10.24425/acs.2024.149670

Archives of Control Sciences
Volume 34(LXX), 2024
No. 3, *pages 513–535*

Necessary optimality conditions for a Lagrange problem governed by a continuous Roesser model with Caputo derivatives

Rafał KAMOCKI

In the paper, we consider a Lagrange problem governed by a continuous Roesser type system with single partial Caputo derivatives. The necessary optimality conditions for such a problem are derived. In our approach, the increment method, as well as a fractional version of Gronwall's type lemma for functions of two variables are used.

Key words: partial fractional integrals and derivatives, Roesser type system, maximum principle, Lagrange type cost functional

1. Introduction

The investigation object of the present paper is the following optimal control problem:

$$\text{minimize} \quad J(u) = \int_P f^0(x, y, z_u^1(x, y), z_u^2(x, y), u(x, y))\,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

subject to

$$\begin{aligned} {}^C\mathcal{D}^{\alpha}_{x+} z^1 &= f^1(x, y, z^1, z^2, u), \\ {}^C\mathcal{D}^{\beta}_{y+} z^2 &= f^2(x, y, z^1, z^2, u), \\ u(x, y) &\in M \subset \mathbb{R}^m \end{aligned} \tag{2}$$

Author (e-mail: rafal.kamocki@wmii.uni.lodz.pl) is with Faculty of Mathematics and Computer Science, University of Lodz, Banacha 22, 90-238 Lodz, Poland.

Received 20.01.2024. Revised 9.08.2024.

a.e. on $P = [0,a] \times [0,b]$ and

$$\begin{aligned} z^1(0,y) &= \delta(y), \quad y \in [0,b] \ a.e., \\ z^2(x,0) &= \gamma(x), \quad x \in [0,a] \ a.e., \end{aligned} \tag{3}$$

where $\alpha, \beta \in (0,1)$, ${}^C\mathcal{D}^{\alpha}_{x+}$, ${}^C\mathcal{D}^{\beta}_{y+}$ denote single partial fractional differential operators in the Caputo sense, $f^0\colon P \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}$, $f^i\colon P \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_i}$, $i = 1,2$, $\delta\colon [0,b] \to \mathbb{R}^{n_1}$, $\gamma\colon [0,a] \to \mathbb{R}^{n_2}$, $z_u = (z^1_u, z^2_u)$ is a unique solution of system (2)–(3), corresponding to any fixed control u.

Systems of the above type reduce to a classical 2D continuous Roesser model ($\alpha = \beta = 1$) which is a counterpart of the 2D discrete model introduced by Roesser in [13]. Such models are used to describe chemical processes occurring in reactors with varying catalyst activity [10, 11, 15]. Many papers are devoted to linear continuous and discrete-time systems described by the fractional Roesser model. In [14], using the 2D Laplace transform, a general response formula for the problem of type (2)–(3) has been derived. In [7], the authors obtained the formula of such a type for fractional discrete-time Roesser model with the aid of $\mathcal{Z}$-transform. Furthermore, in both papers the necessary and sufficient conditions for the positivity and stability have been studied. Existence of solutions, as well as positivity of a fractional hybrid (discrete-continuous) Roesser model have been investigated in [1].

The aim of this paper is to derive the maximum principle for problem (1)–(2). In [2, 8] results of such a type for the classical nD Roesser model and linear systems 2 with the Riemann-Liouville derivatives, respectively, have been obtained. To derive the necessary optimality conditions, a smooth-convex extremum principle by Ioffe–Tikhomirov ([6]) was applied there. In our approach, we use the increment method in which it is necessary to estimate the increments of the trajectory and the cost functional. In contrast to the method used in [2, 8], compactness of the set M is not required (boundedness of M is sufficient). Furthermore, our method allows us to avoid a convexity-type assumption on f^0, f^1, f^2 which is required in a smooth-convex extremum principle. A key role in our approach plays some fractional version of the Gronwall lemma for functions of two variables (Appendix) which enables us to obtain pointwise equiboundedness of trajectories (Proposition 1). In [4], existence of optimal solutions for a Lagrange problem governed by the classical and fractional (with the Riemann-Liouville derivatives) Roesser model has been obtained. Different structures of the control system, sets of controls, as well as different growth conditions imposed on f^0 have been considered there. In [12], a linear–quadratic optimal control problem described by 2D Roesser model with Caputo derivatives is studied. A numerical solution of such a problem by using the Ritz method and the Laplace transform has been obtained there.

The paper is organized as follows. In Section 2, definitions of partial fractional integrals and derivatives are recalled. Section 3 is devoted the main result of the paper, namely the maximum principle for problem (1)–(3). A theoretical illustrative example is contained in Section 4. Finally, in Appendix mentioned Gronwall's lemma, as well as a theorem on the existence of a unique solution to a linear differential system with the right-sided Riemann-Liouville partial derivatives are proved.

2. Preliminaries

In this part of the paper, we recall some necessary definitions and results concerning fractional calculus of functions of two variables (for details, see [5,9]).

Let $R = [c_1, d_1] \times [c_2, d_2] \subset \mathbb{R}^2$ be any bounded rectangle.

We will use the following notation:

- $L_n^r([c,d])$ – the space of all r-summable functions $\varphi\colon [c,d] \to \mathbb{R}^n$, endowed with the norm $\|\varphi\|_{L_n^r([c,d])} = \left(\int_c^d |\varphi(t)|^r \, \mathrm{d}t\right)^{\frac{1}{r}}$ for any $1 \leqslant r < \infty$;
- $L_n^\infty([c,d])$ – the space of all essentially bounded functions $\varphi\colon [c,d] \to \mathbb{R}^n$, endowed with the norm $\|\varphi\|_{L_n^\infty([c,d])} = \operatorname*{ess\,sup}_{t\in[c,d]} |\varphi(t)|$;
- $L_n^r(R)$ – the space of all r-summable functions $\varphi\colon R \to \mathbb{R}^n$, endowed with the norm $\|\varphi\|_{L_n^r(R)} = \left(\int_R |\varphi(x,y)|^r \, \mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{r}}$ for any $1 \leqslant r < \infty$;
- $L_n^\infty(R)$ – the space of all essentially bounded functions $\varphi\colon R \to \mathbb{R}^n$, endowed with the norm $\|\varphi\|_{L_n^\infty(R)} = \operatorname*{ess\,sup}_{(x,y)\in R} |\varphi(x,y)|$.

Let $\alpha > 0$. By the left–sided Riemann–Liouville integrals of a function $w \in L_n^1(R)$ of order α with respect to x and y we shall mean functions

$$(I^\alpha_{c_1+,x} w)(x,y) := \frac{1}{\Gamma(\alpha)} \int_{c_1}^{x} \frac{w(s,y)}{(x-s)^{1-\alpha}} \, \mathrm{d}s, \quad (x,y) \in R \ a.e.$$

$$(I^\alpha_{c_2+,y} w)(x,y) := \frac{1}{\Gamma(\alpha)} \int_{c_2}^{y} \frac{w(x,t)}{(y-t)^{1-\alpha}} \, \mathrm{d}t, \quad (x,y) \in R \ a.e.,$$

respectively, with the convention that
$(I^0_{c_1+,x} w)(x,y) = w(x,y)$ and $(I^0_{c_2+,y} w)(x,y) = w(x,y)$ for a.e. $(x,y) \in R$.

Similarly, we define the right–sided Riemann-Liouville integrals, namely

$$(I^{\alpha}_{d_1-,x}w)(x,y) := \frac{1}{\Gamma(\alpha)}\int_x^{d_1}\frac{w(s,y)}{(s-x)^{1-\alpha}}\,\mathrm{d}s, \quad (x,y)\in R\ a.e.$$

$$(I^{\alpha}_{d_2-,y}w)(x,y) := \frac{1}{\Gamma(\alpha)}\int_y^{d_2}\frac{w(x,t)}{(t-y)^{1-\alpha}}\,\mathrm{d}t, \quad (x,y)\in R\ a.e.,$$

whereby
$(I^{0}_{d_1-,x}w)(x,y) = w(x,y)$ and $(I^{0}_{d_2-,y}w)(x,y) = w(x,y)$ for a.e. $(x,y)\in R$. To simplify the notations, we will use the symbols I^{α}_{x+} and I^{α}_{y+} to denote the left–sided fractional integrals $I^{\alpha}_{0+,x}$, $I^{\alpha}_{0+,y}$, respectively.

Now, we give definitions of partial fractional derivatives in the Caputo sense introduced in [9]. In the rest of this section we assume that $\alpha\in(0,1)$. Let us consider a class of functions $\widetilde{w}\in L^1_n(P)$ such that

(a) $\widetilde{w}(\cdot,y)$ is continuous on $[0,a]$ for a.e. $y\in[0,b]$,

(b) $\widetilde{w}(0,\cdot)\in L^1_n([0,b])$.

