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A computer scientist's perspective on approximation of IFS invariant sets and measures with the random iteration algorithm—proofs and examples

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***Abstract*—We present proofs of the theorems and lemmas demonstrated previously in our paper [1]. We also display some visual examples of minimal absorbing sets and their basins of attractions generated by δ-roundoffs of two-dimensional linear contractions as well as visualizations of DIFS stationary probability measures.**

***Keywords*—IFS; Discrete Space; Markov Chain; Approximation; Invariant Set; Invariant Measure**

I. Proofs of Theorems and Lemmas

In this section we demonstrate proofs of the theorems and lemmas we presented previously in [1].

Proof of Theorem II.1. (c) For each $\tilde{x} \in C$ and each Λ_i from the family of absorbing sets, there is $N(i) \in \mathbb{N}$ such that for all $m \geq N(i)$, $\tilde{w}^{\circ m}(\tilde{x}) \in \Lambda_i$. Hence, for all $m \geq \max_{1\leq i\leq K} N(i)$, $\tilde{w}^{\circ m}(\tilde{x}) \in \bigcap_i \Lambda_i$.

(d) For any $\tilde{x} \in C$, there is $N \in \mathbb{N}$ such that $\{\tilde{w}^{\circ i}(\tilde{x})\}_{i\geq N} \subset \Lambda$. Hence, $\tilde{w}(\{\tilde{w}^{\circ i}(\tilde{x})\}_{i\geq N}) \subset \tilde{w}(\Lambda)$, or equivalently $\{\tilde{w}^{\circ i}(\tilde{x})\}_{i\geq N+1} \subset \tilde{w}(\Lambda)$.

(e) Let $\tilde{x} \in \Lambda$. The set Λ is absorbing in C, so for each $\tilde{x} \in \Lambda$ there exists $M(\tilde{x}) \in \mathbb{N}$ such that $B(\tilde{x}) := \{\tilde{w}^{\circ i}(\tilde{x})\}_{i\geq M(\tilde{x})} \subset \Lambda$. By definition, $\tilde{w}(B(\tilde{x})) \subset B(\tilde{x})$. Therefore, the set $B = \bigcup_{\tilde{x}\in\Lambda} B(\tilde{x})$ has the property that $B \subset \Lambda$ and $\tilde{w}(B) \subset B$. Clearly, B is also absorbing in C, because for any $\tilde{x} \in C$ there is $N \in \mathbb{N}$ such that $\tilde{w}^{\circ N}(\tilde{x}) \in \Lambda$, so for all $i \geq M(\tilde{x}) + N$, $\tilde{w}^{\circ i}(\tilde{x}) \in B$.

(f) Let $\tilde{x} \in D$. Since C is an absorbing set in D, there is $i \in \mathbb{N}$ such that $\tilde{w}^{\circ i}(\tilde{x}) \in C$. Since Λ is an absorbing set in C, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $\tilde{w}^{\circ(i+k)}(\tilde{x}) \in \Lambda$. □

Proof of Theorem II.2. First we show that $\mathcal{M}$ includes all periodic points in C. We have to show that given any periodic point $\tilde{x} \in C$, $\tilde{x}$ belongs to every absorbing set Λ_α for $\tilde{w}$. On the contrary let us assume that $\tilde{x} \notin \Lambda_\alpha$ for some $\alpha \in \mathcal{I}$. Since $\tilde{x}$ is a periodic point, there exists $k \in \mathbb{N}$ such that $\tilde{w}^{\circ k}(\tilde{x}) = \tilde{x}$, and thus $\tilde{w}^{\circ(\alpha k)}(\tilde{x}) = \tilde{x}$ for any $\alpha \in \mathbb{N}$. It follows that for any $N \in \mathbb{N}$ there is $i \geq N$ such that $\tilde{w}^{\circ i}(\tilde{x}) = \tilde{x} \notin \Lambda_\alpha$, which contradicts the assumption of Λ_α being an absorbing set for $\tilde{w}$. Therefore, each absorbing set Λ_α has to include all periodic points in C, and hence they are included in the intersection of the sets.

Now we show that if $\tilde{x} \in \mathcal{M}$, then $\tilde{x}$ must be periodic. On the contrary, let us assume that $\tilde{x} \in \mathcal{M}$ and $\tilde{x}$ is non-periodic, which implies that $\tilde{w}^{\circ i}(\tilde{x}) \neq \tilde{x}$ for all $i \in \mathbb{N}$. Let Λ be any absorbing set such that $\tilde{x} \in \Lambda$. We will show that $\Lambda \setminus \{\tilde{x}\}$ is an absorbing set in C and thus $\tilde{x}$ cannot be in $\mathcal{M}$. Since Λ is absorbing in C, we get that for any $\tilde{y} \in C$ there is $N(\tilde{y}) \in \mathbb{N}$ so that $\tilde{w}^{\circ i}(\tilde{y}) \in \Lambda$ for all $i \geq N(\tilde{y})$. Therefore, if $\tilde{y}$ is such that $\tilde{w}^{\circ i}(\tilde{y}) \neq \tilde{x}$ for all $i \in \mathbb{N}$, then for all $i \geq N(\tilde{y})$, $\tilde{w}^{\circ i}(\tilde{y}) \in \Lambda \setminus \{\tilde{x}\}$. Hence $\Lambda \setminus \{\tilde{x}\}$ absorbs the orbits $\{\tilde{w}^{\circ i}(\tilde{y})\}_{i\in\mathbb{N}}$ that do not intersect $\{\tilde{x}\}$. Now, because $\tilde{x}$ is non-periodic, $\{\tilde{w}^{\circ i}(\tilde{x})\}_{i\in\mathbb{N}}$ does not intersect $\{\tilde{x}\}$, and thus $\tilde{w}^{\circ i}(\tilde{x}) \in \Lambda \setminus \{\tilde{x}\}$ for all $i \geq N(\tilde{x})$. Therefore, for any $\tilde{y} \in C$ such that $\tilde{w}^{\circ k}(\tilde{y}) = \tilde{x}$ for a certain $k \in \mathbb{N}$, we get that $\tilde{w}^{\circ i}(\tilde{y}) \in \Lambda \setminus \{\tilde{x}\}$ for all $i \geq N(\tilde{x}) + k$. Hence $\Lambda \setminus \{\tilde{x}\}$ also absorbs the orbits that do intersect $\{\tilde{x}\}$. Therefore, $\Lambda \setminus \{\tilde{x}\}$ is an absorbing set in C, and it follows that $\tilde{x}$ is not in $\mathcal{M}$, so we have arrived at a contradiction, which completes the proof. □

