where the order of minimization operations is interchanged

$$J_{opt}=\min_{\substack{\mathbf{n}\in\mathbb{R}^e\\ \mathbf{B}^T\mathbf{n}=\mathbf{P}}}\ \min_{\substack{EA_1\ge0,\ldots,EA_e\ge0\\ \sum_{k=1}^{e}EA_k l_k\le\Lambda_o}}\left\{\frac{1}{2}\sum_{k=1}^{e}\frac{(n_k)^2 l_k}{EA_k}-\left(\breve{\mathbf{B}}^{T}\mathbf{n}\right)\cdot\mathbf{U}\right\} \tag{45}$$

4.2. Elimination of the design variables-problem formulation in terms of member forces

The minimization operation over $EA_1,...,EA_e$ can be performed analytically with using the rule (a discrete counterpart of the rule (10))

$$\min_{\mathbf{x}\in\mathbb{R}^e}\left\{\sum_{k=1}^{e}\frac{a_k}{x_k}\ \middle|\ \text{over } x_i\geq 0 \text{ such that } \sum_{k=1}^{e}x_k\leq\Lambda_o\right\} = \frac{1}{\Lambda_o}\left(\sum_{k=1}^{e}\sqrt{a_k}\right)^2 \quad (46)$$

where $a_1,...,a_e$ are given positive numbers. Let us assume

$$a_k=(n_k l_k)^2,\quad x_k=l_k EA_k \quad (47)$$

Since

$$\sum_{k=1}^{e}\frac{(n_k)^2 l_k}{EA_k}=\sum_{k=1}^{e}\frac{a_k}{x_k} \quad (48)$$

we can make use of the equality (46) thus reducing the problem (45) to the form in which all the design variables are eliminated:

$$J_{opt}=\min_{\substack{\mathbf{n}\in\mathbb{R}^e\\ \mathbf{B}^T\mathbf{n}=\mathbf{P}}}\left\{\frac{1}{2\Lambda_o}\left(\sum_{k=1}^{e}|n_k|l_k\right)^2-\left(\breve{\mathbf{B}}^T\mathbf{n}\right)\cdot\mathbf{U}\right\} \quad (49)$$

Let $\left[n_1^*,...,n_e^*\right]$ be the minimizer of this problem. The minimizer of the problem (46) reads (cf. (11) being its continuum counterpart)

$$x_k^*=\Lambda_o\frac{\sqrt{a_k}}{\sum_{i=1}^{e}\sqrt{a_i}} \quad (50)$$

This formula determines the optimal stiffnesses in terms of the quantities $\left[n_1^*,...,n_e^*\right]$

$$EA_k^*=\Lambda_o\frac{\left|n_k^*\right|}{\sum_{i=1}^{e}\left|n_i^*\right|l_i} \quad (51)$$

The formula above means that the absolute values of the stresses in bars are independent of *k*, or the stresses in the optimal truss are made uniform, like in the standard problem concerning the case of $\mathbf{P}\neq\mathbf{0}$, $\mathbf{U}=\mathbf{0}$, see Hemp [18].

4.3. ELIMINATION OF THE DESIGN VARIABLES-PROBLEM FORMULATION IN TERMS OF DISPLACEMENTS

The problem (49) can be rearranged to the form in which the displacements of nodes will be unknowns, namely:

$$J_{opt}=\max_{\mathbf{v}\in\mathbb{R}^s}\left\{\sum_{j=1}^{s}P_j v_j-\frac{\Lambda_o}{2}\left(\max_{1\leq k\leq e}\left|\frac{d_k^U(\mathbf{v})}{l_k}\right|\right)^2\right\} \quad (52)$$

where $d_k^U(\mathbf{v})$ is given by (35). This is just the problem dual to (49).

Proof of (52). The member forces $\left[n_1,...,n_e\right]$ are statically compatible with the load **P**, hence

$$\mathbf{n}\cdot(\mathbf{Bv})=\mathbf{P}\cdot\mathbf{v}\quad\forall\mathbf{v}\in\mathbb{R}^s \quad (53)$$

Let $\mathbf{r}=\breve{\mathbf{B}}^T\mathbf{n}$. Then $\mathbf{n}\cdot\mathbf{d}^U(\mathbf{v})=\mathbf{P}\cdot\mathbf{v}+\mathbf{U}\cdot\mathbf{r}\quad\forall\mathbf{v}\in\mathbb{R}^s$, which can be re-written in the form

$$p(\mathbf{n},\mathbf{v})=0 \quad (54)$$

where

$$p(\mathbf{n},\mathbf{v})=2\sum_{j=1}^{s}P_j v_j+2\sum_{l=1}^{m}\sum_{k=1}^{e}\breve{B}_{kl}n_k U_l-2\sum_{k=1}^{e}n_k d_k^U(\mathbf{v}) \quad (55)$$

Let

$$G(\mathbf{n})=\frac{1}{\Lambda_o}\left(\sum_{k=1}^{e}|n_k|l_k\right)^2-2\sum_{l=1}^{m}\sum_{k=1}^{e}\breve{B}_{kl}n_k U_l \quad (56)$$

and then

$$G(\mathbf{n})+p(\mathbf{n},\mathbf{v})=\frac{1}{\Lambda_o}\left(\sum_{k=1}^{e}|n_k|l_k\right)^2+ +2\sum_{j=1}^{s}P_j v_j-2\sum_{k=1}^{e}n_k d_k^U(\mathbf{v}) \quad (57)$$

Let us express (49) in the form

$$2J_{opt}=\min_{\mathbf{n}\in\mathbb{R}^e}\max_{\mathbf{v}\in\mathbb{R}^s}\left\{G(\mathbf{n})+p(\mathbf{n},\mathbf{v})\right\} \quad (58)$$

By arguments similar to those used in the theory of the *Free Material Design* (see Bołbotowski and Lewiński [13]) one can interchange the orders of min and max operations to arrive at

$$J_{opt}=\max_{\mathbf{v}\in\mathbb{R}^s}\left\{\sum_{j=1}^{s}P_j v_j+\frac{1}{2}Y\left(\mathbf{d}^U(\mathbf{v})\right)\right\} \quad (59)$$

where

$$Y(\boldsymbol{\Delta})=\min_{\mathbf{n}\in\mathbb{R}^e}\left\{\frac{1}{\Lambda_o}\left(\sum_{k=1}^{e}|n_k|l_k\right)^2-2\sum_{k=1}^{e}n_k\Delta_k\right\} \quad (60)$$

and here $\boldsymbol{\Delta}$ is an arbitrary vector in $\mathbb{R}^e$. Let us construct an explicit form of $Y(\boldsymbol{\Delta})$. To this end we insert $n_k=t\tilde{n}_k$, $t\in\mathbb{R}$, $\tilde{\mathbf{n}}\in\mathbb{R}^e$ with

$$t=\sum_{k=1}^{e}|n_k|l_k,\quad \sum_{k=1}^{e}|\tilde{n}_k|l_k=1 \quad (61)$$

Thus,

$$Y(\boldsymbol{\Delta})=\min_{\substack{\tilde{\mathbf{n}}\in\mathbb{R}^e\\ \sum_{k=1}^{e}|\tilde{n}_k|l_k=1}}\ \min_{t\in\mathbb{R}}\left\{\frac{1}{\Lambda_o}t^2-2\left(\sum_{k=1}^{e}\tilde{n}_k\Delta_k\right)t\right\} \quad (62)$$

Let

$$a=\frac{1}{\Lambda_o},\quad b=\sum_{k=1}^{e}\tilde{n}_k\Delta_k \quad (63)$$

Since

$$\min_{t\in\mathbb{R}}\left\{at^2-2bt\right\}=-\frac{b^2}{a} \quad (64)$$

with the minimizer:

$$t = t^* = \frac{b}{a} = \Lambda_o \sum_{k=1}^{e} \tilde{n}_k \Delta_k \tag{65}$$

one can reduce the problem (60) to the form

$$Y(\boldsymbol{\Delta}) = \min_{\substack{\tilde{\mathbf{n}} \in \mathbb{R}^e \\ \sum_{k=1}^{e} |\tilde{n}_k| l_k = 1}} \left\{ -\Lambda_o \left(\sum_{k=1}^{e} \tilde{n}_k \Delta_k \right)^2 \right\} \tag{66}$$

or

$$Y(\boldsymbol{\Delta}) = -\Lambda_o \left(K(\boldsymbol{\Delta}) \right)^2 \tag{67}$$

where

$$K(\boldsymbol{\Delta}) = \max \left\{ \sum_{k=1}^{e} \tilde{n}_k \Delta_k \;\middle|\; \tilde{\mathbf{n}} \in \mathbb{R}^e, \; \sum_{k=1}^{e} |\tilde{n}_k| l_k = 1 \right\}$$

(68)

Let $\tilde{m}_k = \tilde{n}_k l_k$, $\varepsilon_k = \Delta_k / l_k$, $\tilde{\mathbf{m}} = [\tilde{m}_1, ..., \tilde{m}_e]^T$, $\boldsymbol{\varepsilon} = [\varepsilon_1, ..., \varepsilon_e]^T$.