By $C_{x,0}(P,\mathbb{R}^n)$ we shall denote the set of all functions $w\colon P\to\mathbb{R}^n$, for which there exists a function $\widetilde{w}\in L^1_n(P)$ such that $w=\widetilde{w}$ a.e. on P and $\widetilde{w}$ satisfies conditions (a), (b). We shall identify any function $w\in C_{x,0}(P,\mathbb{R}^n)$ with its representant $\widetilde{w}$ described above.

Similarly, we define the set of functions denoted by $C_{y,0}(P,\mathbb{R}^n)$. In this case conditions (a) and (b) are replaced with the following ones:

$(\widetilde{a})$ $\widetilde{w}(x,\cdot)$ is continuous on $[0,b]$ for a.e. $x\in[0,a]$,

$(\widetilde{b})$ $\widetilde{w}(\cdot,0)\in L^1_n([0,a])$.

For a function $w\in C_{x,0}(P,\mathbb{R}^n)$ ($w\in C_{y,0}(P,\mathbb{R}^n)$) we define the left–sided single partial Caputo derivative ${}^{C}\mathcal{D}^{\alpha}_{x+}w$ (${}^{C}\mathcal{D}^{\alpha}_{y+}w$) of order α, with respect to x (y) as follows:

$$({}^{C}\mathcal{D}^{\alpha}_{x+}w)(x,y) = D^{\alpha}_{x+}(w(\cdot,\cdot)-w(0,\cdot))(x,y), \ \ (x,y)\in P\ a.e.$$

$$\big(({}^{C}\mathcal{D}^{\alpha}_{y+}w)(x,y) = D^{\alpha}_{y+}(w(\cdot,\cdot)-w(\cdot,0))(x,y), \ \ (x,y)\in P\ a.e.\big),$$

provided that right-hand side of the above equality exists. Here D^{α}_{x+}, D^{α}_{y+} are single partial fractional differential operators in the Riemann-Liouville sense (cf. [5]).

Let $1 < \frac{1}{\alpha} < p < \infty$. By ${}^{C}AC^{\alpha,p}_{x+}(L^{\infty}_{n}([0,b]))$ we denote the set of all functions $w\colon P \to \mathbb{R}^n$ given by

$$w(x,y) = \eta(y) + (I^{\alpha}_{x+}\varphi)(x,y), \quad (x,y) \in P \ a.e., \tag{4}$$

with some functions $\eta \in L^{\infty}_{n}([0,b])$ and $\varphi \in L^{p}_{n}(P)$. Similarly, we define the set ${}^{C}AC^{\alpha,p}_{y+}(L^{\infty}_{n}([0,a]))$, namely:

$$\begin{aligned}{}^{C}AC^{\alpha,p}_{y+}(L^{\infty}_{n}([0,a])) = \big\{w\colon P \to \mathbb{R}^n;\ \ w(x,y) = \kappa(x) + (I^{\alpha}_{y+}\psi)(x,y),\\ (x,y) \in P \ a.e.\big\},\end{aligned}$$

with some functions $\kappa \in L^{\infty}_{n}([0,a])$ and $\psi \in L^{p}_{n}(P)$.

Remark 1. *From [9, Lemma 3 and Theorem 5] it follows that if $w \in {}^{C}AC^{\alpha,p}_{x+}(L^{\infty}_{n}([0,b]))$ then $w(0,y) = \eta(y)$ and there exists the Caputo derivative ${}^{C}\mathcal{D}^{\alpha}_{x+}w = \varphi$ a.e. on P. Similarly, if $w \in {}^{C}AC^{\alpha,p}_{y+}(L^{\infty}_{n}([0,a]))$ then $w(x,0) = \kappa(x)$ and there exists the Caputo derivative ${}^{C}\mathcal{D}^{\alpha}_{y+}w = \psi$ a.e. on P.*

3. Maximum principle

In this section we derive the necessary optimality conditions for problem (1)–(3).

We introduce the following assumptions on functions f^0 and $f = (f^1, f^2)$:

(A) $f^i(\cdot,\cdot,z^1,z^2,u)$ is measurable on P for all $(z^1,z^2,u) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^m$, $f^i(x,y,z^1,z^2,\cdot)$ is continuous on $\mathbb{R}^m$ for a.e. $(x,y)\in P$ and all $(z^1,z^2) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $f^i(x,y,\cdot,\cdot,u)$ is continuously differentiable on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ for a.e. $(x,y)\in P$ and all $u \in \mathbb{R}^m$, whereby $i = 0,1,2$,

(A_{f^0}) there exist functions $\eta_{f^0} \in C(\mathbb{R}^+_0\times\mathbb{R}^+_0\times\mathbb{R}^+_0,\mathbb{R}^+_0)$, $c_{f^0} \in L^1(P,\mathbb{R}^+_0)$ such that

$$|f^0(x,y,z^1,z^2,u)| \leqslant c_{f^0}(x,y)\eta_{f^0}(|z^1|,|z^2|,|u|) \tag{5}$$

$$|f^0_{z^i}(x,y,z^1,z^2,u)| \leqslant \eta_{f^0}(|z^1|,|z^2|,|u|), \quad i = 1,2 \tag{6}$$

for a.e. $(x,y)\in P$ and all $(z^1,z^2,u) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times M$,

(A_f) there exist a constant $L > 0$ and functions $\eta_{f^i} \in C(\mathbb{R}^+_0,\mathbb{R}^+_0)$, $\gamma_1 \in L^p_1([0,a])$, $\gamma_2 \in L^p_1([0,b])$, with $p > 1$, such that

$$|f^i(x,y,z^1,z^2,u) - f^i(x,y,w^1,w^2,u)| \leqslant L(|z^1-w^1| + |z^2-w^2|),$$

$$|f^1(x,y,0,0,u)| \leqslant \gamma_1(x)\eta_{f^1}(|u|), \qquad |f^2(x,y,0,0,u)| \leqslant \gamma_2(y)\eta_{f^2}(|u|)$$

for a.e. $(x,y)\in P$ and all $u \in M$, $z^i, w^i \in \mathbb{R}^{n_i}$, $i = 1,2$.

Let $\alpha, \beta \in (0,1)$, $1 < \frac{1}{\beta} < p < \infty$, $1 < \frac{1}{\alpha} < p < \infty$, $\delta \in L^{\infty}_{n_1}([0,b])$ and $\gamma \in L^{\infty}_{n_2}([0,a])$. By a solution (trajectory) to control system (2)–(3), corresponding to a control $u \in \mathcal{U}_M$, we mean a function $(z^1, z^2) \in {}^{C}AC^{\alpha,p}_{x+}(L^{\infty}_{n_1}([0,b])) \times {}^{C}AC^{\beta,p}_{y+}(L^{\infty}_{n_2}([0,a]))$ satisfying system (2) a.e. on P and boundary conditions (3), where

$$\mathcal{U}_M := \{u \in L^{\infty}_m(P); \qquad u(x,y) \in M, \quad (x,y) \in P \ a.e.\} \subset L^{\infty}_m(P)$$

(one can show that under assumptions (A) and (A_f), for any fixed $u \in \mathcal{U}_M$ problem (2)–(3) possesses a unique solution[1]). In such a case, the pair $((z^1, z^2), u)$ is called an admissible process, whereby u, (z^1, z^2) are said to be admissible control and state, respectively. A couple $((z^1_*, z^2_*), u_*)$ is called an optimal solution to problem (1)–(3) if it is admissible process and minimizes cost (1) among all admissible processes $((z^1, z^2), u)$.

Now, we prove the following useful result

Proposition 1. *Let* $((z^1, z^2), u) \in \left({}^{C}AC^{\alpha,p}_{x+}(L^{\infty}_{n_1}([0,b])) \times {}^{C}AC^{\beta,p}_{y+}(L^{\infty}_{n_2}([0,a]))\right) \times \mathcal{U}_M$ *be an admissible process. If* $M \subset \mathbb{R}^m$ *is bounded, then there exists a constant* $C > 0$ *(independent on u) such that*

$$|z^1(x,y)|, |z^2(x,y)| \leqslant C, \quad (x,y) \in P \ a.e.$$

for any $u \in \mathcal{U}_M$.