Proof of Theorem II.4. By Theorem II.1 (b) there is at least one absorbing set in C. Moreover, C is bounded and thus finite, so there is at most a finite number of absorbing sets in C. Hence, the the conclusion follows from Theorem II.1 (c). □

Proof of Theorem II.5. By Theorem II.2, $\mathcal{M}[\tilde{w}, C]$ consists of all periodic points of $\tilde{w}$ in C. Naturally, the points remain periodic in every superset of C, in this instance the set D. Hence, every absorbing set in D has to include all periodic points in D (otherwise the set would not be absorbing in D), and we get that the intersection of all absorbing sets in D is nonempty and include $\mathcal{M}[\tilde{w}, C]$. Moreover, $\mathcal{M}[\tilde{w}, C]$ is an absorbing set in C, and by the assumption of the theorem, C is an absorbing set in D, so from Theorem II.1 (f) it follows that $\mathcal{M}[\tilde{w}, C]$ is also an absorbing set in D, and thus $\mathcal{M}[\tilde{w}, C]$ include the intersection of all absorbing sets in D. Since, as we have shown, $\mathcal{M}[\tilde{w}, C]$ is also included in the intersection, this completes the proof. □

Proof of Theorem II.7. Let $\tilde{x}$ be any point of $\mathcal{D}^n(\delta)$. The mapping w is a contraction on $(\mathbb{R}^n, d)$, so $\lim_{i\to\infty} w^{\circ i}(\tilde{w}) =$

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x_f or equivalently

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall i \geq N, d(w^{\circ i}(\tilde{x}), x_f) < \varepsilon.$$

As a consequence, for a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $i \geq N$ we get that

$$\begin{aligned} d(\tilde{w}^{\circ i}(\tilde{x}), x_f) &\leq d(\tilde{w}^{\circ i}(\tilde{x}), w^{\circ i}(\tilde{x})) + d(w^{\circ i}(\tilde{x}), x_f) \\ &< \theta(1-\lambda)^{-1} + \varepsilon \end{aligned}$$

on the basis of inequality (II.2) in [1]. Hence, for all $i \geq N$, $\tilde{w}^{\circ i}(\tilde{x}) \in \Lambda(x_f, r_0 + \varepsilon)$, and because $\tilde{x}$ is any point of $\mathcal{D}^n(\delta)$, we get that, for every $\varepsilon > 0$, $\Lambda(x_f, r_0 + \varepsilon)$ is an absorbing set for $\tilde{w}$.

Now we prove that $\Lambda(x_f, r_0 + \varepsilon)$ also owns the absorbing property for $\varepsilon = 0$. Let $\Lambda^C(x_f, r_0) = \mathcal{D}^n(\delta) \backslash \Lambda(x_f, r_0)$. Since $\Lambda^C(x_f, r_0)$ is countable, the minimum $\varepsilon = \min\{d(\tilde{y}, x_f) : \tilde{y} \in \Lambda^C(x_f, r_0)\}$ exists and $\varepsilon > r_0$. Therefore, $\Lambda(x_f, r_0 + \varepsilon/2)$ does not include any point from $\Lambda^C(x_f, r_0)$, and thus $\Lambda(x_f, r_0 + \varepsilon/2) = \Lambda(x_f, r_0)$.

The last thing to we show is that $\tilde{w}$ maps $\Lambda(x_f, r_0)$ into itself. Let $\tilde{y} \in \Lambda(x_f, r_0)$. We have

$$\begin{aligned} d(\tilde{w}(\tilde{y}), x_f) &\leq d(\tilde{w}(\tilde{y}), w(\tilde{y})) + d(w(\tilde{y}), x_f) \\ &\leq \theta + \lambda d(\tilde{y}, x_f) \\ &\leq \theta\big(1 + \lambda(1-\lambda)^{-1}\big) = \theta(1-\lambda)^{-1} \end{aligned}$$

because $d(\tilde{w}(\tilde{y}), w(\tilde{y})) \leq \theta$ by the definition of a δ-roundoff of a mapping (Def. II.2 in [1]). Therefore, $\tilde{w}(\tilde{y}) \in \Lambda(x_f, r_0)$, which completes the proof. □

Proof of Corollary II.8. On the basis of Theorem II.7, $\Lambda(x_f, r_0)$ is bounded and $\tilde{w}$ maps it into itself, so from Theorem II.4 there exists $\mathcal{M}[\tilde{w}, \Lambda(x_f, r_0)]$. But, by Theorem II.7, $\Lambda(x_f, r_0)$ is an absorbing set for $\tilde{w}$ in $\mathcal{D}^n(\delta)$, and thus $\mathcal{M}[\tilde{w}] = \mathcal{M}[\tilde{w}, \Lambda(x_f, r_0)]$ by Theorem II.5. □

Proof of Theorem III.2. (a) Suppose C is a closed class, so for every $i \in \{1, \ldots, N\}$, $\tilde{w}_i(C) \subset C$. Hence $\bigcup_{i=1}^N \tilde{w}_i(C) \subset C$. Now, suppose that $\bigcup_{i=1}^N \tilde{w}_i(C) \subset C$. The mappings w_i map C into itself, so for any $\tilde{x} \in C$ and every finite sequence $i_1, \ldots, i_m$ of indices from $\{1, \ldots, N\}$, we have $w_{i_m} \circ \cdots \circ w_{i_1}(\tilde{x}) \in C$. Therefore, if $\tilde{y} \notin C$, then $\tilde{y}$ is not accessible from C, and thus C is a closed class. (b) Let $\tilde{x} \in C$, where C is a communication class. Then $\tilde{x}$ is accessible from any state in C, and hence there exist at least one $\tilde{y} \in C$ and $i \in \{1, \ldots, N\}$ such that $\tilde{w}_i(\tilde{y}) = \tilde{x}$. Hence, $\tilde{x} \in \tilde{w}_i(C)$, and as a consequence $\bigcup_{i=1}^N \tilde{w}_i(C) \supset C$. □