The function

$$\rho(\tilde{\mathbf{m}}) = \sum_{k=1}^{e} |\tilde{m}_k| \tag{69}$$

is a norm known from the Michell theory, see Lewiński et al [19]. The function polar to $\rho(\cdot)$ is given by

$$\rho^o(\boldsymbol{\varepsilon}) = \max_{\substack{\tilde{\mathbf{m}} \in \mathbb{R}^e \\ \rho(\tilde{\mathbf{m}}) \le 1}} \tilde{\mathbf{m}} \cdot \boldsymbol{\varepsilon} = \max_{\substack{\tilde{\mathbf{m}} \in \mathbb{R}^e \\ \rho(\tilde{\mathbf{m}}) = 1}} \tilde{\mathbf{m}} \cdot \boldsymbol{\varepsilon} \tag{70}$$

Hence

$$K(\boldsymbol{\Delta}) = \rho^o(\boldsymbol{\varepsilon}) \tag{71}$$

and we know that, see Rockafellar ([20], Sec.15)

$$\rho^o(\boldsymbol{\varepsilon}) = \max_{1 \le k \le e} |\varepsilon_k| \tag{72}$$

Thus, $K(\boldsymbol{\Delta}) = \max_{1 \le k \le e} \left| \frac{\Delta_k}{l_k} \right|$ and, according to (67),

$$Y(\boldsymbol{\Delta}) = -\Lambda_o \left(\max_{1 \le k \le e} \left| \frac{\Delta_k}{l_k} \right| \right)^2 \tag{73}$$

which ends the proof. ■

4.4 THE CONDITIONS OF OPTIMALITY

Let $\mathbf{v}^*$ be the maximizer of (52) and $\boldsymbol{\Delta}^*$ be the associated vector of relative elongations. Let $\mathbf{n}^*$ be the minimizer of (49). These solutions of the mutually dual problems are linked by the conditions of optimality which read:

$$\boldsymbol{\Delta}^* = \mathbf{B}\mathbf{v}^* + \breve{\mathbf{B}}\mathbf{U}, \quad \mathbf{B}^T \mathbf{n}^* = \mathbf{P} \tag{74}$$

$$\begin{aligned}
& n_k^* > 0 \Rightarrow \frac{\Delta_k^*}{l_k} = \frac{1}{\Lambda_o} \sum_{i=1}^{e} |n_i^*| l_i \\
& n_k^* < 0 \Rightarrow \frac{\Delta_k^*}{l_k} = -\frac{1}{\Lambda_o} \sum_{i=1}^{e} |n_i^*| l_i \\
& n_k^* = 0 \Rightarrow -\frac{1}{\Lambda_o} \sum_{i=1}^{e} |n_i^*| l_i \le \frac{\Delta_k^*}{l_k} \le \frac{1}{\Lambda_o} \sum_{i=1}^{e} |n_i^*| l_i
\end{aligned} \tag{75}$$

One can write the optimality conditions (75) in the manner George Rozvany used to write them in case of $\mathbf{P} \ne \mathbf{0}, \mathbf{U} = \mathbf{0}$, namely

$$\frac{\Delta_k^*}{l_k} = \hat{k} \operatorname{sgn}\left(n_k^*\right) \text{ if } \; n_k^* \ne 0 \tag{76}$$

and

$$\left| \frac{\Delta_k^*}{l_k} \right| \le \hat{k} \quad \text{if} \quad n_k^* = 0 \tag{77}$$

where

$$\hat{k} = \frac{1}{\Lambda_o} \sum_{i=1}^{e} |n_i^*| l_i \tag{78}$$

We see that the above conditions determine the values of strains Δ_k^* / l_k in terms of the member forces, if the latter do not vanish. For vanishing bars the virtual strains Δ_k^* / l_k are subject to the lower and upper bounds: $-\hat{k}$, $\hat{k}$. An inverse procedure is not available: one cannot easy adjust the member forces to the values of strains.

The optimization procedure results in the strain values Δ_k^* / l_k saturating both the lower and upper bounds in the bars of non-zero cross sections; these bars are subject to a non-zero stress. Here the bound k is determined by the collection of the values $\{n_j^*\}$.

In the bars of non-zero cross sections(which remain upon the optimization process) the following constitutive equations hold

$$n_k^* = EA_k^* \frac{\Delta_k^*}{l_k} \tag{79}$$

where EA_k^* are given by (51). Indeed, let us insert (51) into the r.h.s. of (79) and compute

$$\Lambda_o \frac{|n_k^*|}{\sum_{i=1}^{e} |n_i^*| l_i} \frac{\Delta_k^*}{l_k} = \Lambda_o \frac{|n_k^*|}{\sum_{i=1}^{e} |n_i^*| l_i} \frac{\sum_{i=1}^{e} |n_i^*| l_i}{\Lambda_o} \operatorname{sgn}\left(n_k^*\right) = \tag{80}$$

$$= |n_k^*| \operatorname{sgn}\left(n_k^*\right) = n_k^*$$

which proves that the equations (79) are fulfilled.

To recapitulate we state that the problem (43) reduces to the two mutually dual problems which can be expressed:

-in terms of membrane forces, (49)

-in terms of displacements of free nodes, (52)

The solution to the problem (49) determines directly the optimal stiffnesses according to (51) and thus determines the topology of the optimal truss, since some values of n_k^* vanish and these bars will disappear. The optimal truss may be geometrically variable.

Remark 4.1

A truss is called internally statically determinate if the member forces can be uniquely computed by solving the equilibrium equations of all nodes, i.e. on the basis of the knowledge of the given loads and reactions. The outlined method of topology optimization of trusses admits vanishing of selected areas of the cross sections of members.

Consequently, the optimum truss has a topology different than the initial design. These initial designs are usually internally statically indeterminate. It occurs, however, that for each optimum design problem there exists at least one optimal truss which is internally statically determinate. In other words, the optimization procedure changes the topology such that in the optimum truss the state of member forces can be uniquely determined by the reactions (created by the optimization) process and by the given set of loads. This property of optimal designs becomes clear while analysing particular examples. Usually, the set of optimal solutions is a singleton.

5. THE CASE OF $\mathbf{P} \neq \mathbf{0}$, $\mathbf{U} = \mathbf{0}$: TRUSSES OF MINIMAL COMPLIANCE

We consider the case of $\mathbf{U} = \mathbf{0}$ and show that the problems (49), (52) reduce to the known problems forming the theory of trusses of optimal compliance. By assuming $\mathbf{U} = \mathbf{0}$ in (49) we obtain

$$J_{opt} = \frac{1}{2\Lambda_o} Z^2 \qquad (81)$$

$$Z = \min_{\substack{\mathbf{n}\in\mathbb{R}^e \\ \mathbf{B}^T\mathbf{n}=\mathbf{P}}} \left\{ \sum_{k=1}^{e} |n_k| l_k \right\} \qquad (82)$$

and we see that the problem assumes the well-known form, see Bendsøe et al. [21], Achtziger[22], Lewiński et al.[19].