Proof. By (A_f) it follows that

$$\begin{aligned} |z^1(x,y)| &\leqslant \|\delta\|_{L^{\infty}_{n_1}([0,b])} + I^{\alpha}_{x+}|{}^{C}\mathcal{D}^{\alpha}_{x+}z^1(x,y)| \\ &= \|\delta\|_{L^{\infty}_{n_1}([0,b])} + I^{\alpha}_{x+}|f^1(x,y,z^1(x,y),z^2(x,y),u(x,y))| \\ &\leqslant \|\delta\|_{L^{\infty}_{n_1}([0,b])} + LI^{\alpha}_{x+}(|z^1(x,y)| + |z^2(x,y)|) + I^{\alpha}_{x+}(\gamma_1(x)\eta_{f^1}(|u(x,y)|)), \end{aligned}$$

for a.e. $(x,y) \in P$. Since M is bounded, therefore there exists a constant $C_i > 0$ (independed on u) such that

$$\eta_{f^i}(|u(x,y)|) \leqslant C_i, \quad (x,y) \in P \ a.e., \ i = 1,2.$$

Hence, with the aid of the Hölder inequality, we assert that

$$|z^1(x,y)| \leqslant LI^{\alpha}_{x+}(|z^1(x,y)| + |z^2(x,y)|) + C_3, \quad (x,y) \in P \ a.e.,$$

[1]The existence result can be obtained for example with the aid of the Banach contraction principle, applied to problem with zero boundary conditions. Next, using an appriopriate substitution the result of such a type can be obtained for problem with nonzero boundary conditions ([9]).

where

$$C_3 = \|\delta\|_{L^\infty_{n_1}([0,b])} + \frac{C_1\|\gamma_1\|_{L^p_1([0,a])} a^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}.$$

Similarly, we obtain the estimation

$$|z^2(x,y)| \leqslant L I^\beta_{y+}(|z^1(x,y)| + |z^2(x,y)|) + C_4, \quad (x,y) \in P \ a.e.,$$

where

$$C_4 = \|\gamma\|_{L^\infty_{n_2}([0,a])} + \frac{C_2\|\gamma_2\|_{L^p_1([0,b])} b^{\beta-\frac{1}{p}}}{\Gamma(\beta)} \left(\frac{p-1}{\beta p-1}\right)^{1-\frac{1}{p}}.$$

Consequently,

$$\begin{aligned}
&|z^1(x,y)| + |z^2(x,y)| \\
&\quad \leqslant L\Big(I^\alpha_{x+}(|z^1(x,y)| + |z^2(x,y)|) + I^\beta_{y+}(|z^1(x,y)| + |z^2(x,y)|)\Big) + C_3 + C_4 \\
&\quad \leqslant L \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\beta)}\right\}\left(\int_0^x \frac{|z^1(s,y)| + |z^2(s,y)|}{(x-s)^{1-\alpha}}\,\mathrm{d}s \right. \\
&\qquad \left. + \int_0^y \frac{|z^1(x,t)| + |z^2(x,t)|}{(y-t)^{1-\beta}}\,\mathrm{d}t\right) + C_3 + C_4,
\end{aligned}$$

for a.e. $(x,y) \in P$. From Corollary 1 (Appendix) it follows that there exists a constant $C_5 > 0$ such that

$$|z^1(x,y)| + |z^2(x,y)| \leqslant (C_3 + C_4)C_5, \quad (x,y) \in P \ a.e.,$$

Putting $C = (C_3 + C_4)C_5$, we conclude

$$|z^i(x,y)| \leqslant |z^1(x,y)| + |z^2(x,y)| \leqslant C, \quad (x,y) \in P \ a.e., \ i = 1,2.$$

Now, we formulate and prove the main result of this paper.

Theorem 1. *Let M be a bounded set. Under assumptions (A), (A_{f^0}), (A_f) if $((z^1_*, z^2_*), u_*) \in \big({}^C AC^{\alpha,p}_{x+}(L^\infty_{n_1}([0,b])) \times {}^C AC^{\beta,p}_{y+}(L^\infty_{n_2}([0,a]))\big) \times \mathcal{U}_M$ is an optimal solution to problem (1)–(3) and $\lambda = (\lambda^1, \lambda^2) \in I^\alpha_{a-,x}(L^\infty_{n_1}(P)) \times$*

$I^{\beta}_{b-,y}(L^{\infty}_{n_2}(P))$[2] is a solution to the conjugated system

$$\begin{cases}(D^{\alpha}_{a-,x}\lambda^1)(x,y) = f^1_{z^1}[x,y]\lambda^1(x,y) + f^2_{z^1}[x,y]\lambda^2(x,y) - f^0_{z^1}[x,y]\\ (D^{\beta}_{b-,y}\lambda^2)(x,y) = f^1_{z^2}[x,y]\lambda^1(x,y) + f^2_{z^2}[x,y]\lambda^2(x,y) - f^0_{z^2}[x,y]\end{cases} \tag{7}$$

a.e. on P with boundary conditions

$$\begin{cases}(I^{1-\alpha}_{a-,x}\lambda^1)(0,y) = 0, & y \in [0,b] \ a.e.\\ (I^{1-\beta}_{b-,y}\lambda^2)(x,0) = 0, & x \in [0,a] \ a.e.,\end{cases} \tag{8}$$

where $f^i_{z^j}[x,y] = f^i_{z^j}(x,y,z^1_*(x,y),z^2_*(x,y),u_*(x,y))$, $i = 0,1,2$, $j = 1,2$, $D^{\alpha}_{a-,x}$ and $D^{\beta}_{b-,y}$ are partial right–sided Riemann-Liouville differential operators (cf. [8, Remark 2]), then

$$\begin{aligned}\sum_{i=1}^{2}\lambda^i(x,y)f^i&\left(x,y,z^1_*(x,y),z^2_*(x,y),u_*(x,y)\right)\\ &- f^0\left(x,y,z^1_*(x,y),z^2_*(x,y),u_*(x,y)\right)\\ &= \max_{v\in M}\left\{\sum_{i=1}^{2}\lambda^i(x,y)f^i(x,y,z^1_*(x,y),z^2_*(x,y),v)\right.\\ &\left.- f^0(x,y,z^1_*(x,y),z^2_*(x,y),v)\right\}\end{aligned} \tag{9}$$

for a.e. $(x,y) \in P$.

Proof. Let $((z^1_*,z^2_*),u_*)$ be an optimal solution to problem (1)–(3) and $((z^1,z^2),u)$ denotes an admissible process. Then the increment

$$(\Delta z^1, \Delta z^2) = (z^1 - z^1_*, z^2 - z^2_*)$$

is a solution to the following system

$$\begin{cases}{}^{C}\mathcal{D}^{\alpha}_{x+}\Delta z^1 = \Delta f^1(x,y,z^1,z^2,u)\\ {}^{C}\mathcal{D}^{\beta}_{y+}\Delta z^2 = \Delta f^2(x,y,z^1,z^2,u)\\ u(x,y) \in M \subset \mathbb{R}^m\end{cases} \tag{10}$$

[2]The sets $I^{\alpha}_{a-,x}(L^{\infty}_{n_1}(P))$ and $I^{\alpha}_{b-,y}(L^{\infty}_{n_2}(P))$ are defined as follows:

$$I^{\alpha}_{a-,x}(L^{\infty}_{n_1}(P)) := \{w\colon P \to \mathbb{R}^{n_1} : \quad w = I^{\alpha}_{a-,x}\varphi \ a.e. \ on \ P, \quad \varphi \in L^{\infty}_{n_1}(P)\},$$

$$I^{\alpha}_{b-,y}(L^{\infty}_{n_2}(P)) := \{w\colon P \to \mathbb{R}^{n_2} : \quad w = I^{\alpha}_{b-,y}\psi \ a.e. \ on \ P, \quad \psi \in L^{\infty}_{n_2}(P)\}.$$

a.e. on P and

$$\begin{cases}\Delta z^1(0,y)=0, & y\in[0,b]\ a.e.\\ \Delta z^2(x,0)=0, & x\in[0,a]\ a.e.,\end{cases} \tag{11}$$

where $\Delta f^i(x,y,z^1,z^2,u)=f^i(x,y,z^1,z^2,u)-f^i(x,y,z^1_*,z^2_*,u_*)$, $i=1,2$. Moreover,

$$\Delta J(u)=J(u)-J(u_*)=\int_P \Delta f^0(x,y,z^1(x,y),z^2(x,y),u(x,y))\,\mathrm{d}x\,\mathrm{d}y,$$

whereby $\Delta f^0(x,y,z^1,z^2,u)=f^0(x,y,z^1,z^2,u)-f^0(x,y,z^1_*,z^2_*,u_*)$. Now, let us fix any function $(\lambda^1,\lambda^2)\in L^\infty_{n_1}(P)\times L^\infty_{n_2}(P)$. Then, the increment $\Delta J(u)$ can be written as follows:

$$\begin{aligned}\Delta J(u)=&\int_P\Big(\Delta f^0(x,y,z^1(x,y),z^2(x,y),u(x,y))\\&+\lambda^1(x,y)\big({}^C\mathcal{D}^\alpha_{x+}\Delta z^1(x,y)-\Delta f^1(x,y,z^1(x,y),z^2(x,y),u(x,y))\big)\\&+\lambda^2(x,y)\big({}^C\mathcal{D}^\beta_{y+}\Delta z^2(x,y)-\Delta f^2(x,y,z^1(x,y),z^2(x,y),u(x,y))\big)\Big)\mathrm{d}x\,\mathrm{d}y\\=&\int_P\big(\lambda^1(x,y)^C\mathcal{D}^\alpha_{x+}\Delta z^1(x,y)+\lambda^2(x,y)^C\mathcal{D}^\beta_{y+}\Delta z^2(x,y)\big)\mathrm{d}x\,\mathrm{d}y\\&-\int_P\Big(H(x,y,z^1(x,y),z^2(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\\&-H(x,y,z^1_*(x,y),z^2_*(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\\&+H(x,y,z^1_*(x,y),z^2_*(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\\&-H(x,y,z^1_*(x,y),z^2_*(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))\Big)\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where: $P\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^m\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\longrightarrow\mathbb{R}$,

$$\begin{aligned}H(x,y,z^1,z^2,u,\lambda^1,\lambda^2)=&\lambda^1f^1(x,y,z^1,z^2,u)+\lambda^2f^2(x,y,z^1,z^2,u)\\&-f^0(x,y,z^1,z^2,u).\end{aligned}$$