Lemma. *Let $\{S; \tilde{w}_1, \ldots, \tilde{w}_N; p_1, \ldots, p_N\}$ be a DIFS. Suppose that for at least one DIFS mapping $\tilde{w}_i$ there exists a minimal absorbing set $\mathcal{M}[\tilde{w}_i, S]$. Let $\mathcal{A}_k$ be a recurrent communication class of the associated Markov chain $\{\tilde{X}_k\}$. If there exists $\tilde{x} \in S$ such that both $\tilde{x} \in \mathcal{A}_k$ and $\tilde{x} \in \mathcal{B}[\mathcal{M}_j[\tilde{w}_i, S]]$, then $\mathcal{A}_k \supset \mathcal{M}_j[\tilde{w}_i, S]$.*

Proof. Suppose that $\tilde{x} \in S$ satisfies the assumptions above. Since $\tilde{x}$ is in the basin of attraction of $\mathcal{M}_j[\tilde{w}_i, S]$, by Def. II.4 there is $m \in \mathbb{N}$ such that $\tilde{w}_i^{\circ m}(\tilde{x}) \in \mathcal{M}_j[\tilde{w}_i, S]$. Moreover, $\mathcal{M}_j[\tilde{w}_i, S]$ is a periodic orbit for w_i in S (Corollary II.3 in [1]). Therefore, writing $p \in \mathbb{N}$ for the period of $\mathcal{M}_j[\tilde{w}_i, S]$, we get that for any $y \in \mathcal{M}_j[\tilde{w}_i, S]$, there is a certain $k < m + p$ such that

$$\Pr(\tilde{X}_k = \tilde{y} \mid \tilde{X}_0 = \tilde{x}) \geq \prod_{l=1}^{k} p_i(\tilde{w}_i^{\circ l}(\tilde{x})) > 0,$$

because $p_i(.)$ is strictly positive over S. Hence, every point of $\mathcal{M}_j[\tilde{w}_i, S]$ is accessible from $\tilde{x}$. Since, by assumption, $\tilde{x}$ is also in $\mathcal{A}_k$ and the set is a closed class, as a result we get that $\mathcal{A}_k \supset \mathcal{M}_j[\tilde{w}_i, S]$ as required. □

Proof of Theorem III.3. (a) Let $\mathcal{A}_k \in \mathcal{A}_\mathcal{F}$ and let $\mathcal{M}[\tilde{w}_i, S] \in \mathcal{M}_\mathcal{F}$. The basins of attraction $\mathcal{B}[\mathcal{M}_j[\tilde{w}_i, S]]$, $j \in \{1, \ldots, \mathcal{M}_\#[\tilde{w}_i, S]\}$ forms a countable partition of S. Therefore, $\mathcal{A}_k$ is a subset of a union of a certain number of the basins, $\mathcal{A}_k = \mathcal{A}_k \cap \bigcup_j \mathcal{B}[\mathcal{M}_j[\tilde{w}_i, S]]$. On the basis of the above lemma, if $\mathcal{A}_k \cap \mathcal{B}[\mathcal{M}_j[\tilde{w}_i, S]] \neq \varnothing$, then $A_k \supset \mathcal{M}_j[\tilde{w}_i, S]$. Hence, each set in $\mathcal{A}_\mathcal{F}$ includes at least one component of the set $\mathcal{M}[\tilde{w}_i, S]$. Moreover, for any $\mathcal{A}_m \in \mathcal{A}_\mathcal{F}$ such that $A_m \supset \mathcal{M}_j[\tilde{w}_i, S]$, we have $\mathcal{A}_k = \mathcal{A}_m$, because the recurrent communication classes are disjoint. Hence, for any $\mathcal{M}[\tilde{w}_i, S] \in \mathcal{M}_\mathcal{F}$, the number of sets in $\mathcal{A}_\mathcal{F}$ cannot exceed the number of the components of $\mathcal{M}[\tilde{w}_i, S]$, which completes this part of the proof.

(b) If $\mathcal{A}_\mathcal{F}$ is empty, the conclusion of the theorem follows trivially. Assume that $\mathcal{A}_\mathcal{F}$ is nonempty. The basins of attractions of the components of the set $\mathcal{M}[\tilde{w}_i, S]$ constitute a countable partition of S, so by the above lemma, each $\mathcal{A}_k \in \mathcal{A}_\mathcal{F}$ includes at least one of the components of $\mathcal{M}[\tilde{w}_i, S]$. But by the assumption of the theorem, all the components belong to the basin of attraction $\mathcal{B}[\mathcal{M}_k[\tilde{w}_j, S]]$ and thus, again by the lemma above, $\mathcal{M}_k[\tilde{w}_j, S]$ is a subset of every set $\mathcal{A}_k \in \mathcal{A}_\mathcal{F}$. Since $\mathcal{A}_\mathcal{F}$ is a family of disjoint sets, $\mathcal{A}_\mathcal{F}$ consists of a single set as required. □

Proof of Theorem III.6. First we show that the DIFS transformations map S into itself. Suppose that $\tilde{x}$ is any point of S. We need to show that, for any $i \in \{1, \ldots, N\}$, $\tilde{w}_i(\tilde{x}) \in S$.Using the triangle inequality along with the Lipschitz continuity of w_i's and the definition of a δ-roundoff of a mapping (Def. II.2 in [1]), we get, for any $i \in \{1, \ldots, N\}$ and $\varepsilon > 0$, that

$$\begin{aligned} d(\tilde{w}_i(\tilde{x}), o) &\leq d(w_i(\tilde{x}), o) + d(\tilde{w}_i(\tilde{x}), w_i(\tilde{x})) \\ &\leq d(w_i(\tilde{x}), x_f^{(i)}) + d(x_f^{(i)}, o) + \theta \\ &\leq \lambda_{max}\, d(\tilde{x}, x_f^{(i)}) + r_{max} + \theta \\ &\leq \lambda_{max}\, \big(d(\tilde{x}, o) + d(x_f^{(i)}, o)\big) + r_{max} + \theta \\ &< \lambda_{max}\, (\alpha r_{max} + \theta(1-\lambda_{max})^{-1} + \varepsilon + r_{max}) \\ &\quad + r_{max} + \theta \\ &= \alpha r_{max} + \lambda_{max}(\theta(1-\lambda_{max})^{-1} + \varepsilon) \\ &< \alpha r_{max} + \theta(1-\lambda_{max})^{-1} + \varepsilon \end{aligned}$$

as required.

