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Positive stable linear systems with desired poles and zeros

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A new method of the decomposition of the fractional descriptor linear continuoustime and discrete-time systems into dynamical and static parts is proposed. Conditions for the decomposition of the fractional descriptor linear systems are established and procedures for compositions of the matrices of dynamical and static parts are given. The procedures are illustrated by numerical examples.

Key words: design method, linear, continuous-time, discrete-time, nilpotent, system, pole, zero, procedure, transfer matrix

1. Introduction

The concepts of controllability and observability introduced by Kalman [11, 12] have been the basic notions of the modern control theory. It is well-known that if the linear system is controllable then using of state feedbacks it is possible to modify the dynamical properties of the closed-loop systems [1, 2, 5, 6, 10, 13–16]. If the linear system is observable then it is possible to design an observer which reconstructs the state vector of the system [1, 5, 6, 15, 16]. The realization problem for linear systems has been considered in many books [1, 5, 6, 9, 15, 16]. Transformations of the matrices of linear systems to their canonical form with desired eigenvalues has been given in [7] and the transfer matrices with positive coefficients of descriptor linear systems has been addressed in [8]. A new method

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for design of linear systems with desired poles and zeros of the transfer matrices has been proposed in [4].

In this paper the new method of the design of linear systems with desired stable poles and zeros of the poles continuous-time and discrete-time linear systems is proposed. In Section 2 some basic definitions and theorems concerning continuous-time and discrete-time linear systems and of the matrix equations with non-square matrices are recalled. The proposed method for continuous-time linear systems is presented in Section 3 and for discrete-time linear systems in Section 4. In Section 5 the reduction of the discrete-time systems to the systems with nilpotent matrices is analyzed. Concluding remarks are given in Section 6.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times n}$ – the set of $n\times n$ real matrices with nonnegative entries, M_n – the set of $n\times n$ real Metzler matrices, I_n – the $n\times n$ identity matrix.

2. Positive system

2.1. Continuous-time linear systems

Consider the linear continuous-time syste

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx + Du, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Theorem 1. [1, 5, 6, 16] *The solution of the equation (1a)*

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau, \quad x_0 = x(0). \tag{2}$$

Definition 1. [6] *The system (1) is called positive if the state vector $x(t) \in \Re_+^n$, output vector $y(t) \in \Re_+^p$ for $t \geqslant 0$ and all initial conditions $x(0) \in \Re_+^n$, and all inputs $u(t) \in \Re_+^m$ for $t \geqslant 0$.*

Definition 2. [6] *A real matrix $A = [a_{ij}] \in \Re^{n\times n}$ is called Metzler matrix if its off diagonal entries are nonnegative, i.e. $a_{ij} \geqslant 0$ for $i \neq j$.*

Lemma 1. [6] *Let $A \in \Re^{n\times n}$. Then*

$$e^{At}x_0 \in \Re_+^{n\times n}, \quad t \geqslant 0 \tag{3}$$

if and only if A is the Metzler matrix.

Theorem 2. [6] *The linear system (1) is positive if and only if*

$$A \in M_n\,, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}, \tag{4}$$

where M_n is the set of Metzler matrices.

Definition 3. [6] *The positive system (1) is called asymptotically stable if and only if solution of (1a) satisfies the condition*

$$\lim_{t\to\infty} x(t) = 0 \quad \textit{for every} \quad x(0) \in \Re_+^n. \tag{5}$$

Theorem 3. [1, 5, 6] *The system (1) is asymptotically stable if and only if all eigenvalues s_k, $k = 1, \ldots, n$ of the matrix A satisfy the condition*

$$\mathrm{Re}\, s_k < 0 \quad \textit{for} \quad k = 1, \ldots, n. \tag{6}$$

Theorem 4. [6] *The positive system (1) if asymptotically stable if and only if all coefficients a_i, $i = 0, 1, \ldots, n-1$ of the characteristic polynomial*

$$\det\left[I_n s - A\right] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{7}$$

are positive.

Definition 4. [6] *The positive system (1) is called reachable in time $[0, t_f]$ if there exists an input $u(t) \in \Re_+^m$ for $t \in [0, t_f]$ which steers the state of the system from the zero initial condition $x(0) = 0$ to the final state $x_f = x(t_f) \in \Re_+^n$.*

Definition 5. [6] *A square matrix is called monomial if its every column and its every row has only one positive entry and the remaining entries are zero.*

Theorem 5. [6] *The positive system (1) is reachable if and only if the matrix*

$$R_f = \int_0^t e^{At} B B^T e^{At}\, \mathrm{d}\tau, \quad t_f > 0 \tag{8}$$

is monomial.