Now we shall perform a similar reduction of the problem (52). If $\mathbf{U} = 0$ then $d_k^U(\mathbf{v}) = d_k(\mathbf{v})$, see (35). Let $v_j = t\tilde{v}_j$, j=1,…,s and then $d_k(\mathbf{v}) = t d_k(\tilde{\mathbf{v}})$. Let

$$\max_{1\le k\le e} \left| \frac{d_k(\tilde{\mathbf{v}})}{l_k} \right| = 1 \qquad (83)$$

Then

$$\max_{1\le k\le e} \left| \frac{d_k(\mathbf{v})}{l_k} \right| = |t| \max_{1\le k\le e} \left| \frac{d_k(\tilde{\mathbf{v}})}{l_k} \right| = |t| \qquad (84)$$

The problem (52) can be written as below

$$2J_{opt} = \max_{\substack{\tilde{\mathbf{v}}\in\mathbb{R}^s \\ \max_{1\le k\le e}\left|\frac{d_k(\tilde{\mathbf{v}})}{l_k}\right|=1}} \max_{t\in\mathbb{R}} \left\{ 2t\mathbf{P}\cdot\tilde{\mathbf{v}} - \Lambda_o t^2 \right\} \qquad (85)$$

Let $b = \mathbf{P}\cdot\tilde{\mathbf{v}}$, $a = \Lambda_o$. We have $\max_{t\in\mathbb{R}} \left\{ 2bt - at^2 \right\} = b^2 / a$ and the maximizer equals:

$$t^* = \frac{b}{a} = \frac{1}{\Lambda_o} \mathbf{P}\cdot\tilde{\mathbf{v}} \qquad (86)$$

Thus (87) assumes the form

$$2J_{opt} = \frac{1}{\Lambda_o} \max \left\{ (\mathbf{P}\cdot\tilde{\mathbf{v}})^2 \;\middle|\; \tilde{\mathbf{v}}\in\mathbb{R}^s,\ \max_{1\le k\le e} \left| \frac{d_k(\tilde{\mathbf{v}})}{l_k} \right| = 1 \right\} \qquad (87)$$

or

$$J_{opt} = \frac{1}{2\Lambda_o} Z^2 \qquad (88)$$

$$Z = \max \left\{ \mathbf{P}\cdot\tilde{\mathbf{v}} \;\middle|\; \tilde{\mathbf{v}}\in\mathbb{R}^s,\ \max_{1\le k\le e} \left| \frac{d_k(\tilde{\mathbf{v}})}{l_k} \right| = 1 \right\} \qquad (89)$$

The equality in the curly brackets in (89) can be replaced by $\le$. The problems (82) and (89) are mutually dual, which is known from the theory of truss optimization, see Hemp[18], Achtziger[22] and Lewiński et al [19].

Let $\mathbf{n}^*$ be the minimizer of (82) and let $\tilde{\mathbf{v}}^*$ be the maximizer of (89). These vectors are linked by the optimality conditions:

$$\mathbf{B}^T\mathbf{n}^* = \mathbf{P}$$

$$\tilde{\mathbf{\Delta}}^* = \mathbf{B}\tilde{\mathbf{v}}^*$$

$$\begin{aligned} n_k^* > 0 &\Rightarrow \frac{\tilde{\Delta}_k^*}{l_k} = 1 \\ n_k^* < 0 &\Rightarrow \frac{\tilde{\Delta}_k^*}{l_k} = -1 \\ n_k^* = 0 &\Rightarrow -1 \le \frac{\tilde{\Delta}_k^*}{l_k} \le 1 \end{aligned} \qquad (90)$$

Note that

$$\begin{aligned} &\frac{\tilde{\Delta}_k^*}{l_k} = \operatorname{sgn}(n_k^*) \quad \text{if} \quad n_k^* \neq 0 \\ &\left| \frac{\tilde{\Delta}_k^*}{l_k} \right| \le 1 \qquad\quad \text{if} \quad n_k^* = 0 \end{aligned} \qquad (91)$$

and we see that now the bound for the relative elongation $\left| \tilde{\Delta}_k^* / l_k \right|$ does not depend on the values of the membrane forces: $\left[n_1^*, ..., n_e^* \right]$.

In the case considered among the optimal trusses at least one is statically determinate; the proof can be found in Achtziger [22].

6. THE CASE OF $\mathbf{P} = \mathbf{0}$, $\mathbf{U} \neq \mathbf{0}$

We shall now derive the particular forms of the problems (49), (52) in case of a truss with prescribed displacements of supports: $\mathbf{P} = \mathbf{0}$, $\mathbf{U} \neq \mathbf{0}$. The optimal value of the total potential energy equals

$$J_{opt} = -\frac{1}{2} \Lambda_o Z^2 \qquad (92)$$

$$Z = \max \left\{ \left(\breve{\mathbf{B}}^T \tilde{\mathbf{n}} \right) \cdot \mathbf{U} \;\middle|\; \tilde{\mathbf{n}} \in Ker(\mathbf{B}^T),\ \sum_{k=1}^{e} |\tilde{n}_k| l_k = 1 \right\} \qquad (93)$$

or

$$Z = \min_{\mathbf{v}\in\mathbb{R}^s} \left\{ \max_{1\le k\le e} \left| \frac{1}{l_k} \left(\sum_{j=1}^{s} B_{kj} v_j + \sum_{l=1}^{m} \breve{B}_{kl} U_l \right) \right| \right\} \qquad (94)$$

and the problems (93) and (94) are mutually dual. The solution $\tilde{\mathbf{n}}^*$ to the problem (93) defines the optimal stiffnesses by

$$EA_k^* = \Lambda_o \tilde{n}_k^* \tag{95}$$

Those bars in which $\tilde{n}_k^* = 0$ disappear.

Proof.

The virtual vector $\mathbf{n}$ in (49) satisfies the homogeneous equation: $\mathbf{B}^T\mathbf{n} = \mathbf{0}$, hence one can represent the components of $\mathbf{n}$ as $n_k = t\tilde{n}_k$ with $t \in \mathbb{R}$ and

$$\sum_{k=1}^{e} |\tilde{n}_k| l_k = 1, \quad t = \sum_{k=1}^{e} |n_k| l_k \tag{96}$$

Now we re-write (49) in the form

$$2J_{opt} = \min_{\substack{\tilde{\mathbf{n}} \in Ker(\mathbf{B}^T) \\ \sum_{k=1}^{e} |\tilde{n}_k| l_k = 1}} \min_{t \in \mathbb{R}} \left\{ \frac{1}{\Lambda_o} t^2 - 2t \left(\breve{\mathbf{B}}^T \tilde{\mathbf{n}} \right) \cdot \mathbf{U} \right\} \tag{97}$$

Let

$$a = \frac{1}{\Lambda_o}, \quad b = \left(\breve{\mathbf{B}}^T \tilde{\mathbf{n}} \right) \cdot \mathbf{U} \tag{98}$$

but

$$\min_{t \in \mathbb{R}} \left\{ at^2 - 2bt \right\} = -\frac{b^2}{a} \tag{99}$$

and the minimizer equals

$$t = t^* = \frac{b}{a} = \Lambda_o \left(\breve{\mathbf{B}}^T \tilde{\mathbf{n}} \right) \cdot \mathbf{U} \tag{100}$$

The new form of (6.2) reads

$$2J_{opt} = \min_{\substack{\tilde{\mathbf{n}} \in Ker(\mathbf{B}^T) \\ \sum_{k=1}^{e} |\tilde{n}_k| l_k = 1}} \left\{ -\Lambda_o \left(\left(\breve{\mathbf{B}}^T \tilde{\mathbf{n}} \right) \cdot \mathbf{U} \right)^2 \right\} \tag{101}$$

which confirms (92), (93). Let $\tilde{\mathbf{n}}^*$ be the maximizer of this problem. Then

$$t^* = \Lambda_o \left(\breve{\mathbf{B}}^T \tilde{\mathbf{n}}^* \right) \cdot \mathbf{U} \tag{102}$$

and the optimal forces in bars are

$$\mathbf{n}^* = t^* \tilde{\mathbf{n}}^* \tag{103}$$

The optimal member forces determine the optimal stiffnesses by (51). Thus, they can be expressed by (95), since the denominator in (51) equals t^*. The stress-based formulation is now justified.