From the Mean Value Theorem it follows that for a.e. $(x,y)\in P$ there exists $\theta(x,y)\in(0,1)$ such that

$$\begin{aligned}&H(x,y,z^1,z^2,u,\lambda^1,\lambda^2)-H(x,y,z^1_*,z^2_*,u,\lambda^1,\lambda^2)\\&\quad=H_{z^1}(x,y,z^1_*+\theta(x,y)\Delta z^1,z^2_*+\theta(x,y)\Delta z^2,u,\lambda^1,\lambda^2)\Delta z^1\\&\qquad+H_{z^2}(x,y,z^1_*+\theta(x,y)\Delta z^1,z^2_*+\theta(x,y)\Delta z^2,u,\lambda^1,\lambda^2)\Delta z^2.\end{aligned}$$

Consequently,

$$
\begin{aligned}
\Delta J(u) = &\int_P \big(\lambda^1(x,y)\,{}^C\mathcal{D}^{\alpha}_{x+}\Delta z^1(x,y) + \lambda^2(x,y)\,{}^C\mathcal{D}^{\beta}_{y+}\Delta z^2(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&- \int_P \Big(\Delta_{u_*} H_{z^1}(x,y,z^1_*(x,y),z^2_*(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\Delta z^1(x,y) \\
&+ \Delta_{u_*} H_{z^2}(x,y,z^1_*(x,y),z^2_*(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\Delta z^2(x,y) \\
&+ \Delta_{u_*} H(x,y,z^1_*(x,y),z^2_*(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y)) \\
&+ H_{z^1}(x,y,z^1_*(x,y),z^2_*(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))\Delta z^1(x,y) \\
&+ H_{z^2}(x,y,z^1_*(x,y),z^2_*(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))\Delta z^2(x,y) \\
&+ R(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

where

$$
\Delta_{u_*} G(x,y,z^1_*,z^2_*,u,\lambda^1,\lambda^2) = G(x,y,z^1_*,z^2_*,u,\lambda^1,\lambda^2) - G(x,y,z^1_*,z^2_*,u_*,\lambda^1,\lambda^2),
$$

$$
\begin{aligned}
R(x,y) = &\Big(H_{z^1}\big(x,y,(z^1_*(x,y),z^2_*(x,y)) \\
&+ \theta(x,y)(\Delta z^1(x,y),\Delta z^2(x,y)),u(x,y),\lambda^1(x,y),\lambda^2(x,y)\big) \\
&- H_{z^1}(x,y,z^1_*(x,y),z^2_*(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\Big)\Delta z^1(x,y) \\
&+ \Big(H_{z^2}\big(x,y,(z^1_*(x,y),z^2_*(x,y)) \\
&+ \theta(x,y)(\Delta z^1(x,y),\Delta z^2(x,y)),u(x,y),\lambda^1(x,y),\lambda^2(x,y)\big) \\
&- H_{z^2}(x,y,z^1_*(x,y),z^2_*(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\Big)\Delta z^2(x,y).
\end{aligned}
$$

Since $(\Delta z^1, \Delta z^2)$ is a solution to (10), therefore

$$
\Delta z^1 = I^{\alpha}_{x+}\,{}^C\mathcal{D}^{\alpha}_{x+}\Delta z^1, \qquad \Delta z^2 = I^{\beta}_{y+}\,{}^C\mathcal{D}^{\beta}_{y+}\Delta z^2.
$$

Thus, using Fubini's theorem, we conclude

$$
\begin{aligned}
&\int_P H_{z^1}(x,y,z^1_*(x,y),z^2_*(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))\Delta z^1(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_P I^{\alpha}_{a-,x} H_{z^1}(x,y,z^1_*(x,y),z^2_*(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))\,{}^C\mathcal{D}^{\alpha}_{x+}\Delta z^1(x,y)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

and

$$\int_P H_{z^2}(x,y,z_*^1(x,y),z_*^2(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))\Delta z^2(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_P I^{\beta}_{b-,y}H_{z^2}(x,y,z_*^1(x,y),z_*^2(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))^C\mathcal{D}^{\beta}_{y+}\Delta z^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Now, let $(\lambda^1,\lambda^2)\in I^{\alpha}_{a-,x}(L^{\infty}_{n_1}(P))\times I^{\beta}_{b-,y}(L^{\infty}_{n_2}(P))$ be a solution to conjugated system (7)–(8). Then (by definition of $I^{\alpha}_{a-,x}(L^{\infty}_{n_1}(P))\times I^{\beta}_{b-,y}(L^{\infty}_{n_2}(P))$), it satisfies the following integral system

$$\begin{cases}\lambda^1(x,y)=I^{\alpha}_{a-,x}H_{z^1}(x,y,z_*^1(x,y),z_*^2(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))\\ \lambda^2(x,y)=I^{\beta}_{b-,y}H_{z^2}(x,y,z_*^1(x,y),z_*^2(x,y),u_*(x,y),\lambda^1(x,y),\lambda^2(x,y))\end{cases}$$

a.e. on P. Consequently,

$$\begin{aligned}\Delta J(u)=-\int_P\Big(&\Delta_{u_*}H_{z^1}(x,y,z_*^1(x,y),z_*^2(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\Delta z^1(x,y)\\ &+\Delta_{u_*}H_{z^2}(x,y,z_*^1(x,y),z_*^2(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))\Delta z^2(x,y)\\ &+\Delta_{u_*}H(x,y,z_*^1(x,y),z_*^2(x,y),u(x,y),\lambda^1(x,y),\lambda^2(x,y))+R(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Now, let us fix any $v\in M$ and denote by $\mathcal{L}_v$ a set of the Lebesgue points $(\xi,\zeta)\in[0,a)\times[0,b)$ of functions

$$\begin{aligned}&(x,y)\longrightarrow f^i(x,y,z_*^1(x,y),z_*^2(x,y),u_*(x,y)),\\ &(x,y)\longrightarrow f^i(x,y,z_*^1(x,y),z_*^2(x,y),v),\quad i=0,1,2.\end{aligned}$$

For a fixed $(\xi,\zeta)\in\mathcal{L}_v$ and sufficiently small $\varepsilon>0$ (such that $\xi+\varepsilon<a$ and $\zeta+\varepsilon<b$) we define an admissible control u_ε in the following way

$$u_\varepsilon(x,y)=\begin{cases}v; & (x,y)\in P_\varepsilon=[\xi,\xi+\varepsilon)\times[\zeta,\zeta+\varepsilon)\\ u_*(x,y); & (x,y)\in P\setminus P_\varepsilon.\end{cases}$$

Then

$$\begin{aligned}\Delta J(u_\varepsilon)=-\int_{P_\varepsilon}\Big(&\Delta_{u_*}H_{z^1}(x,y,z_*^1(x,y),z_*^2(x,y),v,\lambda^1(x,y),\lambda^2(x,y))\Delta z^1_\varepsilon(x,y)\\ &+\Delta_{u_*}H_{z^2}(x,y,z_*^1(x,y),z_*^2(x,y),v,\lambda^1(x,y),\lambda^2(x,y))\Delta z^2_\varepsilon(x,y)\\ &+\Delta_{u_*}H(x,y,z_*^1(x,y),z_*^2(x,y),v,\lambda^1(x,y),\lambda^2(x,y))+R_{\varepsilon,v}(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}\tag{12}$$

where $(z^1_\varepsilon, z^2_\varepsilon)$ is an admissible trajectory (a solution of (2)–(3), corresponding to u_ε) and

$$
\begin{aligned}
&R_{\varepsilon,v}(x,y)\\
&= \Big(H_{z^1}(x,y,(z^1_*(x,y),z^2_*(x,y)) + \theta(x,y)(\Delta z^1_\varepsilon(x,y),\Delta z^2_\varepsilon(x,y)),v,\lambda^1(x,y),\lambda^2(x,y))\\
&\quad - H_{z^1}(x,y,z^1_*(x,y),z^2_*(x,y),v,\lambda^1(x,y),\lambda^2(x,y))\Big)\Delta z^1_\varepsilon(x,y)\\
&\quad + \Big(H_{z^2}(x,y,(z^1_*(x,y),z^2_*(x,y)) + \theta(x,y)(\Delta z^1_\varepsilon(x,y),\Delta z^2_\varepsilon(x,y)),v,\lambda^1(x,y),\lambda^2(x,y))\\
&\quad - H_{z^2}(x,y,z^1_*(x,y),z^2_*(x,y),v,\lambda^1(x,y),\lambda^2(x,y))\Big)\Delta z^2_\varepsilon(x,y).
\end{aligned}
$$