2.2. Discrete-time linear systems

Consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i\,, \tag{9a}$$

$$y_i = Cx_i + Du_i\,, \tag{9b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

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Definition 6. [6] *The system (9) is called (internally) positive if* $x_i \in \Re_+^n$ *and* $y_i \in \Re_+^p$, $i \in Z_+$ *for any initial conditions* $x_0 \in \Re_+^n$ *and all inputs* $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 6. [6] *The system (9) is positive if and only if*

$$A \in \Re_+^{n\times n}, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{10}$$

Definition 7. [1,5,6] *The system (9) is called asymptotically stable if* $\lim_{i\to\infty} x_i = 0$ *for* $u_i = 0$ *and any initial* $x_0 \neq 0$.

Theorem 7. [1, 5, 6] *The system (9) is asymptotically stable if and only if all eigenvalues of matrix A satisfy the condition*

$$|z_k| < 1 \quad for \quad k = 1, \ldots, n. \tag{11}$$

Theorem 8. [6] *The positive (9) is asymptotically stable if and only if*

1) *all coefficients of the polynomial*

$$p_A(z) = \det\left[I_n(z+1) - A\right] = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \tag{12}$$

are positive, i.e. $a_i > 0$ *for* $i = 0, 1, \ldots, n-1$.

2) *there exists strictly positive vector* $\lambda^T = \begin{bmatrix}\lambda_1 & \cdots & \lambda_n\end{bmatrix}^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ *such that*

$$A\lambda < 0 \quad or \quad \lambda^T A < 0. \tag{13}$$

2.3. Matrix equations with non-square matrices and their solutions

Consider the matrix equation

$$PX = Q, \tag{14}$$

where $P \in \Re^{n\times m}$, $Q \in \Re^{n\times p}$ are given and $X \in \Re^{m\times p}$ is unknown matrix.

Theorem 9. *The matrix equation (14) has a solution X if and only if*

$$\text{rank}\begin{bmatrix}P & Q\end{bmatrix} = \text{rank}P. \tag{15}$$

Proof follows immediately from the Kronecker-Cappelli Theorem [3].

Theorem 10. *If the condition (15) is satisfied, then the solution X of the equation (14) is given by*

$$X = P_r Q, \tag{16}$$

where $P_r \in \Re^{m\times n}$ *is the right inverse of the matrix P given by*

$$P_r = P^T\left[PP^T\right]^{-1} + \left(I_n - P^T\left[PP^T\right]^{-1}P\right)K_1, \quad K_1 \in \Re^{m\times n} \tag{17a}$$

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or

$$P_r = K_2\,[PK_2]^{-1}, \quad K_2 \in \Re^{m\times n} \tag{17b}$$

matrix K_1 *is arbitrary and* K_2 *is chosen so that* $\det\left[AK_2\right] \neq 0$.

Proof. From (14) and (17a) we have

$$X = P^T\left[PP^T\right]^{-1}Q + \left(I_n - P^T\left[PP^T\right]^{-1}P\right)K_1Q. \tag{18}$$

Substituting (18) into (17a) we obtain

$$PX = PP^T\left[PP^T\right]^{-1}Q + \left(P - PP^T\left[PP^T\right]^{-1}P\right)K_1Q = Q. \tag{19}$$

Proof of (17b) is similar. □

Consider the matrix equation

$$\overline{X}\,\overline{P} = \overline{Q}, \tag{20}$$

where $\overline{P} \in \Re^{m\times n}$, $\overline{Q} \in \Re^{p\times n}$ are given and $\overline{X} \in \Re^{p\times m}$ is unknown matrix.

Theorem 11. *The matrix equation (20) has a solution X if and only if*

$$\operatorname{rank}\begin{bmatrix}\overline{P}\\ \overline{Q}\end{bmatrix} = \operatorname{rank}\overline{P}. \tag{21}$$

Proof is similar (dual) to the proof of Theorem 10.

Theorem 12. *If the condition (21) is satisfied, then the solution of the equation (20) is given by*

$$\overline{X} = \overline{Q}\,\overline{P}_l\,, \tag{22}$$

where the left inverse of the matrix $\overline{P}$ *is given by*

$$\overline{P}_l = \left[\overline{P}^T\overline{P}\right]^{-1}\overline{P}^T + K_1\left(I_m - \overline{P}\left[\overline{P}^T\overline{P}\right]^{-1}\overline{P}^T\right), \quad K_1 \in \Re^{n\times m}\text{-arbitrary} \tag{23a}$$

or

$$\overline{P}_l = \left[K_2\overline{P}\right]^{-1}K_2\,, \quad K_2 \in \Re^{m\times m}\text{-arbitrary} \tag{23b}$$

and the matrix K_2 *is chosen so that* $\det\left[K_2\overline{P}\right] \neq 0$.

Proof is similar (dual) to the proof of Theorem 10.