Now we shall find the problem dual to (92), (93) with using (52). We insert $\mathbf{P} = \mathbf{0}$ and obtain

$$2J_{opt} = -\Lambda_o \left(\min_{\mathbf{v} \in \mathbb{R}^s} \left\{ \max_{1 \le k \le e} \left| \frac{d_k^U(\mathbf{v})}{l_k} \right| \right\} \right)^2 \tag{104}$$

where $d_k^U(.)$ is given by the equation (35). We arrive at (92) with

$$Z = \min_{\mathbf{v} \in \mathbb{R}^s} \left\{ \max_{1 \le k \le e} \left| \frac{d_k^U(\mathbf{v})}{l_k} \right| \right\} \tag{105}$$

which is equivalent to (94).

The displacement-based formulation of the problem is now derived.

As follows from the derivation, the problems (93) and (94) are mutually dual with zero duality gap between them. ■

The problem (94) has a very clear meaning: the aim is to minimize the state of strain in the norm $\rho^o(\cdot)$ given by (72). By solving this problem one obtains the layout of the optimum structure, but to find its optimal stiffnesses one should solve the stress-based problem (93).

Remark 6.1

It is thought appropriate to derive the formula (94) directly from (93). To this end we introduce

$$\Delta_U = \breve{\mathbf{B}}\mathbf{U}, \ \mathbf{L} = diag\{l_1, ..., l_e\}, \ \boldsymbol{\varepsilon}_U = \mathbf{L}^{-1}\Delta_U,$$

$$(\boldsymbol{\varepsilon}(\mathbf{v}))_k = \frac{1}{l_k} d_k(\mathbf{v}), \ \tilde{\mathbf{m}} = \mathbf{L}\tilde{\mathbf{n}}$$

The condition $\tilde{\mathbf{n}} \in Ker(\mathbf{B}^T)$ implies $\tilde{\mathbf{n}} \cdot (\mathbf{B}\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbb{R}^s$.

Thus, the problem (93) can be written and then rearranged as below

$$\begin{aligned}
Z &= \max_{\substack{\tilde{\mathbf{n}} \in Ker(\mathbf{B}^T) \\ \rho(\tilde{\mathbf{m}}) \le 1}} \tilde{\mathbf{m}} \cdot \boldsymbol{\varepsilon}_U = \\
&= \max_{\substack{\tilde{\mathbf{n}} \in \mathbb{R}^e \\ \rho(\mathbf{L}\tilde{\mathbf{n}}) \le 1}} \min_{\mathbf{v} \in \mathbb{R}^s} \left\{ (\mathbf{L}\tilde{\mathbf{n}}) \cdot \boldsymbol{\varepsilon}_U + \tilde{\mathbf{n}} \cdot (\mathbf{B}\mathbf{v}) \right\} \\
&= \max_{\substack{\tilde{\mathbf{n}} \in \mathbb{R}^e \\ \rho(\mathbf{L}\tilde{\mathbf{n}}) \le 1}} \min_{\mathbf{v} \in \mathbb{R}^s} \left\{ (\mathbf{L}\tilde{\mathbf{n}}) \cdot \boldsymbol{\varepsilon}_U + (\mathbf{L}\tilde{\mathbf{n}}) \cdot (\mathbf{L}^{-1}\mathbf{B}\mathbf{v}) \right\} = \\
&= \max_{\substack{\boldsymbol{\tau} \in \mathbb{R}^e \\ \rho(\boldsymbol{\tau}) \le 1}} \min_{\mathbf{v} \in \mathbb{R}^s} \left\{ \boldsymbol{\tau} \cdot \boldsymbol{\varepsilon}_U + \boldsymbol{\tau} \cdot (\mathbf{L}^{-1}\mathbf{B}\mathbf{v}) \right\} \\
&= \min_{\mathbf{v} \in \mathbb{R}^s} \max_{\substack{\boldsymbol{\tau} \in \mathbb{R}^e \\ \rho(\boldsymbol{\tau}) \le 1}} \left\{ \boldsymbol{\tau} \cdot \left(\boldsymbol{\varepsilon}_U + (\mathbf{L}^{-1}\mathbf{B}\mathbf{v}) \right) \right\} = \\
&= \min_{\mathbf{v} \in \mathbb{R}^s} \rho^o\left((\mathbf{L}^{-1}\mathbf{B}\mathbf{v}) + \boldsymbol{\varepsilon}_U \right) = \min_{\mathbf{v} \in \mathbb{R}^s} \rho^o\left(\boldsymbol{\varepsilon}(\mathbf{v}) + \boldsymbol{\varepsilon}_U \right)
\end{aligned} \tag{106}$$

where swapping the operations max and min is justified, like in the *free material design* problems, see Bołbotowski and Lewiński [13]. We conclude that the problems (93) and (94) are mutually dual. ■

Let us derive now the optimality conditions for the problem considered. Let $\mathbf{v}^*$ be the maximizer of (94) and $\tilde{\mathbf{n}}^*$ be the minimizer of (93). These vectors are linked by the optimality conditions as below

$$\begin{aligned}
&\mathbf{B}^T \tilde{\mathbf{n}}^* = \mathbf{0} \\
&\boldsymbol{\Delta}^* = \mathbf{B}\mathbf{v}^* + \breve{\mathbf{B}}\mathbf{U} \\
&\tilde{n}_k^* > 0 \Rightarrow \frac{\Delta_k^*}{l_k} = Z \\
&\tilde{n}_k^* < 0 \Rightarrow \frac{\Delta_k^*}{l_k} = -Z \\
&\tilde{n}_k^* = 0 \Rightarrow -Z \le \frac{\Delta_k^*}{l_k} \le Z
\end{aligned} \tag{107}$$

and

$$Z = \left(\breve{\mathbf{B}}^T \tilde{\mathbf{n}}^* \right) \cdot \mathbf{U} \tag{108}$$

Thus

$$\frac{\Delta_k^*}{l_k} = \mathrm{sgn}(\tilde{n}_k^*) Z, \qquad \left| \frac{\Delta_k^*}{l_k} \right| \le Z \quad \text{if} \quad \tilde{n}_k^* = 0 \tag{109}$$

According to (109) in all members the absolute values of the strains are the same and equal Z. A remarkable feature of this design is that the bound Z depends on the collection of values: $\tilde{n}_k^*,\ k=1,\dots,e$. Let us remind that in the case of $\mathbf{P}\neq\mathbf{0},\ \ \mathbf{U}=\mathbf{0}$ this bound was equal to 1, hence independent of the values of the member forces.

7. NUMERICAL CONSTRUCTION OF THE STATICALLY ADMISSIBLE MEMBER FORCES

The solution $\mathbf{n}=(n_1,n_2,\dots,n_e)\in\mathbb{R}^e$ of the equilibrium equation (19)$_2$, i.e. $\mathbf{B}^T\mathbf{n}=\mathbf{P}$, can be written as follows

$$\mathbf{n}=\bar{\mathbf{n}}+\sum_{k=1}^{r}\alpha_k\mathbf{h}_k \tag{110}$$

where $\bar{\mathbf{n}}=\begin{pmatrix}\bar{n}_1 & \bar{n}_2 & \dots & \bar{n}_e\end{pmatrix}^T\in\mathbb{R}^e$ is an arbitrary solution of the equilibrium equation $\mathbf{B}^T\mathbf{n}=\mathbf{P}$, $r=\dim Ker(\mathbf{B}^T)$ is the dimension of the kernel of the matrix $\mathbf{B}^T$ and $\mathbf{h}_k=\begin{pmatrix}h_{k_1} & h_{k_2} & \dots & h_{k_e}\end{pmatrix}^T\in\mathbb{R}^e$ are the basis vectors that span the vector subspace $Ker(\mathbf{B}^T)$; α_k (k = 1,2,…,r) are arbitrary real numbers. For the given position of the nodes, bars and the known load $\mathbf{P}\in\mathbb{R}^s$, any statically admissible member forces $\mathbf{n}\in\mathbb{R}^e$ can be identified with the vector