Now, we investigate the behavior of increments Δz^1_ε, Δz^2_ε on P_ε. First, let us note that since $(\Delta z^1_\varepsilon, \Delta z^2_\varepsilon)$ is a solution to (10)–(11), corresponding to u_ε, therefore problem (10)–(11) splits into two systems:

$$
\begin{cases}
{}^C\mathcal{D}^\alpha_{x+}\Delta z^1_\varepsilon = 0,\\
{}^C\mathcal{D}^\beta_{y+}\Delta z^2_\varepsilon = 0.\\
u(x,y) \in M \subset \mathbb{R}^m
\end{cases}
\tag{13}
$$

a.e. on $P \setminus P_\varepsilon$ with boundary conditions

$$
\begin{cases}
\Delta z^1_\varepsilon(0,y) = 0, & y \in [0,b] \ a.e.,\\
\Delta z^2_\varepsilon(x,0) = 0, & x \in [0,a] \ a.e.
\end{cases}
\tag{14}
$$

and

$$
\begin{cases}
{}^C\mathcal{D}^\alpha_{x+}\Delta z^1_\varepsilon = f^1(x,y,z^1_\varepsilon,z^2_\varepsilon,v) - f^1(x,y,z^1_*,z^2_*,u_*),\\
{}^C\mathcal{D}^\beta_{y+}\Delta z^2_\varepsilon = f^2(x,y,z^1_\varepsilon,z^2_\varepsilon,v) - f^2(x,y,z^1_*,z^2_*,u_*),\\
u(x,y) \in M \subset \mathbb{R}^m
\end{cases}
\tag{15}
$$

a.e. on P_ε. It is clear that the solution $(\Delta z^1_\varepsilon, \Delta^2_\varepsilon)$ of problem (13)–(14), corresponding to u_ε, satisfies the following conditions:

- $\Delta z^1_\varepsilon(x,y) = 0$ for all $x \in [0,\xi)$ and a.e. $y \in [0,b]$,
- $\Delta z^2_\varepsilon(x,y) = 0$ for a.e. $x \in [0,a]$ and all $y \in [0,\zeta)$.

In view of continuity of Δz^1_ε with respect to x and Δz^2_ε with respect to y we conclude that $\Delta z^1_\varepsilon(\xi,y) = 0$ for a.e. $y \in [0,b]$ and $\Delta z^2_\varepsilon(x,\zeta) = 0$ for a.e. $x \in [0,a]$. Consequently, problem (15) can be considered with the boundary conditions

$$\begin{cases} \Delta z_\varepsilon^1(\xi, y) = 0, & y \in [\zeta, \zeta + \varepsilon) \ a.e., \\ \Delta z_\varepsilon^2(x, \zeta) = 0, & x \in [\xi, \xi + \varepsilon) \ a.e. \end{cases} \tag{16}$$

Now, we estimate pointwise on P_ε the solution to (15)–16. Using (A_f) and Proposition 1 we obtain

$$\begin{aligned} |\Delta z_\varepsilon^1(x,y)| &\leqslant I^\alpha_{\xi+,x}\big(|f^1(x,y,z_\varepsilon^1(x,y),z_\varepsilon^2(x,y),v) - f^1(x,y,z_*^1(x,y),z_*^2(x,y),v)| \\ &\quad + |f^1(x,y,z_*^1(x,y),z_*^2(x,y),v) - f^1(x,y,z_*^1(x,y),z_*^2(x,y),u_*(x,y))|\big) \\ &\quad \leqslant LI^\alpha_{\xi+,x}(|\Delta z_\varepsilon^1(x,y)| + |\Delta z_\varepsilon^2(x,y)|) + 2LI^\alpha_{\xi+,x}(|z_*^1(x,y)| + |z_*^2(x,y)|) \\ &\quad + 2C_1(I^\alpha_{\xi+,x}\gamma_1)(x) \leqslant LI^\alpha_{\xi+,x}(|\Delta z_\varepsilon^1(x,y)| + |\Delta z_\varepsilon^2(x,y)|) + D_1\varepsilon^{\alpha-\frac{1}{p}}, \end{aligned}$$

for a.e. $(x, y) \in P_\varepsilon$, where

$$D_1 = \varepsilon^{\frac{1}{p}} \frac{4LC}{\Gamma(\alpha+1)} + \frac{2C_1\|\gamma_1\|_{L_1^p([0,a])}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-\frac{1}{p}}.$$

Similarly,

$$|\Delta z_\varepsilon^2(x,y)| \leqslant LI^\beta_{\zeta+,y}(|\Delta z_\varepsilon^1(x,y)| + |\Delta z_\varepsilon^2(x,y)|) + D_2\varepsilon^{\beta-\frac{1}{p}},$$

for a.e. $(x, y) \in P_\varepsilon$, where

$$D_2 = \varepsilon^{\frac{1}{p}} \frac{4LC}{\Gamma(\beta+1)} + \frac{2C_2\|\gamma_2\|_{L_1^p([0,b])}}{\Gamma(\beta)} \left(\frac{p-1}{\beta p - 1}\right)^{1-\frac{1}{p}}$$

(here C, C_1, C_2 are constants from Proposition 1). Consequently,

$$\begin{aligned} &|\Delta z_\varepsilon^1(x,y)| + |\Delta z_\varepsilon^2(x,y)| \\ &\quad \leqslant L\Big(I^\alpha_{\xi+,x}(|\Delta z_\varepsilon^1(x,y)| + |\Delta z_\varepsilon^2(x,y)|) + I^\beta_{\zeta+,y}(|\Delta z_\varepsilon^1(x,y)| + |\Delta z_\varepsilon^2(x,y)|)\Big) \\ &\quad + D_1\varepsilon^{\alpha-\frac{1}{p}} + D_2\varepsilon^{\beta-\frac{1}{p}} \\ &\quad \leqslant L\max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\beta)}\right\}\left(\int_\xi^x \frac{|\Delta z_\varepsilon^1(s,y)| + |\Delta z_\varepsilon^2(s,y)|}{(x-s)^{1-\alpha}}\,\mathrm{d}s\right. \\ &\quad \left. + \int_\zeta^y \frac{|\Delta z_\varepsilon^1(x,t)| + |\Delta z_\varepsilon^2(x,t)|}{(y-t)^{1-\beta}}\,\mathrm{d}t\right) + \max\{D_1, D_2\}(\varepsilon^{\alpha-\frac{1}{p}} + \varepsilon^{\beta-\frac{1}{p}}), \end{aligned}$$

for a.e. $(x, y) \in P_\varepsilon$. From Corollary 1 it follows that there exists a constant $D > 0$ (independent on u) such that for $i = 1, 2$

$$|\Delta z^i_\varepsilon(x, y)| \leqslant |\Delta z^1_\varepsilon(x, y)| + |\Delta z^2_\varepsilon(x, y)| \leqslant D(\varepsilon^{\alpha-\frac{1}{p}} + \varepsilon^{\beta-\frac{1}{p}}), \quad (x, y) \in P_\varepsilon \; a.e.$$

Now, we calculate $\Delta J(u_\varepsilon)$ given by (12). First, let us note that by Proposition 1, for $i = 1, 2$ we have

$$|z^i_*(x, y) + \theta(x, y)\Delta z^i_\varepsilon(x, y)| \leqslant 2|z^i_*(x, y)| + |z^i_\varepsilon(x, y)| \leqslant 3C, \quad (x, y) \in P_\varepsilon.$$

Hence

$$\int_{P_\varepsilon} |R_{\varepsilon,v}(x, y)| \mathrm{d}x\mathrm{d}y \leqslant 4LD(\|\lambda^1\|_{L^\infty_{n_1}} + \|\lambda^2\|_{L^\infty_{n_2}} + E)(\varepsilon^{\alpha-\frac{1}{p}} + \varepsilon^{\beta-\frac{1}{p}}) \int_{P_\varepsilon} \mathrm{d}x\mathrm{d}y$$
$$= 4LD(\|\lambda^1\|_{L^\infty_{n_1}} + \|\lambda^2\|_{L^\infty_{n_2}} + E)(\varepsilon^{\alpha-\frac{1}{p}} + \varepsilon^{\beta-\frac{1}{p}})\varepsilon^2,$$

where $E = \max\{\eta_{f^0}(r_1, r_2, r_3) : \; |r_1|, |r_2| \leqslant 3C, \; |r_3| \leqslant \nu\}$, $\nu > 0$ is a constant such that $|w| \leqslant \nu$ for all $w \in M$.