Now we show that the Markov chain associated with the DIFS possesses a set $\mathcal{A}$ of recurrent states. By Theorem III.2 (a) in [1], the set S is a closed class and, moreover, S is by definition finite, so due to the second part of Theorem

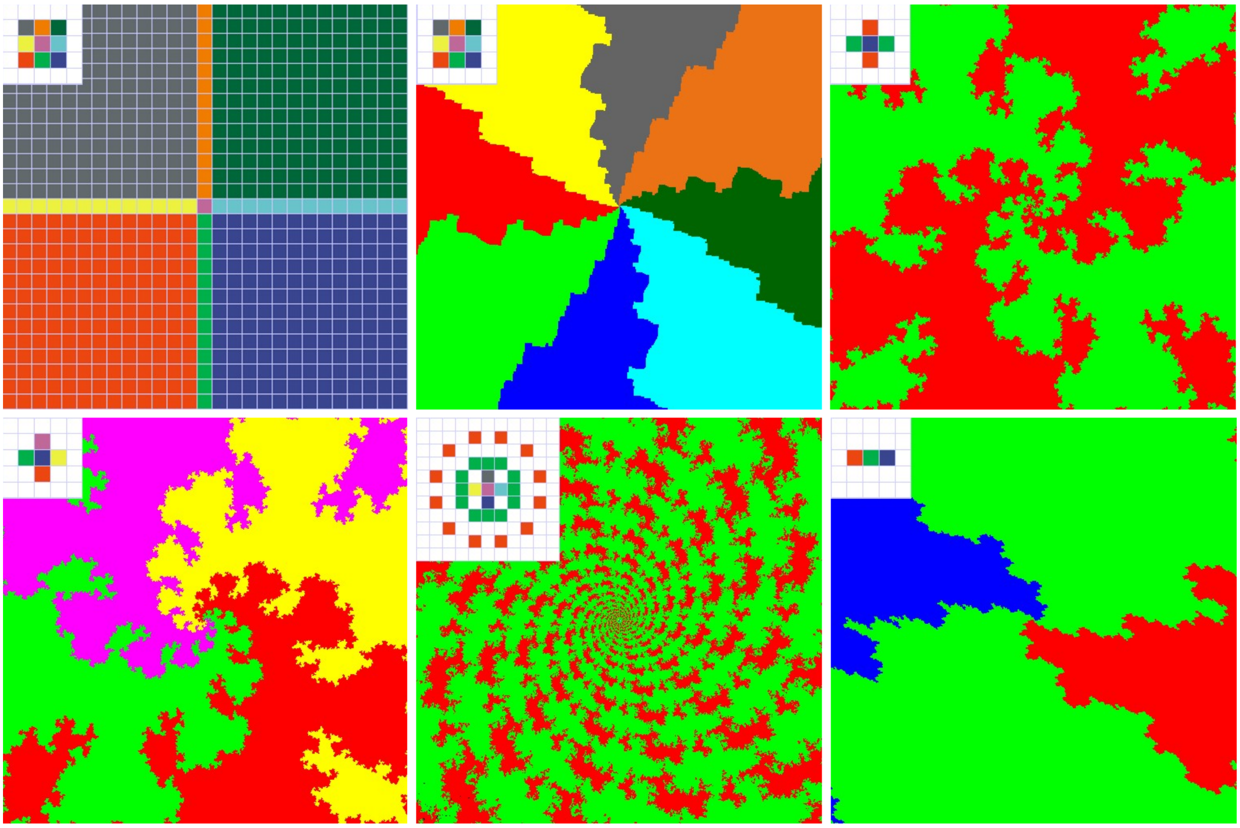

Fig. 1. Examples of minimal absorbing sets and their basins of attractions generated by δ-roundoffs of two-dimensional linear contractions. From left to right, top to bottom, the first five pictures are derived from similarities with scaling factors and rotations respectively: 0.6 and 0°, 0.6 and 5°, 0.6 and 150°, 0.6 and 30°, 0.9 and 30°. The last picture is derived from a linear mapping specified by the matrix $\left[\begin{smallmatrix} 0.5 & 0.3 \\ -0.1 & 0.4 \end{smallmatrix}\right]$.

III.1 in [1], the Markov chain possesses at least one nonempty recurrent communication class within S, and hence $\mathcal{A} \neq \varnothing$.

To complete the proof we need to show that $\mathcal{A} \subset S$. By the assumption of the theorem, $\tilde{w}_i$ are roundoffs of contractions, so by Corollary II.8 in [1], for each $\tilde{w}_i$, there is a finite minimal absorbing set $\mathcal{M}[\tilde{w}_i]$ (with respect to the whole space $\mathcal{D}^n(\delta)$). Now, fix some $i \in \{1, \dots, N\}$ and observe that if $\tilde{x} \in \mathcal{A}$, then $\tilde{x}$ is located in one of the basins of attraction of $\mathcal{M}[\tilde{w}_i]$. Therefore, for *each* $\tilde{x} \in \mathcal{A}$, there is a state $\tilde{y} \in \mathcal{M}[\tilde{w}_i]$ accessible from $\tilde{x}$, and thus also $\tilde{y} \in \mathcal{A}$, because $\mathcal{A}$ is a closed class. In addition, $\mathcal{A}$ is composed of communication classes $\mathcal{A}_k$. From this we conclude that any set which is a closed class and, at the same time, includes $\mathcal{M}[\tilde{w}_i]$ has to contain $\mathcal{A}$ too. The set S is a closed class, so to finish the proof it suffices to show that $S \supset \mathcal{M}[\tilde{w}_i]$. On the basis of Theorem II.7 in [1], for every point $\tilde{x} \in \mathcal{M}[\tilde{w}_i]$, $d(\tilde{x}, x_f^{(i)}) \leq \theta(1-\lambda_i)^{-1}$, and hence for any $\varepsilon > 0$,

$$\begin{aligned} d(\tilde{x}, o) &\leq d(x_f^{(i)}, o) + d(\tilde{x}, x_f^{(i)}) \\ &\leq r_{max} + \theta(1-\lambda_{max})^{-1} \\ &< \alpha r_{max} + \theta(1-\lambda_{max})^{-1} + \varepsilon, \end{aligned}$$

because $\alpha \geq 1$. This completes the proof. □

Proof of Corollary III.7. The only thing to show is that for every $\tilde{X}_0 = \tilde{x} \in \mathcal{D}^n(\delta)$, $\Pr(\exists i \in \mathbb{N} : \tilde{X}_i \notin \mathcal{T}) = 1$. Using Theorem III.6 in [1], for any $\tilde{x} \in \mathcal{T}$ one can construct a finite closed class S so that $\tilde{x} \in S$ and $\mathcal{A} \subset S$. Therefore, the "escape-from-transient-class" conclusion follows from the second part of Theorem III.1. □