$$\boldsymbol{\alpha}=\begin{pmatrix}\alpha_0 & \alpha_1 & \dots & \alpha_{r-1}\end{pmatrix}^T\in\mathbb{R}^r \tag{111}$$

upon establishing the particular solution $\bar{\mathbf{n}}$ and the vectors $\mathbf{h}_k$ (k = 1,2,…,r) spanning the kernel of the equilibrium matrix. In other words, any statically admissible solution $\mathbf{n}\in\mathbb{R}^e$ can be identified with a certain vector $\boldsymbol{\alpha}\in\mathbb{R}^r$. Therefore, the minimization of any functional over statically admissible forces $\mathbf{n}$ can be reduced to its minimization over all vectors $\boldsymbol{\alpha}$. Thus, according to (49), the minimized function J can be equivalently defined as the function $J:\mathbb{R}^r\to\mathbb{R}$ (we do not change the notation J adopted in (36)):

$$J(\boldsymbol{\alpha})=\frac{1}{2\Lambda_0}\left(\sum_{k=1}^{e}\underbrace{\sqrt{\left(n_k(\boldsymbol{\alpha})\right)^2}}_{|n_k(\boldsymbol{\alpha})|}\,l_k\right)^2-\left(\breve{\mathbf{B}}^T\mathbf{n}(\boldsymbol{\alpha})\right)\cdot\mathbf{U} \tag{112}$$

where $\boldsymbol{\alpha}=(\alpha_0,\alpha_1,\dots\alpha_{r-1})\in\mathbb{R}^r$,

$\mathbf{n}(\boldsymbol{\alpha})=\left(n_1(\boldsymbol{\alpha}),n_2(\boldsymbol{\alpha}),\dots,n_e(\boldsymbol{\alpha})\right)^T\in\mathbb{R}^e$ and

$$n_k(\boldsymbol{\alpha})=\bar{n}_k+\sum_{i=1}^{r}\alpha_{i-1}h_{i_k}\ ,\ k=1,2,\dots,e \tag{113}$$

Except for such $\boldsymbol{\alpha}$ for which $n_k(\boldsymbol{\alpha})=0$ for some index k, the gradient of the function (112) is defined by the vector:

$$\nabla J(\boldsymbol{\alpha})=\left(\frac{\partial J}{\partial\alpha_0}(\boldsymbol{\alpha})\ \ \dots\ \ \frac{\partial J}{\partial\alpha_i}(\boldsymbol{\alpha})\ \ \dots\ \ \frac{\partial J}{\partial\alpha_{r-1}}(\boldsymbol{\alpha})\right)^T\in\mathbb{R}^r \tag{114}$$

where

$$\frac{\partial J}{\partial\alpha_{i-1}}(\boldsymbol{\alpha})=\left(\sum_{k=1}^{e}|n_k(\boldsymbol{\alpha})|\,l_k\right)\left(\sum_{k=1}^{e}\frac{n_k(\boldsymbol{\alpha})}{|n_k(\boldsymbol{\alpha})|}\,l_kh_{i_k}\right)-\left(\breve{\mathbf{B}}^T\mathbf{h}_i\right)\cdot\mathbf{U},\quad i=1,2,\dots,r \tag{115}$$

Finally, the minimization of the function (49) over statically admissible member forces $\mathbf{n}\in\mathbb{R}^e,\ \mathbf{B}^T\mathbf{n}=\mathbf{P}$ is reduced to its minimization over all vectors $\boldsymbol{\alpha}=(\alpha_0,\alpha_1,\dots\alpha_{r-1})$, or

$$J_{opt}=\min_{\boldsymbol{\alpha}=(\alpha_0,\alpha_1,\dots\alpha_{r-1})\in\mathbb{R}^r}J(\boldsymbol{\alpha}) \tag{116}$$

thus arriving at a fully non-constrained minimization problem.

8. OPTIMUM DESIGN OF PLANAR AND SPATIAL TRUSSES. CASE STUDIES

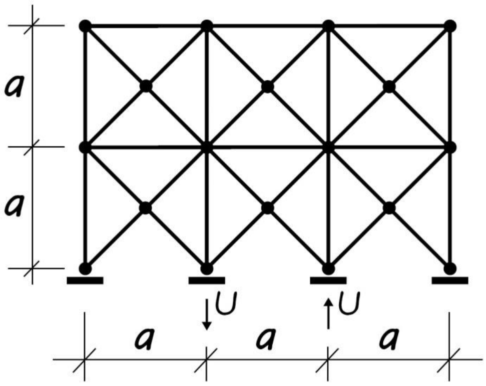
The Singular Value Decomposition algorithm (SVD) along with the routines (available in Press et al [23]) implementing algorithms of non-linear mathematical programming (e.g. Fletcher-Reeves (FR), Polak-Ribiere (PR) or Broyden-Fletcher-Goldfarb-Shanno (BFGS)) will be used in the process of constructing the statically admissible representation (110) and the optimal solution of the problem (116). The Young modulus E and the initial cross-sectional area A_{k-1}^{init}, k = 1,2,…, e of each element of the truss (planar or spatial) to be optimized are assumed as equal to E = 7.2·10^6 [N/cm^2] and 1.0 [cm^2], respectively. In all examples, the bound in the resource condition (42) is assumed according to the rule

$$\Lambda_0=\sum_{k=1}^{e}EA_{k-1}^{init}l_{k-1}[N\,cm] \tag{117}$$

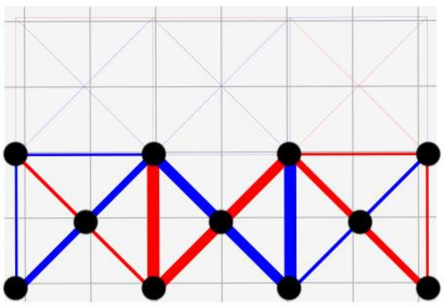
where now $l_{k-1},\ k=1,2,\dots,e$, the lengths of the truss elements are numbered from 0. Red/blue colour indicates the bar in tension/compression state, respectively. In all the figures the green stars ★ show the position of the supporting nodes.

8.1 PLANAR 38-BAR TRUSS

The 38-bar truss (Fig.1) is supported at the four lower nodes and loaded kinematically (skew-symmetrically with respect to the vertical axis of symmetry) with vertical downward and upward displacements of magnitudes U of the left and right middle nodes, respectively (see Fig. 1).



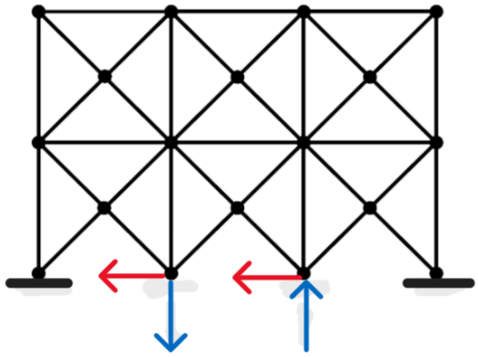
a)



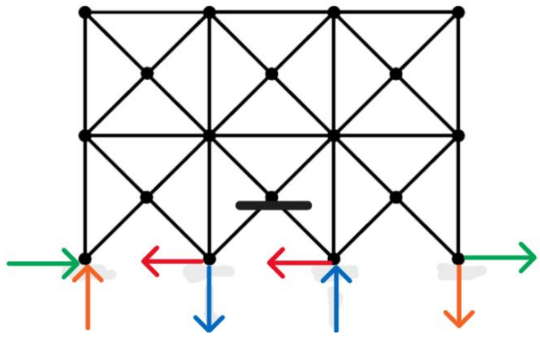
b)

Fig. 1. a) 38-bar truss with marked vertical displacements of supports with the same values but opposite signs. b) Optimal (skew-symmetric with respect to the vertical axis) layout. Red/blue means the bar is in tension/compression. The optimal layout of bars is internally statically determinate.