Similarly, for $i = 1, 2$, we obtain

$$\int_{P_\varepsilon} \left|\Delta_{u_*} H_{z^i}(x, y, z^1_*(x, y), z^2_*(x, y), v, \lambda^1(x, y), \lambda^2(x, y))\Delta z^i_\varepsilon(x, y)\right| \mathrm{d}x\mathrm{d}y$$
$$\leqslant 2LD(\|\lambda^1\|_{L^\infty_{n_1}} + \|\lambda^2\|_{L^\infty_{n_2}} + E)(\varepsilon^{\alpha-\frac{1}{p}} + \varepsilon^{\beta-\frac{1}{p}})\varepsilon^2.$$

Consequently, since $((z^1_*, z^2_*), u_*)$ is an optimal solution to problem (1)–(3), therefore

$$0 \leqslant J(u_\varepsilon) \leqslant -\int_{P_\varepsilon} \Delta_{u_*} H(x, y, z^1_*(x, y), z^2_*(x, y), v, \lambda^1(x, y), \lambda^2(x, y)) \mathrm{d}x\mathrm{d}y$$
$$+ 8LD(\|\lambda^1\|_{L^\infty_{n_1}} + \|\lambda^2\|_{L^\infty_{n_2}} + E)(\varepsilon^{\alpha-\frac{1}{p}} + \varepsilon^{\beta-\frac{1}{p}})\varepsilon^2,$$

so

$$\frac{1}{\varepsilon^2} \int_{P_\varepsilon} \Delta_{u_*} H(x, y, z^1_*(x, y), z^2_*(x, y), v, \lambda^1(x, y), \lambda^2(x, y)) \mathrm{d}x\mathrm{d}y$$
$$\leqslant 8LD(\|\lambda^1\|_{L^\infty_{n_1}} + \|\lambda^2\|_{L^\infty_{n_2}} + E)(\varepsilon^{\alpha-\frac{1}{p}} + \varepsilon^{\beta-\frac{1}{p}}) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Using the Lebesgue Differentiation Theorem, we assert that

$$\Delta_{u_*} H(\xi, \zeta, z^1_*(\xi, \zeta), z^2_*(\xi, \zeta), v, \lambda^1(\xi, \zeta), \lambda^2(\xi, \zeta)) \leqslant 0, \quad (\xi, \zeta) \in P \; a.e.,$$

so condition (9) is satisfied. The proof is completed.

4. Theoretical example

Example 1. *Let us consider problem (1)–(3), where*
$\alpha_1 = \alpha_2 = \frac{1}{2}$, $p > 2$, $P = [0,2] \times [0,2]$, $M = \left(-\frac{\pi}{2}, \frac{3}{2}\pi\right)$, $\gamma,\ \delta\colon [0,2] \to \mathbb{R}^2$,

$$\gamma(x) = (x^2, x), \quad x \in [0,2], \qquad \delta(y) = (y, y^3), \quad y \in [0,2],$$

$f^0\colon P \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$,

$$f^0(x,y,z^1,z^2,u) = f^0(x,y,(z^1_1,z^1_2),(z^2_1,z^2_2),u) = z^1_1 - 2z^1_2 + z^2_1 + z^2_2 + \cos u,$$

$f^i\colon P \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$,

$$f^i(x,y,z^1,z^2,u) = A_i z^i - B_i \cos u, \quad i = 1,2,$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

It is easily to check that all assumptions of Theorem 1 are satisfied. Consequently, if $(z^1_*, z^2_*, u_*) \in ({}^CAC^{\frac{1}{2},p}_{x+}(L^\infty_2([0,2])) \times {}^CAC^{\frac{1}{2},p}_{y+}(L^\infty_2([0,2]))) \times \mathcal{U}_M$ is an optimal solution to problem (1)–(3) and $(\lambda^1, \lambda^2) \in I^{\frac{1}{2}}_{2-,x}(L^\infty_2(P)) \times I^{\frac{1}{2}}_{2-,y}(L^\infty_2(P))$ is a solution to the conjugate system

$$\begin{cases} \left(D^{\frac{1}{2}}_{2-,x}\lambda^1\right)(x,y) = A_1^T \lambda^1(x,y) + \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ \left(D^{\frac{1}{2}}_{2-,y}\lambda^2\right)(x,y) = A_2^T \lambda^2(x,y) + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \end{cases} \tag{17}$$

for a.e. $(x,y) \in P$ and

$$\left(I^{\frac{1}{2}}_{2-,x}\lambda^1\right)(2,y) = 0, \quad y \in [0,2] \ a.e., \tag{18}$$

$$\left(I^{\frac{1}{2}}_{2-,y}\lambda^2\right)(x,2) = 0, \quad x \in [0,2] \ a.e. \tag{19}$$

then

$$\begin{aligned} &\left(\lambda^1(x,y)B_1 + \lambda^2(x,y)B_2 + 1\right)(-\cos u_*(x,y)) \\ &\quad = \max_{u \in (-\frac{\pi}{2}, \frac{3}{2}\pi)} \left\{(\lambda^1(x,y)B_1 + \lambda^2(x,y)B_2 + 1)(-\cos u)\right\} \end{aligned} \tag{20}$$

for a.e. $(x,y) \in P$.

It is easy to verify that the solution (λ_1, λ_2) to system (17)–(19) is given by

$$\begin{bmatrix} \lambda_1(x,y) \\ \lambda_2(x,y) \end{bmatrix} = \begin{bmatrix} -\dfrac{(2-x)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} & x-2+\dfrac{2(2-x)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \\ -\dfrac{(2-y)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}+y-2 & -\dfrac{(2-y)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \end{bmatrix}, \quad (x,y) \in P \ a.e.$$

Consequently, condition (20) is equivalent to the following one

$$(x+y-5)\cos u_*(x,y) = \max_{u \in (-\frac{\pi}{2}, \frac{3}{2}\pi)} \left\{(x+y-5)\cos u\right\}$$

for a.e. $(x,y) \in P$.

Thus, u_* is of the form

$$u_*(x,y) = \pi, \quad (x,y) \in P \ a.e.$$

It means that for a.e. $(x,y) \in P$

$$\begin{bmatrix} z_*^1(x,y) \\ z_*^2(x,y) \end{bmatrix} = \begin{bmatrix} x^2 - x - \dfrac{2x^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} & x - \dfrac{x^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \\ y - \dfrac{y^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} & y^3 - y + \dfrac{y^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \end{bmatrix},$$

so the pair

$$(z_*, u_*) = \left(\left(z_*^1, z_*^2\right), u_*\right)$$

is only one candidate to be the optimal solution to problem (1) − (3).

5. Conclusions

In the paper we considered a Lagrange type problem described by a fractional Roesser model with Caputo derivatives. Using the increment method the necessary optimality conditions in the form of a Pontryagin maximum principle for such a problem were derived. Let us note that in the above example the set M is not compact, as well as functions f^0, f^1, f^2 are not convex with respect to u. The aim of a forthcoming work will be studying of the sufficient optimality conditions for problem (1)–(3).

Appendix

In the first part of this section we formulate and prove a some version of Gronwall's lemma for functions of two variables.

Let $\alpha, \beta > 0$ and $w \in L_n^1(R)$, $R = [c_1, d_1] \times [c_2, d_2]$. The left–sided mixed Riemann-Liouville integral of order (α, β) of the function w is defined by

$$(I^{\alpha,\beta}_{c_1+,x,c_2+,y} w)(x,y) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_{c_1}^{x} \int_{c_2}^{y} \frac{w(s,t)}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} \mathrm{d}s \mathrm{d}t, \quad (x,y) \in R \ a.e.$$

One can show ([5]) that

$$(I^{\alpha,\beta}_{c_1+,x,c_2+,y} w)(x,y) = (I^{\alpha}_{c_1+,x} I^{\beta}_{c_2+,y} w)(x,y) = (I^{\beta}_{c_2+,y} I^{\alpha}_{c_1+,x} w)(x,y), \quad (x,y) \in R \ a.e. \tag{21}$$

Lemma 1 (Gronwall's lemma)**.** *If $g, h \in L_1^1(R)$ are nonnegative, $N > 0$ and*

$$g(x,y) \leqslant h(x,y) + N\left(\int_{c_1}^{x} \frac{g(s,y)}{(x-s)^{1-\alpha}} \mathrm{d}s + \int_{c_2}^{y} \frac{g(x,t)}{(y-t)^{1-\beta}} \mathrm{d}t \right), \quad (x,y) \in R \ a.e. \tag{22}$$

then

$$g(x,y) \leqslant \Psi(h)(x,y), \quad (x,y) \in R \ a.e., \tag{23}$$

where $\Psi \colon L_1^1(R) \to L_1^1(R)$ is a linear and bounded operator, depending on R, N, α, β.

Proof. The proof of this result is analogous to the proof of [3, Lemma 3.1]. For the convenience of a reader, we present a sketch of the proof of Lemma 1.