Proof of Lemma III.8. The mutual existence of the orbits $\{\tilde{x}_k = \tilde{w}_{i_k}(\tilde{x}_{k-1})\}$ and $\{x_k = w_{i_k}(x_{k-1})\}$ of the chain $\tilde{X}$ and X, respectively, is trivially provided by the strict positivity of probability functions $p_i(.)$ and probability weights q_i. Therefore, all we need to show is that all points of the orbits satisfy inequality (III.6) in [1]. The proof is by induction. For $k = 1$, we have

$$\begin{aligned} d(x_1, \tilde{x}_1) &\leq d(x_1, w_{i_1}(\tilde{x}_0)) + d(w_{i_1}(\tilde{x}_0), \tilde{x}_1) \\ &\leq d(w_{i_1}(\tilde{x}_0), \tilde{w}_{i_1}(\tilde{x}_0)) \leq \theta, \end{aligned}$$

where the last inequality follows from the definition of a δ-roundoff of a mapping (Def. II.2 in [1]). Now assume that inequality III.6 in [1] is true for k. Then

$$\begin{aligned} d(x_{k+1}, \tilde{x}_{k+1}) &\leq d(x_{k+1}, w_{i_{k+1}}(\tilde{x}_k)) + d(w_{i_{k+1}}(\tilde{x}_k), \tilde{x}_{k+1}) \\ &\leq d(w_{i_{k+1}}(x_k), w_{i_{k+1}}(\tilde{x}_k)) \\ &\qquad + d(w_{i_{k+1}}(\tilde{x}_k), \tilde{w}_{i_{k+1}}(\tilde{x}_k)) \\ &\leq \lambda_{max}\, d(x_k, \tilde{x}_k) + \theta \\ &\leq \lambda_{max}\, (1-\lambda_{max})^{-1}\theta + \theta \\ &= \theta(1-\lambda_{max})^{-1} \end{aligned}$$

as required. □

Proof of Theorem III.9. The Hausdorff distance between $\mathcal{A}_k^+$ and A_∞ is $h(\mathcal{A}_k^+, A_\infty) = \inf\{\varepsilon \geq 0 : A_\infty \subset$

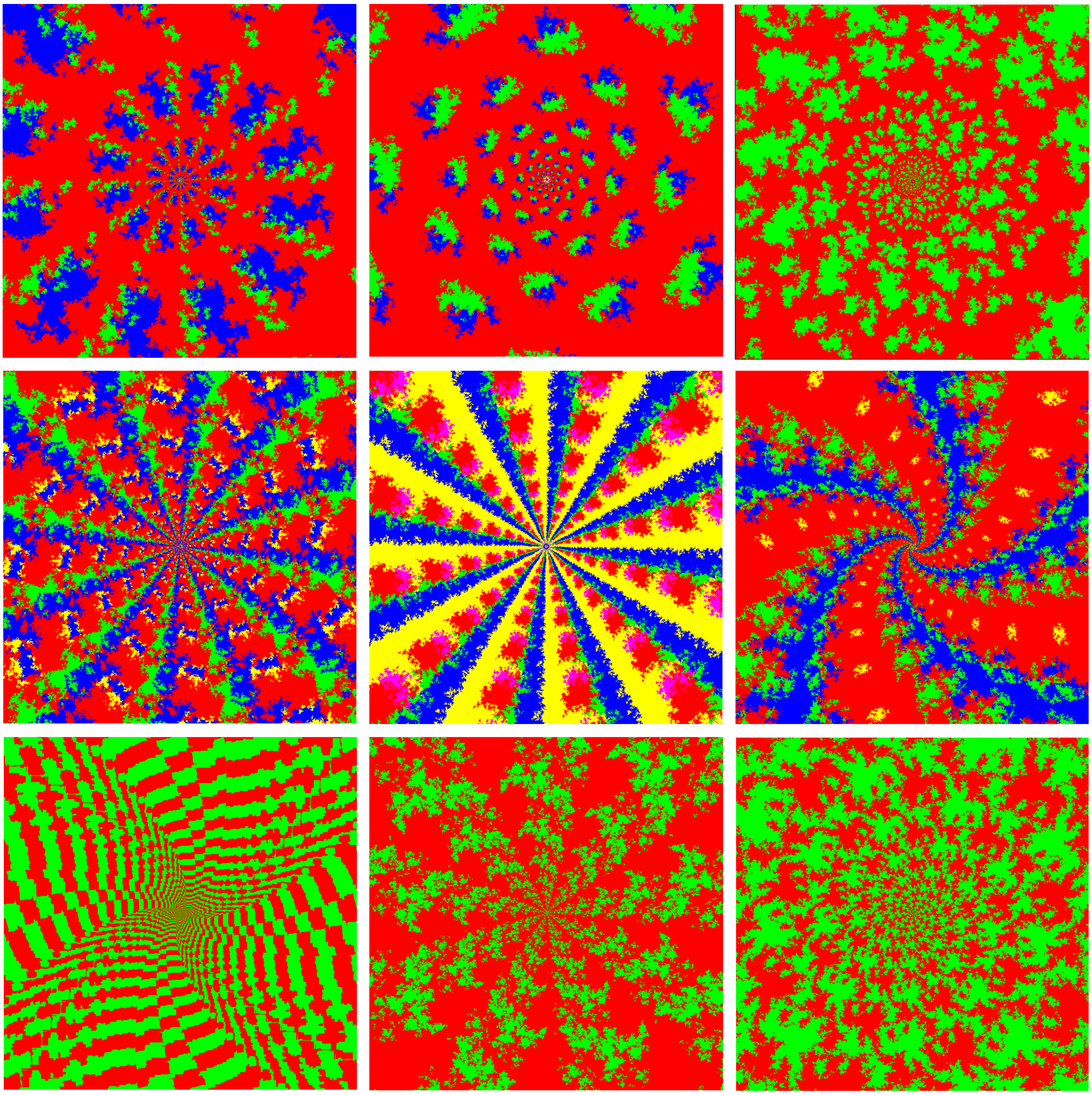

Fig. 2. Examples of basins of attractions generated by δ-roundoffs of two-dimensional affine maps being contractions with respect to the "weighted" maximum metric $d_\infty^{(p)}(x,y) = \max(p|x_1 - y_1|, |x_2 - y_2|)$, $p > 1$.