The consequence of the optimum designing process is vanishing of 20 bars such that the emerging optimal truss becomes *internally statically determinate*, cf. Remark 4.1. Thus, the reactions which emerge in the optimal truss determine the state of member forces uniquely.
Let us consider now the consequences of removing the two middle pin supports and replacing them, in an equivalent way, with support reactions appearing in the optimized truss. These reactions are then treated as known static loads and numerical calculations are again performed to find the optimal cross-sections to minimize the compliance of this 38-bar truss supported this time only at the two extreme nodes, see Fig. 2a).



a)



b)

Fig. 2. a) The equivalent to the skew-symmetric kinematic load: the skew-symmetric static load is applied to the two central nodes where the truss was supported; b) Equivalent to the skew-symmetric kinematic load, the self-balanced skew-symmetric static load is applied to the lower four nodes. The truss is supported only at the central node in the second row from the bottom, remaining externally geometrically variable.

The optimal areas of the cross-sections of the bars turn out to be identical to those found in the previous case, see Fig. 1b), which confirms correctness of the algorithm.

Let us remove now all supports, only adding (to ensure stability in numerical calculations) a pin non-movable support at the lower central node (see Fig. 2b)). After replacing the removed constraints with support reactions appearing in the optimal truss, these reactions are treated, similarly to the previous case, as a known static load. Numerical calculations are again carried out in order to find the optimal bar sections minimizing the compliance of this 38-bar truss, supported at the only one node (the structure is externally geometrically variable) and loaded with a self-balanced static load (composed of the previous reactions) applied to the four lower nodes of the 38-bar truss. Of course, the reaction in the only supporting node is (due to the load being self-balanced) equal to zero. As in the previous case, the optimal cross-sections of the bars are obtained identical as before (up to high accuracy). However, it should be emphasized that during the numerical simulations, the calculations were found to be very sensitive even to very small changes in the values of the reaction forces (with an accuracy of up to several decimal places).

Finally (only for testing purposes), the process of optimization of this 38-bar truss is carried out for the kinematic load corresponding to a small rigid body rotation around the point S, see Fig.3. Let us stress that this kinematic load corresponds to a rigid rotation only within the linearized theory used here.

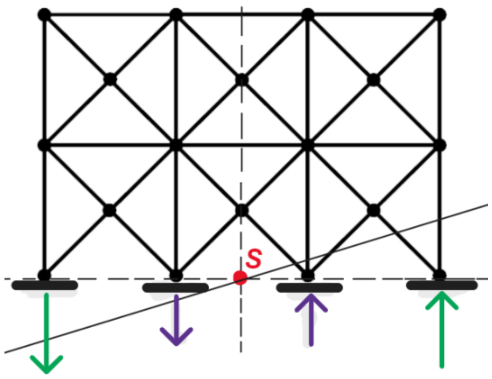


Fig. 3. A kinematic load applied to the 38-bar truss that realizes (within the geometrically linear theory) a rigid rotation of the entire structure around the point S.

The optimal value of the function J^* turns out to be exactly equal to zero and the values of all forces in the bars and the reactions in the four lower support nodes are also equal to zero, which confirms correctness of the algorithms.

8.2 LATTICE SHELL FORMED ON AN ELLIPTIC PARABOLOID

Consider the one-layer lattice shell formed on the elliptic paraboloid

$$z(x,y) = -\beta\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right) \tag{118}$$

covering the rectangle $(-L_x/2, L_x/2)\times(-L_y/2, L_y/2)$; our data will be: a = b = 50 [cm], L_x = L_y = 1500 [cm]. The structure is pin-supported at all 16 boundary nodes, see Fig. 4.

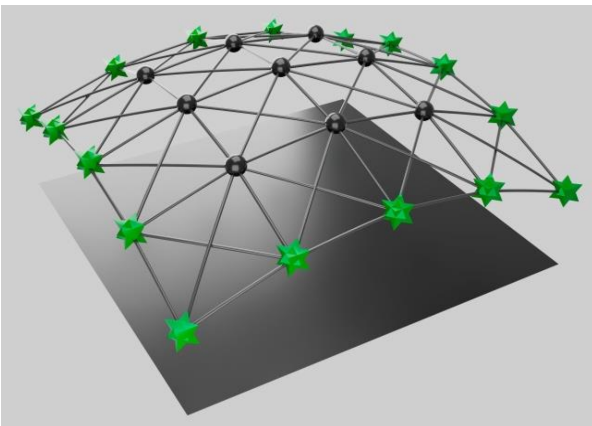


Fig. 4. The lattice shell pin-supported at all 16 boundary nodes (green stars).

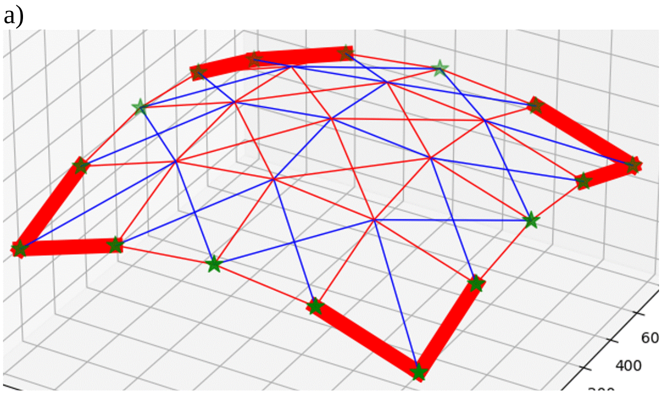
The diagonal bars pass each other in the middle of the cells, i.e. they do not have common nodes there. Two kinds of loads are considered. Either all the internal nodes of the truss are subjected to the vertical concentrated forces of the same magnitude P = –200000 [N] (the negative sign means that they are directed downwards) or the truss is kinematically loaded: all the boundary nodes are displaced downwards according to the interpolation rule

$$U(x,y) = -\alpha\left(\left(\frac{x}{c}\right)^2 + \left(\frac{y}{d}\right)^2\right) \tag{119}$$

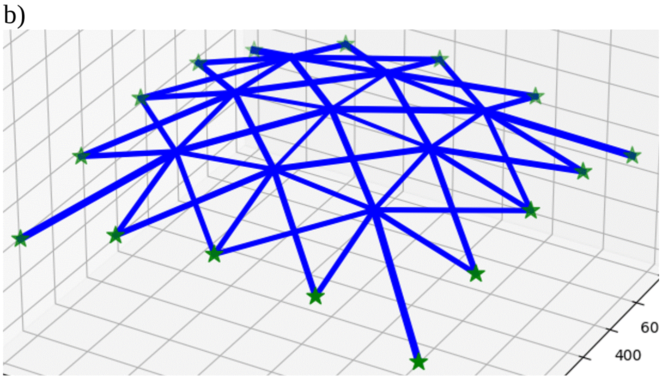
where we have chosen: c = d = 300 [cm]; the parameter α is determined from the condition that U = –12.5 [cm] at four corners; then U = –6.25 [cm] at four middle edge nodes.

The optimization results for the purely kinematic loading (**P** = **0**) read : J^* = –2.65305·10^6 [N cm].
The optimal cross-sections of 8 corner bars are: $A_k^* = 10.24\left[\text{cm}^2\right]$; the optimal cross-sections of the remaining bars are numerically equal to 0, see Fig.5a.
To explain this phenomenon let us note that the kinematic load acts along the boundary while the surface (118) on which the nodes of the structure are placed has a positive Gauss curvature; consequently, the stiffest members are placed around the corners.
The optimization results for the purely static loading (**U** = **0**) read: J^* = 282453,5[N cm]
The maximal optimal area of the cross-section of bars is: $A_k^* = 1.68792\left[\text{cm}^2\right]$, the optimal cross-sections of the remaining bars are slightly smaller and all the cross-sections of the bars on the edges are exactly equal to 0. A view of the optimal truss is shown in Fig. 5b.
The optimization results for the static and kinematic loadings acting simultaneously are: J^* = 273504.5[N cm].
The maximal optimal area of the cross-section of bars is: $A_k^* = 8.57222\left[\text{cm}^2\right]$, the optimal cross-sections of the remaining bars are smaller or vanish.
Thus, when both the loads are applied the solution represents a certain compromise and the design depends heavily on the values of the ratios $P:U$. A view of the optimal truss is shown in Fig. 5c.

a)



b)



c)

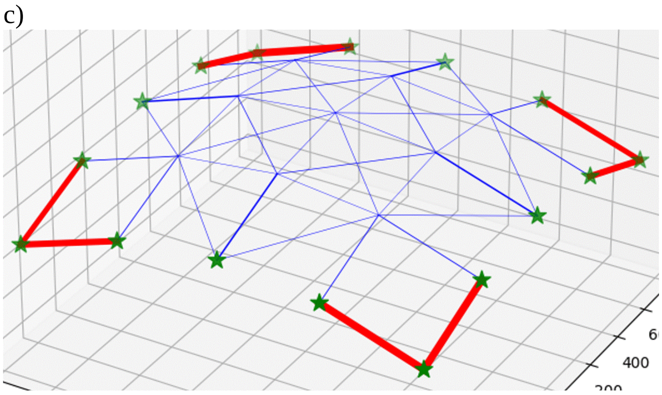


Fig. 5. Optimal layouts of the lattice shell truss from Fig. 4 in: a) purely kinematic-, b) purely static-, and c) kinematic and static case of loading.