Assumption (22) can be written as follows:

$$g(x,y) \leqslant h(x,y) + G(I^{\alpha}_{c_1+,x} + I^{\beta}_{c_2+,y}) g(x,y), \quad (x,y) \in R \ a.e.,$$

where $G = N \max\{\Gamma(\alpha), \Gamma(\beta)\}$. Hence, for $n \geqslant 1$, we obtain

$$g(x,y) \leqslant \sum_{k=0}^{n-1} G^k (I^{\alpha}_{c_1+,x} + I^{\beta}_{c_2+,y})^k h(x,y) + G^n (I^{\alpha}_{c_1+,x} + I^{\beta}_{c_2+,y})^n g(x,y), \quad (x,y) \in R \ a.e., \tag{24}$$

whereby (see (21))

$$(I^{\alpha}_{c_1+,x} + I^{\beta}_{c_2+,y})^k w(x,y) = \sum_{i=0}^{k} \binom{k}{i} I^{i\alpha}_{c_1+,x} I^{(k-i)\beta}_{c_2+,y} w(x,y)$$
$$= \sum_{i=0}^{k} \binom{k}{i} I^{i\alpha,(k-i)\beta}_{c_1+,x,c_2+,y} w(x,y)$$

for any nonnegative function $w \in L^1_1(R)$. Let k_0 be the smallest integer such that

$$\left[\frac{k_0}{2}\right] \min\{\alpha,\beta\} > \arg\min\{\Gamma(\mu); \quad \mu > 0\}.$$

Then, for $k \geqslant k_0$, we have

$$(I^{\alpha}_{c_1+,x} + I^{\beta}_{c_2+,y})^k w(x,y)$$
$$\leqslant \sum_{\substack{0\leqslant i\leqslant k,\\ i\alpha\geqslant 1,\\ (k-i)\beta\geqslant 1}} \binom{k}{i} \frac{\max\{1,(d_1-c_1)^{k\alpha-1}\}\max\{1,(d_2-c_2)^{k\beta-1}\}}{\Gamma\left(\left[\frac{k}{2}\right]\min\{\alpha,\beta\}\right)E} \left(I^{1,1}_{c_1+,x,c_2+,y} w\right)(x,y)$$
$$+ \sum_{\substack{0\leqslant i\leqslant k,\\ i\alpha\geqslant 1,\\ (k-i)\beta<1}} \binom{k}{i} \frac{\max\{1,(d_1-c_1)^{k\alpha-1}\}}{\Gamma\left(\left[\frac{k}{2}\right]\min\{\alpha,\beta\}\right)} \left(I^{1,(k-i)\beta}_{c_1+,x,c_2+,y} w\right)(x,y)$$
$$+ \sum_{\substack{0\leqslant i\leqslant k,\\ i\alpha<1,\\ (k-i)\beta\geqslant 1}} \binom{k}{i} \frac{\max\{1,(d_2-c_2)^{k\beta-1}\}}{\Gamma\left(\left[\frac{k}{2}\right]\min\{\alpha,\beta\}\right)} \left(I^{i\alpha,1}_{c_1+,x,c_2+,y} w\right)(x,y),$$

where $E = \min\{\Gamma(\mu); \quad \mu > 0\}$. Let us define the linear operator
$B \colon L^1_1(R) \to L^1_1(R)$ as follows:

$$B(w) := \left(I^{1,1}_{c_1+,x,c_2+,y} + \sum_{\substack{0\leqslant j\leqslant k,\\ j\beta<1}} I^{1,j\beta}_{c_1+,x,c_2+,y} + \sum_{\substack{0\leqslant i\leqslant k,\\ i\alpha<1}} I^{i\alpha,1}_{c_1+,x,c_2+,y} \right) w.$$

Since B consists of a finite number of terms, therefore it is bounded. Then, for $k > k_0$

$$G^k (I^{\alpha}_{c_1+,x} + I^{\beta}_{c_2+,y})^k w(x,y) \leqslant d_k B(w)(x,y),$$

where

$$d_k = \frac{(2G)^k}{\Gamma\left(\left[\frac{k}{2}\right]\min\{\alpha,\beta\}\right)} \times \left(\frac{\max\{1,(d_1-c_1)^{k\alpha-1}\}\max\{1,(d_2-c_2)^{k\beta-1}\}}{E} + \max\{1,(d_1-c_1)^{k\alpha-1}\} + \max\{1,(d_2-c_2)^{k\beta-1}\}\right).$$

It is easy to check that the sequence $(d_k)_{k\in\mathbb{N}}$ is convergent to 0, as well as, the series $\sum_{k=k_0+1}^{\infty} d_k$ is convergent. Consequently, (24) gives

$$g(x,y) \leqslant \sum_{k=0}^{\infty} G^k (I^{\alpha}_{c_1+,x} + I^{\beta}_{c_2+,y})^k h(x,y) \leqslant A(h)(x,y) + dB(h)(x,y), \quad (x,y)\in R \ a.e.,$$

where $d = \sum_{k=k_0+1}^{\infty} d_k$ and $A\colon L^1_1(R) \to L^1_1(R)$ is a linear bounded operator given by

$$A(h) = \sum_{k=0}^{k_0} G^k (I^{\alpha}_{c_1+,x} + I^{\beta}_{c_2+,y})^k h.$$

Putting

$$\Psi(h) = A(h) + dB(h),$$

we get (23).

From the above Gronwall's lemma, we immediately obtain the following useful result

Corollary 1. *If* $h(x,y) \equiv C > 0$ *then there exists a constant* $D > 0$, *depending on* R, N, α, β, *such that*

$$g(x,y) \leqslant CD, \quad (x,y)\in R \ a.e.$$

In the second part of this section, we shall study the existence and uniqueness of a solution to the following linear problem

$$\begin{aligned} (D^{\alpha}_{a-,x}\lambda^1)(x,y) &= A_{11}(x,y)\lambda^1(x,y) + A_{12}(x,y)\lambda^2(x,y) + B_1(x,y), \\ (D^{\beta}_{b-,y}\lambda^2)(x,y) &= A_{21}(x,y)\lambda^1(x,y) + A_{22}(x,y)\lambda^2(x,y) + B_2(x,y) \end{aligned} \tag{25}$$

a.e. on $P = [0, a] \times [0, b]$ with boundary conditions

$$\begin{aligned} (I^{1-\alpha}_{a-,x}\lambda^1)(0, y) &= 0, \quad y \in [0, b] \ a.e., \\ (I^{1-\beta}_{b-,y}\lambda^2)(x, 0) &= 0, \quad x \in [0, a] \ a.e., \end{aligned} \tag{26}$$

where $\alpha, \beta \in (0, 1)$, $A_{ij} \colon P \to \mathbb{R}^{n_i \times n_j}$, $B_i \colon P \to \mathbb{R}^{n_i}$, $i, j = 1, 2$.

By a solution of the above problem we mean a function

$$\lambda = (\lambda^1, \lambda^2) \in I^{\alpha}_{a-,x}(L^{\infty}_{n_1}(P)) \times I^{\beta}_{b-,y}(L^{\infty}_{n_2}(P)).$$

It is easy to check that the existence of the solution to problem (25)–(26) in $I^{\alpha}_{a-,x}(L^{\infty}_{n_1}(P)) \times I^{\beta}_{b-,y}(L^{\infty}_{n_2}(P))$ is equivalent to the existence of a solution to the following integral problem in $L^{\infty}_{n_1}(P) \times L^{\infty}_{n_2}(P)$

$$\begin{aligned} \psi^1(x, y) &= A_{11}(x, y)(I^{\alpha}_{a-,x}\psi^1)(x, y) + A_{12}(x, y)(I^{\beta}_{b-,y}\psi^2)(x, y) + B_1(x, y), \\ \psi^2(x, y) &= A_{21}(x, y)(I^{\alpha}_{a-,x}\psi^1)(x, y) + A_{22}(x, y)(I^{\beta}_{b-,y}\psi^2)(x, y) + B_2(x, y) \end{aligned} \tag{27}$$

a.e. on $P = [0, a] \times [0, b]$. In such a case $(\lambda^1, \lambda^2) = (I^{\alpha}_{a-,x}\psi^1, I^{\beta}_{b-,y}\psi^2)$.

We have

Theorem 2. *If $A_{ij} \in L^{\infty}_{n_i \times n_j}(P)$, $B_i \in L^{\infty}_{n_i}(P)$, $i = 1, 2$ then problem (25)–(26) has a unique solution $\lambda = (\lambda^1, \lambda^2) \in I^{\alpha}_{a-,x}(L^{\infty}_{n_1}(P)) \times I^{\beta}_{b-,y}(L^{\infty}_{n_2}(P))$.*

Proof. It is sufficient to prove that the operator

$$T = (T^1, T^2) \colon L^{\infty}_{n_1}(P) \times L^{\infty}_{n_2}(P) \to L^{\infty}_{n_1}(P) \times L^{\infty}_{n_2}(P),$$

defined by

$$\begin{aligned} T^i \colon (\psi^1, \psi^2) \to A_{i1}(x, y)(I^{\alpha}_{a-,x}\psi^1)(x, y) + A_{i2}(x, y)(I^{\beta}_{b-,y}\psi^2)(x, y) \\ + B_i(x, y), \quad i = 1, 2, \end{aligned}$$

possesses a unique fixed point. Of course, T is well defined. Let us consider in $L^{\infty}_{n_1}(P) \times L^{\infty}_{n_2}(P)$ the Bielecki norm given by

$$\|\sigma^1, \sigma^2\|_{r, n_1 \times n_2} = \|\sigma^1\|_{r, n_1} + \|\sigma^2\|_{r, n_2},$$

where

$$\|\sigma^i\|_{r, n_i} = \operatorname*{ess\,sup}_{(x,y) \in P} e^{-r(a-x+b-y)} |\sigma^i(x, y)|, \quad i = 1, 2$$

and $r > 0$ is a fixed constant. Due to the relation

$$e^{-r(a+b)} \|\sigma^1, \sigma^2\|_{L^{\infty}_{n_1}(P) \times L^{\infty}_{n_1}(P)} \leqslant \|\sigma^1, \sigma^2\|_{r, n_1 \times n_2} \leqslant \|\sigma^1, \sigma^2\|_{L^{\infty}_{n_1}(P) \times L^{\infty}_{n_1}(P)},$$

where $\|\sigma^1,\sigma^2\|_{L^\infty_{n_1}(P)\times L^\infty_{n_1}(P)} = \operatorname*{ess\,sup}_{(x,y)\in P}|\sigma^1(x,y)| + \operatorname*{ess\,sup}_{(x,y)\in P}|\sigma^2(x,y)|)$, we assert that the space $L^\infty_{n_1}(P)\times L^\infty_{n_1}(P)$ with the norm $\|\cdot,\cdot\|_{r,n_1\times n_2}$ is complete.