$N(\mathcal{A}_k^+, \varepsilon)$ *and* $\mathcal{A}_k^+ \subset N(A_\infty, \varepsilon)\}$, where $N(A, \varepsilon) := \{x \in \mathbb{R}^n : d(x,a) < \varepsilon,\ a \in A\}$ denotes the (open) ε-neighbourhood of a set A. First we show that for any $\varepsilon > 0$, $A_\infty \subset N(\mathcal{A}_k^+, \theta(1-\lambda_{max})^{-1} + \varepsilon)$. We need to show that for any $a \in A_\infty$, there is a certain $\tilde{x} \in \mathcal{A}_k^+$ such that $d(a, \tilde{x}) < \theta(1-\lambda_{max})^{-1} + \varepsilon$. Let a be any point of A_∞. The attractor is the support of the IFS invariant measure π, so for any $\varepsilon > 0$, $\pi(B(a,\varepsilon)) > 0$. Moreover, by Elton's ergodic theorem (Eq. III.4 in [1]) π is ergodic, and hence, for any initial point $x_0 \in \mathbb{R}^n$, almost every orbit $\{x_i\}_{i=0}^\infty$ of the Markov chain generated by the IFS visits $B(a,\varepsilon)$ infinitely often. Hence, for any $x_0 \in \mathbb{R}^n$, there is a finite sequence of indices $i_m, \ldots, i_1 \in \{1, \ldots, N\}$ such that $d(w_{i_m} \circ \cdots \circ w_1(x_0), a) < \varepsilon$. On that basis, putting $x_0 \in \mathcal{A}_k^+$ and using the previous lemma, we get that there is a finite sequence $\mathbf{i} = (i_m, \ldots, i_1)$ of indices such that

$$\begin{aligned} d(\tilde{w}_\mathbf{i}(x_0), a) &\le d(\tilde{w}_\mathbf{i}(x_0), w_\mathbf{i}(x_0)) + d(w_\mathbf{i}(x_0), a) \\ &< \theta(1-\lambda_{max})^{-1} + \varepsilon, \end{aligned}$$

where $w_\mathbf{i}(.) := w_{i_m} \circ \cdots \circ w_1(.)$. But $\mathcal{A}_k^+$ is a closed class, and thus $\tilde{w}_\mathbf{i}(x_0) \in \mathcal{A}_k^+$. Hence, $A_\infty \subset N(\mathcal{A}_k^+, \theta(1-\lambda_{max})^{-1} + \varepsilon)$ as required.

Now we show that for any $\varepsilon > 0$, $\mathcal{A}_k^+ \subset N(A_\infty, \theta(1-\lambda_{max})^{-1} + \varepsilon)$. Let $\tilde{x}$ be any point of $\mathcal{A}_k^+$. Since $\mathcal{A}_k^+$ is a recurrent class, $\tilde{x}$ is a recurrent state and thus there is a finite sequence of indices $\mathbf{i} = (i_m, \ldots, i_1) \in \{1, \ldots, N\}^m$ such that $\tilde{w}_\mathbf{i}(\tilde{x}) = \tilde{x}$, and hence for any $j \in \mathbb{N}$, $\tilde{w}_\mathbf{i}^{\circ j}(\tilde{x}) = \tilde{x}$. Now let $a \in A_\infty$. Since the IFS maps w_i are contractions, we have, for any $j \in \mathbb{N}$,

$$d(w_{\mathbf{i}}^{\circ j}(\tilde{x}), w_{\mathbf{i}}^{\circ j}(a)) \leq \lambda_{max}^{m \cdot j}\, d(\tilde{x}, a)$$

and hence for any $\varepsilon > 0$, there is $M \in \mathbb{N}$ such that

$$d(w_{\mathbf{i}}^{\circ M}(\tilde{x}), w_{\mathbf{i}}^{\circ M}(a)) \leq \lambda_{max}^{M}\, d(\tilde{x}, a) < \varepsilon,$$

because $\lambda_{max} \in [0, 1)$. On the basis of above, using the previous lemma we conclude that

$$\begin{aligned} d(\tilde{x}, w_{\mathbf{i}}^{\circ M}(a)) &= d(\tilde{w}_{\mathbf{i}}^{\circ M}(\tilde{x}), w_{\mathbf{i}}^{\circ M}(a)) \\ &\leq d(\tilde{w}_{\mathbf{i}}^{\circ M}(\tilde{x}), w_{\mathbf{i}}^{\circ M}(\tilde{x})) + d(w_{\mathbf{i}}^{\circ M}(\tilde{x}), w_{\mathbf{i}}^{\circ M}(\tilde{a})) \\ &< \theta(1-\lambda_{max})^{-1} + \varepsilon. \end{aligned}$$

But $w_{\mathbf{i}}^{\circ M}(a) \in A_\infty$, because w_i's map A_∞ into itself. It follows that $\mathcal{A}_k^+ \subset N(A_\infty, \theta(1-\lambda_{max})^{-1}+\varepsilon)$ as required.

Since both $A_\infty \subset N(\mathcal{A}_k^+, \theta(1-\lambda_{max})^{-1}+\varepsilon)$ and $\mathcal{A}_k^+ \subset N(A_\infty, \theta(1-\lambda_{max})^{-1}+\varepsilon)$ for any $\varepsilon > 0$, we get that the infimum in the Hausdorff distance $h(\mathcal{A}_k^+, A_\infty)$ is bounded from above by the value of $\theta(1-\lambda_{max})^{-1}$. This completes the proof. □

Proof of Corollary III.10. By Theorem III.6 in [1], for any $\delta > 0$, the set of all recurrent states of the associated Markov chain is nonempty and finite (and thus compact), and so are the set's subsets $\mathcal{A}_k^+(\delta)$. Therefore, for any $\delta > 0$, $\mathcal{A}_k^+(\delta)$ is an element of $\mathcal{H}(\mathbb{R}^n)$, the family of all nonempty and compact subsets of $\mathbb{R}^n$. By a standard argument for hyperbolic IFSs, $A_\infty \in \mathcal{H}(\mathbb{R}^n)$ too. Since $\mathrm{diam}_d(C_\delta) \to 0$ as $\delta \to 0$, and the Hausdorff distance h is a metric on $\mathcal{H}(\mathbb{R}^n)$, the conclusion follows from inequality III.7 in [1]. □

Proof of Theorem III.11. In [2] Peruggia showed that a very similar conclusion holds under the assumption that the fixed point of one of the IFS mappings coincides with $\mathbf{0} \in \mathcal{D}^2(\delta)$, the zero vector of the discrete (pixel) space, which naturally stays intact while changing the value of the discretization parameter δ (cf. [2], Theorem 4.38, pp. 129–131). Although our theorem does not impose such a restriction, the proof is founded on similar arguments as those used in the proof by Peruggia.