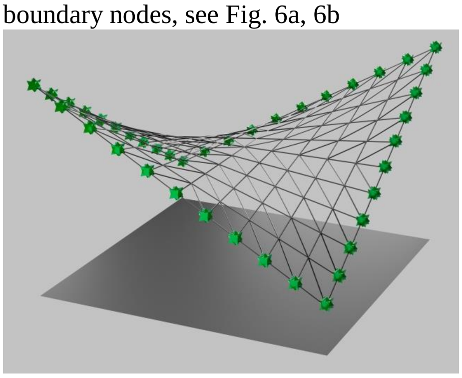
8.3 LATTICE SHELL FORMED ON A HYPERBOLIC PARABOLOID

Consider two one-layer lattice shells formed on the hyperbolic paraboloid :

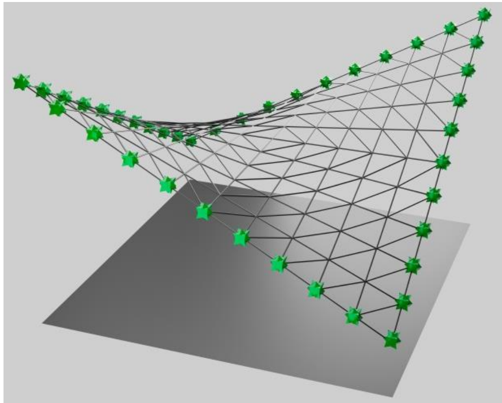
$$z(x,y)=\gamma\, x\, y \qquad (120)$$

covering the rectangular domain:

$(-L_x/2, L_x/2)\times(-L_y/2, L_y/2)$, for the data: $\gamma=10^{-3}\text{cm}^{-1}$, $L_x = L_y = 1500$ [cm]. The structures are pin-supported at all 40 boundary nodes, see Fig. 6a, 6b



a)



b)

Fig. 6. The two lattice shells a) and b), pin-supported at all 40 boundary nodes (green stars).

Two kinds of loads are considered. The static vertical load is applied to all truss nodes. In each node the vertical force is directed downwards and has the same value P = –2000 [N]. The kinematic vertical load is applied to all boundary nodes. The vertical displacements directed downwards are equal U = –5.625 [cm] at four corners and U = 0 [cm] at four middle edge nodes. The vertical directed downward displacements U of the remaining edge nodes are interpolated by the equation $U(x,y)=-|\eta\, x\, y|$ and we choose $\eta=10^{-5}\text{cm}^{-1}$.

The optimization results for the purely kinematic loading (**P**=**0**) are the same for the a) and b) cases: J^* = –2.83431·10^6 [N cm]. The optimal cross-sections of all boundary bars: $A_k^*=8.1\left[\text{cm}^2\right]$, the optimal cross-sections of the remaining bars are (numerically) equal to 0. A view of the optimal truss is shown in Fig. 7a).

The optimization results for the purely static loading (**U**=**0**) : J^* = 242903.5 [N cm] and J^* = 17691 [N cm], for a) and b) case, respectively.

The maximal optimal cross-sections of bars: $A_k^*=2.32128\left[\text{cm}^2\right]$ or $A_k^*=1.66245\left[\text{cm}^2\right]$ for a) or b) cases respectively, and the optimal cross-sections of the remaining bars are smaller, other cross-sections of the bars on the edges are (at least numerically) equal to 0. A view of the optimal trusses is shown in Fig. 7b, 7c).

The optimization results for the static and the kinematic loading acting simultaneously : J^* =–62236 [N cm] and J^* = –1.09688·10^6 [N cm] for a) and b) case respectively.

The maximal optimal cross-sections of bars: $A_k^*=6.82712\left[\text{cm}^2\right]$ and $A_k^*=6.72158\left[\text{cm}^2\right]$ for a) and b) cases, respectively, and optimal cross-sections of the remaining bars are smaller, other cross-sections of the bars are (at least numerically) equal to 0. A view of the optimal truss is shown in Fig. 7d, 7e.

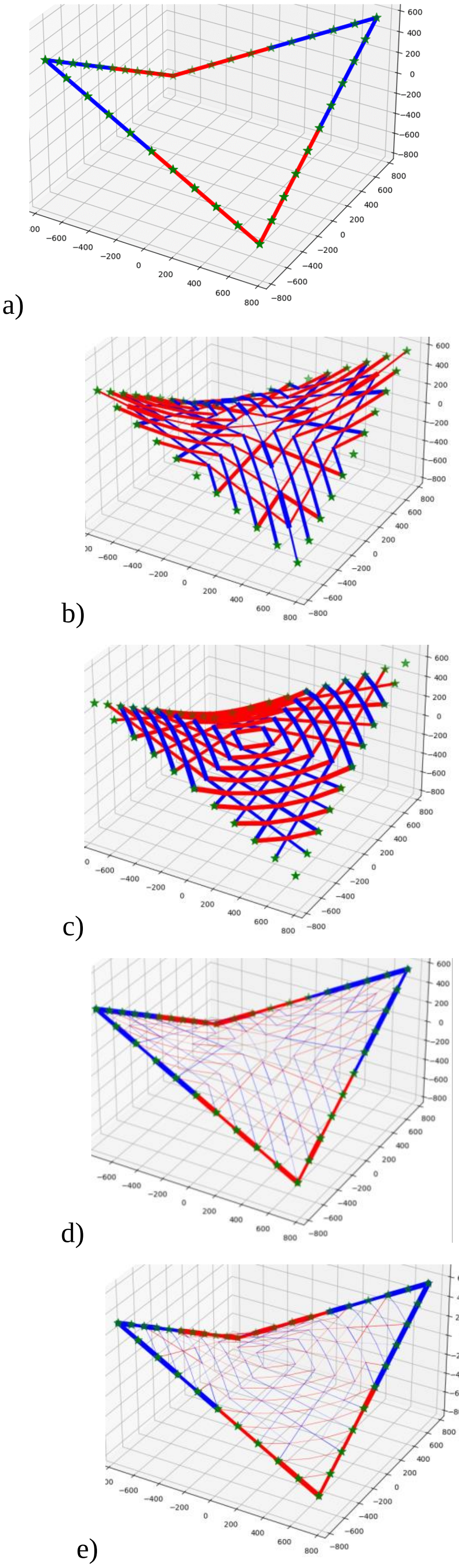


Fig. 7. The optimal layouts of the lattice shell trusses from Fig. 6a) and 6b) in: a) pure kinematic-, b), c) pure static- and d), e) kinematic and static case of loading, respectively.

The kinematic load leads to the identical optimal designs for both layouts (see Fig.6a, 6b) reducing to an empty frame surrounding the structure, see Fig.7a. The optimum designs for the static loads do depend on the initial layout of members, see Fig.7b,c. If both the loads are applied, the optimal designs assume a compromise topology depending on the initial layout of bars, see (d), (e) in Fig.7, and on the ratio $P:U$. The specific topologies in (a,d,e) are consequences of the surface (120) having a negative Gauss curvature.