Now, we show that the operator T is contraction. Indeed, let

$$c_i = \max\{\|A_{i1}\|_{L^\infty_{n_1}(P)}, \|A_{i2}\|_{L^\infty_{n_2}(P)}\}, \quad i=1,2.$$

Using the fact that

$$\int_x^a \frac{e^{-r(s-x)}}{(s-x)^{1-\alpha}}\mathrm{d}s = \frac{1}{r^\alpha}\int_0^{r(a-x)} e^{-\tau}\tau^{\alpha-1}\mathrm{d}\tau \leqslant \frac{1}{r^\alpha}\int_0^\infty e^{-\tau}\tau^{\alpha-1}\mathrm{d}\tau = \frac{\Gamma(\alpha)}{r^\alpha}$$

and

$$\int_y^b \frac{e^{-r(t-y)}}{(t-y)^{1-\beta}}\mathrm{d}t \leqslant \frac{\Gamma(\beta)}{r^\beta},$$

we have

$$\begin{aligned}
&\left\|T^i(\psi^1,\psi^2) - T^i(\varphi^1,\varphi^2)\right\|_{r,n_i} \\
&\quad\leqslant c_i\left(\|I^\alpha_{a-,x}(\psi^1-\varphi^1)\|_{r,n_1} + \|I^\beta_{b-,y}(\psi^2-\varphi^2)\|_{r,n_2}\right) \\
&\quad\leqslant c_i\left(\operatorname*{ess\,sup}_{(x,y)\in P}\frac{1}{\Gamma(\alpha)}\int_x^a e^{-r(s-x)}e^{-r(a-s+b-y)}\frac{|\psi^1(s,y)-\varphi^1(s,y)|}{(s-x)^{1-\alpha}}\mathrm{d}s\right. \\
&\qquad\left. + \operatorname*{ess\,sup}_{(x,y)\in P}\frac{1}{\Gamma(\beta)}\int_y^b e^{-r(t-y)}e^{-r(a-x+b-t)}\frac{|\psi^2(x,t)-\varphi^2(x,t)|}{(t-y)^{1-\beta}}\mathrm{d}t\right) \\
&\quad\leqslant c_i\left(\|\psi^1-\varphi^1\|_{r,n_1}\operatorname*{ess\,sup}_{(x,y)\in P}\frac{1}{\Gamma(\alpha)}\int_x^a \frac{e^{-r(s-x)}}{(s-x)^{1-\alpha}}\mathrm{d}s\right. \\
&\qquad\left. + \|\psi^2-\varphi^2\|_{r,n_2}\operatorname*{ess\,sup}_{(x,y)\in P}\frac{1}{\Gamma(\beta)}\int_y^b \frac{e^{-r(t-y)}}{(t-y)^{1-\beta}}\mathrm{d}t\right) \\
&\quad\leqslant c_i\max\{r^{-\alpha},r^{-\beta}\}\left(\|\psi^1-\varphi^1\|_{r,n_1} + \|\psi^2-\varphi^2\|_{r,n_2}\right) \\
&\quad= c_i\max\{r^{-\alpha},r^{-\beta}\}\|(\psi^1,\psi^2)-(\varphi^1,\varphi^2)\|_{r,n_1\times n_2}
\end{aligned}$$

for $i=1,2$ and any $(\psi_1,\psi_2),(\varphi_1,\varphi_2)\in L^\infty_{n_1}(P)\times L^\infty_{n_2}(P)$.

Consequently,

$$\begin{aligned}\left\|T(\psi^1,\psi^2)-T(\varphi^1,\varphi^2)\right\|_{r,n_1\times n_2} \\ &= \left\|T^1(\psi^1,\psi^2)-T^1(\varphi^1,\varphi^2)\right\|_{r,n_1} + \left\|T^2(\psi^1,\psi^2)-T^2(\varphi^1,\varphi^2)\right\|_{r,n_2} \\ &\leqslant \max\{c_1,c_2\}\max\{r^{-\alpha},r^{-\beta}\}\left\|(\psi^1,\psi^2)-(\varphi^1,\varphi^2)\right\|_{r,n_1\times n_2}.\end{aligned}$$

Let us choose r such that $(\max\{c_1,c_2\}\max\{r^{-\alpha},r^{-\beta}\}) \in (0,1)$. Then T is a contraction, so from the Banach contraction principle it follows that T has a unique fixed point. The proof is completed.

References

[1] M.A. Ghezzar, A. Berilha, K. Benyettou and D. Bouagada: On the positivity of 2D fractional linear Roesser model using tthe conformable derivative. *UPB Scientific Bulletin*, Series A **85**(2), (2023), 103–114.

[2] D. Idczak: Necessary optimality conditions for a nonlinear continuous n-dimensional Roesser model. *Mathematics and Computers in Simulation*, **41**(1-2), (1996), 87–94.

[3] D. Idczak: A Gronwall lemma for functions of two variables and its application to partial differential equations of fractional order. *Mathematical Control and Related Fields*, **12**(1), (2022), 225–243. DOI: 10.3934/mcrf.2021019

[4] D. Idczak, R. Kamocki, M. Majewski and S. Walczak: Existence of optimal solutions to Lagrange problems for Roesser type systems of the first and fractional orders. *Applied Mathematics and Compuattion*, **266** (2015), 809–819. DOI: 10.1016/j.amc.2015.05.142

[5] D. Idczak, R. Kamocki and M. Majewski: Nonlinear continuous Fornasini-Marchesini model of fractional order with nonzero initial conditions. *Journal of Integral Equations Applications*, **32**(1), (2020), 19–34. DOI: 10.1216/JIE.2020.32.19

[6] A.D. Ioffe and B.M. Tichomirow: *Theory of Extremal Problems*. North-Holland Pub. Co: Amsterdam, New York, Oxford, 1979.

[7] T. Kaczorek and K. Rogowski: Positivity and stabilization of fractional 2D linear systems described by the Roesser model. *International Journal of Applied Mathematics and Computer Science*, **20**(1), (2010), 85–92. DOI: 10.2478/v10006-010-0006-6

[8] R. Kamocki: Necessary and sufficient optimality conditions for fractional nonhomogeneous Roesser model. *Optimal Control Applications and Methods*, **37**(4), (2016), 574–589. DOI: 10.1002/oca.2180

[9] R. Kamocki and C. Obczyński: On the single partial Caputo derivatives for functions of two variables. *Periodica Mathematica Hungarica*, **87** (2023), 324–339. DOI: 10.1007/s10998-023-00520-x

[10] E. Markin: On antagonistic games for a class of dynamic systems, describing relations in chemical reactors. *Vestnic Moskovskogo Universiteta, Seriya, Comput. Math. Cybern.*, **1** (1978), 44–52. (in Russian).

[11] E. Markin and A.S. Strekalovki: On the existence, uniqueness and stability of the solution for a class of dynamic systems describing chemical processes. *Vestnic Moskovskogo Universiteta, Seriya, Comput. Math. Cybern.*, **4** (1977), 3–11. (in Russian).

[12] A. Nemati and K. Mamehrashi: The use of the Ritz method and Laplace transform for solving 2D fractional-order optimal control problems described by the Roesser model. *Asian Journal of Control*, **21**(3), (2019), 1189–1201. DOI: 10.1002/asjc.1791

[13] R.A. Roesser: Discrete state-space model for linear image processing. *IEEE Transactions on Automatic Control*, **AC-20**(1), (1975), 1–10. DOI: 10.1109/TAC.1975.1100844

[14] K. Rogowski: General response formula for fractional 2D continuous-time linear systems described by the Roesser model. *Acta Mechanica et Automatica*, **5**(2), (2011), 112–116.

[15] A.N. Tikhonov and A.A. Samarski: *Equations of Mathematical Physics*. Science, Nauka, Moskva, 1972. (in Russian).