First, observe that the summation on the left hand side of Eq. III.8 in [1] can be restricted to the support of the measure $\pi_k(\delta)$, $\mathrm{supp}(\pi_k(\delta)) = \mathcal{A}_k^+(\delta)$, and, as we pointed out earlier, the corresponding Markov chain $\{\tilde{X}_i^\delta(\tilde{x}_0) : \tilde{X}_0^\delta(\tilde{x}_0) = \tilde{x}_0 \in \mathcal{A}_k^+(\delta)\}$ generated by the DIFS (for fixed δ) is (Birkhoff's) ergodic on $\mathcal{A}_k^+(\delta)$. In addition, by Elton's ergodic theorem, the IFS invariant measure π_∞ is ergodic for the Markov chain $\{X_i(x_0) : X_0 = x_0 \in \mathbb{R}^n\}$ generated by the IFS on $\mathbb{R}^n$. Putting these facts together, we get that for any $\delta > 0$, with probability one,

$$\begin{aligned} &\Big| \sum_{\tilde{x} \in \mathcal{A}_k^+(\delta)} f(\tilde{x}) \pi_k(\delta)\{\tilde{x}\} - \int_{\mathbb{R}^n} f(x) d\pi_\infty \Big| \\ &\quad = \lim_{m\to\infty} \frac{1}{m} \Big| \sum_{i=0}^{m-1} \big(f(\tilde{X}_i^\delta(\tilde{x}_0)) - f(X_i(\tilde{x}_0)) \big) \Big| \qquad \text{(I.1)} \\ &\quad \leq \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \big| f(\tilde{X}_i^\delta(\tilde{x}_0)) - f(X_i(\tilde{x}_0)) \big| \end{aligned}$$

where $\tilde{x}_0 \in \mathcal{A}_k^+(\delta)$. But $\tilde{X}_i^\delta(\tilde{x}_0) = \tilde{w}_{I_i}\big(\tilde{X}_{i-1}^\delta(\tilde{x}_0)\big)$ and $X_i(\tilde{x}_0) = w_{I_i}\big(X_{i-1}(\tilde{x}_0)\big)$, that is, both Markov chains are driven by the same sequence $\{I_i\}_{i\in\mathbb{N}}$ of the i.i.d. random variables I_i distributed as $[p_1, \ldots, p_N]$. Hence, from Lemma III.8 in [1],

$$d(X_i(\tilde{x}_0), \tilde{X}_i^\delta(\tilde{x}_0)) \leq \theta(1-\lambda_{max})^{-1} \qquad \text{(I.2)}$$

for every $i \in \mathbb{N}$, where λ_{max} is the maximum contractivity factor of the IFS mappings w_i. Now crucial is the observation, which will be shown to be true at the end of the proof, that there exists a compact set $E \subset \mathbb{R}^n$ such that, for every $\delta \in (0, R)$, where $R > 0$ is a certain real number, $E \supset \{X_i(\tilde{x}_0)\}$ and $E \supset \{\tilde{X}_i^\delta(\tilde{x}_0)\}$ for any $\tilde{x}_0 \in \mathcal{A}_k^+(\delta)$, that is, none of the chains for $\delta \in (0, R)$ moves out of E. Then, on the basis of the Heine–Cantor theorem, f's are uniformly continuous on E. Since the right-hand side of inequality (I.2) converges to 0 as $\delta \to 0$, from this we conclude that for any $\varepsilon > 0$, there is $\delta(\varepsilon) \in (0, R)$ such that for any $\delta \in (0, \delta(\varepsilon))$, $\big|f(\tilde{X}_i^\delta(\tilde{x}_0)) - f(X_i(\tilde{x}_0))\big| < \varepsilon$ for all $i \in \mathbb{N}$. Therefore,

$$\lim_{\delta\to 0} \Big(\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \big| f(\tilde{X}_i^\delta(\tilde{x}_0)) - f(X_i(\tilde{x}_0)) \big| \Big) = 0$$

and hence, taking limits as $\delta \to 0$ on both sides of inequality (I.1), we get

$$\lim_{\delta\to 0} \sum_{\tilde{x} \in \mathcal{A}_k^+(\delta)} f(\tilde{x}) \pi_k(\delta)\{\tilde{x}\} = \int_{\mathbb{R}^n} f(x) d\pi_\infty.$$

Since the sets $\mathcal{A}_k^+(\delta)$ are the supports of the measures $\pi_k(\delta)$, the summation on the left-hand side of the above formula equals the summation over the whole space $\mathcal{D}^n(\delta)$, and thus we have arrived at the conclusion of the theorem.

The remaining thing to show is the existence of a compact set E, in which the Markov chains $\{X_i(\tilde{x}_0)\}$ and $\{\tilde{X}_i^\delta(\tilde{x}_0)\}$ reside for any $\tilde{x}_0 \in \mathcal{A}_k^+(\delta)$ and $\delta \in (0, R)$, so as to assure the uniform continuity of the functions f. To this end, we can apply the following construction: Fix $R > 0$ and observe that by inequality III.7 in [1], for any $\delta \in (0, R)$, $\mathcal{A}_k^+(\delta) \subset N(A_\infty, r_0)$, $r_0 = \frac{1}{2}\mathrm{diam}_d(C_R)(1-\lambda_{max})^{-1}$, and because A_∞ is compact, so is the closure $\overline{N}(A_\infty, r_0)$ of the neighbourhood. Hence, for all $\delta \in (0, R)$ and $\tilde{x}_0 \in \mathcal{A}_k^+(\delta)$, all Markov chains $\{\tilde{X}_i^\delta(\tilde{x}_0)\}$ reside in $\overline{N}(A_\infty, r_0)$ (because $\mathcal{A}_k^+(\delta)$'s are closed classes). Next we extend $\overline{N}(A_\infty, r_0)$ so as to additionally encompass all Markov chains $\{X_i(\tilde{x}_0)\}$ for $\delta \in (0, R)$ and $\tilde{x}_0 \in \mathcal{A}_k^+(\delta)$. Due to inequality (I.2), it is easily done by doubling the radius of $\overline{N}(A_\infty, r_0)$, so the required compact set is $\overline{N}(A_\infty, 2r_0)$. This completes the proof. □

II. MINIMAL ABSORBING SETS—EXAMPLES

In Fig. 1 we present examples of minimal absorbing sets and their basins of attractions generated by δ-roundoffs of two-dimensional affine contractions with respect to the Euclidean metric. In turn, Fig. 2 shows some examples of basins of attractions of affine contractions with respect to the "weighted" maximum metric defined as $d_\infty^{(p)}(x, y) := \max(p|x_1 - y_1|, |x_2 - y_2|)$, with weight $p > 1$ as a parameter. It is worth noting what intricate dynamics an iteration of a single affine map can exhibit when quantized and viewed in a discrete space.