9. REMARKS ON THE PROGRAMMING PROBLEMS

The chosen numerical method for solving the optimum design problems considered reduces to the minimization of functions without constraints. For solving this problem the three different C-codes have been used: powell(...), frprmn(...), dfpmin(...), expounded in Press et al [23], which allowed in most cases to find the solution (i.e. the minimizer $\boldsymbol{\alpha}^*$) with a reliable accuracy. The answer to the question whether in each problem the numerical solution is unique turned out to be negative. Moreover, some doubts arise whether the obtained numerical solutions can always be treated as global ones in each case of the numerical search for the minimum, especially if both the loads: static and kinematic are applied simultaneously. It often turned out that after calling the three functions powell(...), frprmn(...), dfpmin(...), the minima values were significantly different in numerical terms, although the layouts of optimal topologies obtained on their bases were very similar to each other. The simplest explanation is such that the minimum of the difference of two functions has been sought and the absolute values of both of these functions reach big and often very high values. The second reason is the non-differentiability of the function (112) when the force disappears in at least one truss element, which usually appears at the end of iteration. This obviously suggests the use of algorithms using the concept of a subgradient instead of a gradient, but not necessary for e.g. the non-gradient powell(...) method. However, powell(…) method quite often returned optimal value that was worse (i.e. greater) than those returned by algorithms using the gradients e.g. frprmn(...), dfpmin(...). To increase the probability of finding the best optimal value of the function (112) often all three procedures mentioned above, together with the own optimization procedure, have been called (in different orders) in the program.

10. FINAL REMARKS

The problem of minimization of the total potential energy of a truss has been reduced, upon eliminating all the design variables, to the two mutually dual problems: the stress-based (49) and the displacement-based (52). The solution to the stress-based problem determines directly the optimal stiffnesses of the truss members, and, consequently, the layout of the optimal truss. The optimum design process loosens the initial layout of bars thus leading to internally statically determinate layouts: the loads and the reactions found by the optimization process determine the state of axial forces in bars directly by the equilibrium equations. Thus, minimization of the total potential energy relaxes the state of stress.

The theory developed is a prerequisite of the theory of optimum design of distortions in truss structures.

ACKNOWLEDGEMENTS

The authors express their cordial thanks to Dr. Karol Bołbotowski for his valuable comments and verification of selected results, and to Andrzej M. Lewiński for his help in preparing the figures 4, 6. The paper was prepared within the Research Grant no 2019/33/B/ST8/00325 financed by the National Science Centre (Poland), entitled: Merging the optimum design problems of structural topology and of the optimal choice of material characteristics. The theoretical foundations and numerical methods

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APPENDIX A

The aim is to prove that the statements: (a1) and (a2) in Theorem 2.1 are equivalent.

Let $\mathbf{P} \in \mathrm{Im}(\mathbf{B}^T)$.

1.Let $\mathbf{u}$ satisfy (23) We rearrange $L(\mathbf{v})$ as follows

$$\begin{aligned} L(\mathbf{v}) &= L(\mathbf{v}-\mathbf{u}+\mathbf{u}) = \mathbf{P}\cdot(\mathbf{v}-\mathbf{u}) + \mathbf{P}\cdot\mathbf{u} \\ &-\frac{1}{2}\left(\mathbf{B}(\mathbf{v}-\mathbf{u})+\mathbf{Bu}-\tilde{\boldsymbol{\Delta}}^o\right)\cdot\left[\mathbf{E}(\mathbf{B}(\mathbf{v}-\mathbf{u})+\mathbf{Bu}-\tilde{\boldsymbol{\Delta}}^o)\right] = \\ &= L(\mathbf{u}) + \underline{\mathbf{P}\cdot(\mathbf{v}-\mathbf{u}) - (\mathbf{B}(\mathbf{v}-\mathbf{u}))\cdot\left[\mathbf{E}(\mathbf{Bu}-\tilde{\boldsymbol{\Delta}}^o)\right]} \\ &-\frac{1}{2}(\mathbf{B}(\mathbf{v}-\mathbf{u}))\cdot\left[\mathbf{E}(\mathbf{B}(\mathbf{v}-\mathbf{u}))\right] \end{aligned} \tag{A1}$$

Note that

$$\mathbf{E}(\mathbf{Bu}-\tilde{\boldsymbol{\Delta}}^o) = \mathbf{N} \tag{A2}$$

and due to $\mathbf{P} \in \mathrm{Im}(\mathbf{B}^T)$ we have

$$(\mathbf{B}\hat{\mathbf{v}})\cdot\mathbf{N} = \mathbf{P}\cdot\hat{\mathbf{v}} \qquad \forall \hat{\mathbf{v}} \in \mathbb{R}^s \tag{A3}$$

Substitution of $\hat{\mathbf{v}} = \mathbf{v}-\mathbf{u}$ gives the equality which makes zero the component underscored in (A1). Thus

$$L(\mathbf{v}) = L(\mathbf{u}) - \frac{1}{2}(\mathbf{B}(\mathbf{v}-\mathbf{u}))\cdot\left[\mathbf{E}(\mathbf{B}(\mathbf{v}-\mathbf{u}))\right] \le L(\mathbf{u}) \tag{A4}$$

and the equality takes place only if $\mathbf{v} = \mathbf{u}$.

2.Assume now that $\mathbf{u}$ is the maximizer of (27). Then the term underscored in (A1) must vanish. Due to $\mathbf{v}$ being arbitrary one has

$$\mathbf{B}^T\hat{\mathbf{N}} = \mathbf{P}, \quad \hat{\mathbf{N}} = \mathbf{E}(\mathbf{Bu}-\tilde{\boldsymbol{\Delta}}^o) \tag{A5}$$

but then $\hat{\mathbf{N}}$, $\mathbf{u}$, $\tilde{\boldsymbol{\Delta}} = \mathbf{Bu}$ satisfy (23).

APPENDIX B

The aim is to prove that the statements: (b1) and (b2) in Theorem 2.2 are equivalent.

1.Let $\mathbf{n} \in \mathbb{R}^e$, $\mathbf{P} \in \mathrm{Im}(\mathbf{B}^T)$, $\mathbf{B}^T\mathbf{n} = \mathbf{P}$. Let $\mathbf{N}$ satisfy (23). We shall prove that

$$\Upsilon(\mathbf{n}) \ge \Upsilon(\mathbf{N}) \tag{B1}$$

Indeed, let us write

$$\begin{aligned} \Upsilon(\mathbf{n}) &= \Upsilon(\mathbf{n}-\mathbf{N}+\mathbf{N}) = \frac{1}{2}(\mathbf{n}-\mathbf{N})\cdot\left(\mathbf{E}^{-1}(\mathbf{n}-\mathbf{N})\right) + \Upsilon(\mathbf{N}) \\ &+ \underline{(\mathbf{n}-\mathbf{N})\cdot\left(\mathbf{E}^{-1}\mathbf{N}+\tilde{\boldsymbol{\Delta}}^o\right)} \end{aligned}$$

(B2)

Since $\mathbf{N}$ satisfies (23), or

$$\begin{aligned} &(\mathbf{n}-\mathbf{N})\cdot\left(\mathbf{E}^{-1}\mathbf{N}+\tilde{\boldsymbol{\Delta}}^o\right) = (\mathbf{n}-\mathbf{N})\cdot(\mathbf{Bu}) = \\ &= \left(\mathbf{B}^T\mathbf{n}-\mathbf{B}^T\mathbf{N}\right)\cdot\mathbf{u} = 0 \end{aligned} \tag{B3}$$

we see that (B1) holds; if $\mathbf{n}-\mathbf{N} \in \mathrm{Im}(\mathbf{E})$ then $\Upsilon(\mathbf{n}) < +\infty$.

2. Let now $\mathbf{N}$ be the minimizer of (28). Then the term underscored in (B2) vanishes, which implies $\mathbf{E}^{-1}\mathbf{N}+\tilde{\boldsymbol{\Delta}}^o = \mathbf{Bv}$, $\mathbf{v} \in \mathbb{R}^s$ and $\mathbf{B}^T\mathbf{N} = \mathbf{P}$, hence $\mathbf{v} = \mathbf{u}$ or $\mathbf{N}$ is the solution of (23).