III. DIFS Stationary Probability Measures—Examples

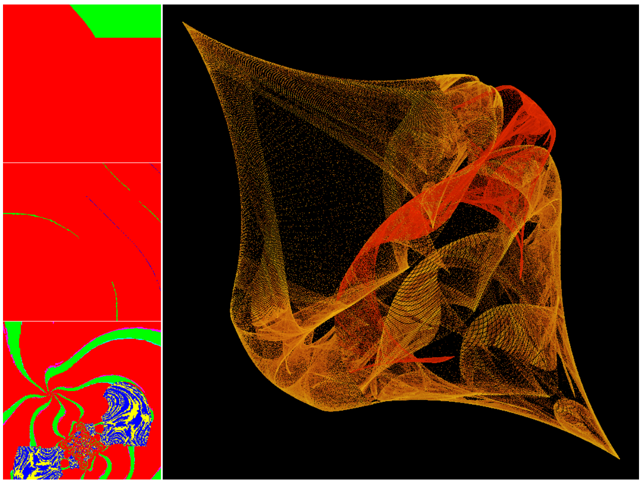

Fig. 3. An example of a DIFS stationary distribution visualized with the use of the random iteration algorithm. The DIFS is composed of three maps with minimal absorbing sets consisting of 2, 3 and 6 components, respectively. The basins of attractions of the maps are depicted on the left-hand side of the picture

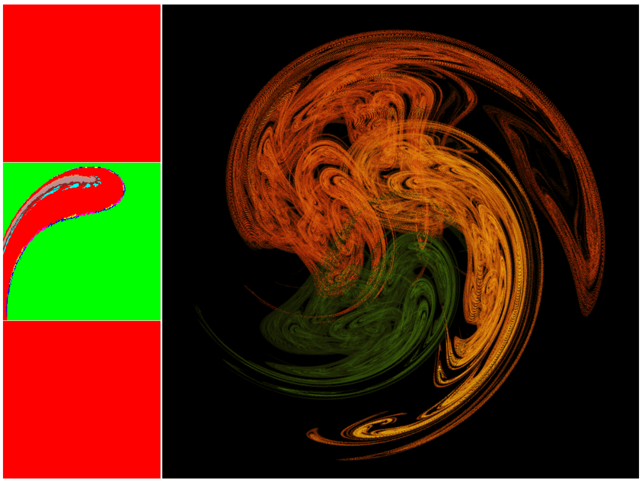

Fig. 4. Another example of a DIFS stationary distribution visualized with the use of the random iteration algorithm. The minimal absorbing sets consist of 1, 13 and 1 components, respectively

In Fig. 3–5 we present three examples of stationary probability measures of Discrete Iterated Function Systems defined on a squared subset $C \subset \mathcal{D}^2(\delta)$ of 1000×1000 resolution. Each DIFS consists of three mappings, which were constructed and stored in 1000×1000 arrays. Each array W_i represented a single DIFS map $\tilde{w}_i$ in the form of a pair of integer numbers as $(W_i)_{kl} = \tilde{w}(k\delta, l\delta)/\delta$. In other words, each W_i held information about a directed graph with vertices (k, l) being states of the associated Markov chain and edges generated by $\tilde{w}_i$, and thus the outdegree of each vertex being 1. For each

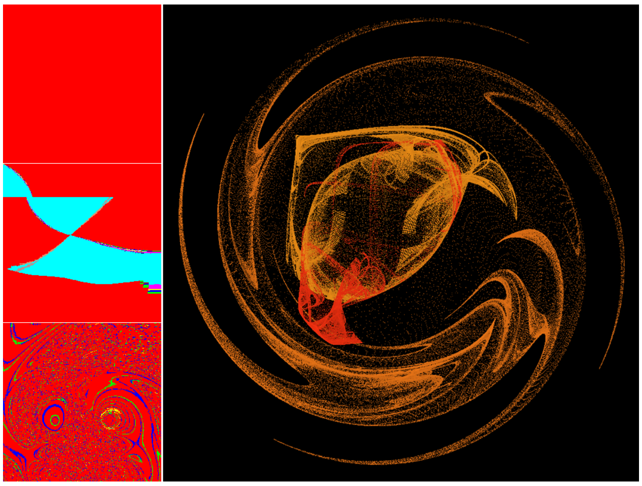

Fig. 5. One more example of a DIFS stationary distribution visualized with the aid of the random iteration algorithm. The minimal absorbing sets consist of 1, 11 and 5 components, respectively

DIFS, the arrays were initialized with three δ-roundoffs of contractive similarities describing a Sierpiński's triangle, and then connections between the states were processed (changed) several times with the aid of nonlinear mappings (a collection of interesting transformations can be found in [3]), with care not to create a connection with a state lying outside C. Finally, the arrays (graphs) were additionally delicately "smoothed" with a 3×3 Gaussian filter. Analogously, the distributions of the DIFS place-dependent probabilities were stored in a 1000×1000 array D such that $(D)_{kl}$ was the distribution at state (k, l). The array was initiated with uniform distributions, which were then perturbed with a nonlinear mapping. The renderings were obtained by means of the random iteration algorithm that generated an orbit in subset C with the aid of the arrays W_i and D, so that given state (k, l), the next state was determined as $(W_i)_{kl}$ with i drawn from the distribution $(D)_{kl}$. Since W_i's and D function as lookup tables, the orbit was generated in an extremely efficient manner, based only on fetches from the arrays and a pseudorandom generator. In the images the value of the measure of a state is interpreted as brightness, and the coloring reflects participation of a DIFS map in conveying a measure to a state, according to the formula $c_{new} = (c_{old} + c_i)/2$, where c_{new} and c_{old} are a new color and, respectively, a current color assigned to a visited state (with c_{old} initialized with black at the beginning of the algorithm), and c_i is a color assigned to the ith DIFS map.

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