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3. The new method of analysis of the continuous-time linear systems

Let $\overline{x}(t) \in \Re^n$, $\overline{u}(t) \in \Re^m$, $\overline{y}(t) \in \Re^p$ be the new state, input and output vectors of the continuous-time system (1) and

$$\begin{bmatrix} \dot{\overline{x}}(t) \\ \overline{y}(t) \end{bmatrix} = M \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix}, \quad \text{rank} M = n + p \tag{24a}$$

and

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = N \begin{bmatrix} \overline{x}(t) \\ \overline{u}(t) \end{bmatrix}, \quad \text{rank} N = n + m, \tag{24b}$$

where $M \in \Re^{(n+p)\times(n+p)}$, $N \in \Re^{(n+m)\times(n+m)}$.

From (24) we have

$$\begin{aligned} \begin{bmatrix} \dot{\overline{x}}(t) \\ \overline{y}(t) \end{bmatrix} &= M \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = M \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &= M \begin{bmatrix} A & B \\ C & D \end{bmatrix} N \begin{bmatrix} \overline{x}(t) \\ \overline{u}(t) \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} \begin{bmatrix} \overline{x}(t) \\ \overline{u}(t) \end{bmatrix}, \end{aligned} \tag{25a}$$

where

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = M \begin{bmatrix} A & B \\ C & D \end{bmatrix} N. \tag{25b}$$

Two following cases will be considered.

Case 1. $M = I_{n+p}$. In this case the equation (25b) has the form

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} N. \tag{26}$$

and it has the solution if the condition of Theorem 10 is satisfied.

Case 2. $N = I_{n+m}$. In this case the equation (25b) has the form

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = M \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{27a}$$

and after transposition we obtain

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix}^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T M^T. \tag{27b}$$

The solution M^T of (27b) can be found using of Theorem 10. Therefore, Case 2 has been reduced to Case 1.

Knowing the matrices A, B, C, D and the desired matrices $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ of the asymptotically stable system we may compute the matrix N using of the following procedure.

Procedure 1.
Step 1. Knowing the matrices A, B, C, D find (compute) the matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{28}$$

Step 2. Knowing the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ find the matrix

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix}. \tag{29}$$

Step 3. Using (28), (29) and (26) compute matrix N.

The details of this approach will be shown in the following simple numerical example.

Example 1. Consider the system (1) with the matrices

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}. \tag{30}$$

The desired asymptotically stable system is asymptotically stable with the matrices

$$\overline{A} = \begin{bmatrix} -1 & 0.3 \\ 0.5 & -2 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0 \end{bmatrix}. \tag{31}$$

Compute the matrix N.

Using Procedure 1 and (26), (30), (31) we obtain the following:

Step 1. Using (30) and (28) we obtain

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{32}$$

Step 2. Using (31) and (29) we obtain

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} -1 & 0.3 & 0 \\ 0.5 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{33}$$

Step 3. From (26) and (32), (33) we have

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0.3 & 0 \\ 0.5 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.3 & 0 \\ -0.5 & -2 & 1 \end{bmatrix}. \tag{34}$$

4. The new method of analysis of the discrete-time linear systems

Consider the discrete-time system (9) and the new system

$$\overline{x}_{i+1} = \overline{A}\overline{x}_i + \overline{B}\overline{u}_i\,, \tag{35a}$$

$$\overline{y}_i = \overline{C}\overline{x}_i + \overline{D}\overline{u}_i\,, \tag{35b}$$

where $\overline{x}_i \in \Re^n$, $\overline{u}_i \in \Re^m$, $\overline{y}_i \in \Re^p$ are the state, input and output vectors and $\overline{A} \in \Re^{n\times n}$, $\overline{B} \in \Re^{n\times m}$, $\overline{C} \in \Re^{p\times n}$, $\overline{D} \in \Re^{p\times m}$.

The state, input and output vectors of the systems (9) and (35) are related by

$$\begin{bmatrix} \overline{x}_{i+1} \\ \overline{y}_i \end{bmatrix} = M \begin{bmatrix} x_{i+1} \\ y_i \end{bmatrix}, \quad \det M \neq 0, \quad i = 0, 1, \ldots \tag{36}$$

and

$$\begin{bmatrix} x_i \\ u_i \end{bmatrix} = N \begin{bmatrix} \overline{x}_i \\ \overline{u}_i \end{bmatrix}, \quad \det N \neq 0, \quad i = 0, 1, \ldots \tag{37}$$

where $M \in \Re^{(n+p)\times(n+p)}$, $N \in \Re^{(n+m)\times(n+m)}$.

In a similar way as for the continuous-time systems it can be shown that the matrices A, B, C, D of (9) and $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ of (35) are related by

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = M \begin{bmatrix} A & B \\ C & D \end{bmatrix} N. \tag{38}$$

If $M = I_{n+p}$ then from (38) we have

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} N \tag{39}$$

and if $N = I_{n+m}$ then

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = M \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{40}$$

By transposition from (40) we have

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix}^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T M^T. \tag{41}$$

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Therefore, in both cases the problem has been reduced to the solution of the equation (40) or (41) using of Theorem 10.

To find the solution of the equation (40) and (41) Procedure 1 can be applied.

Example 2. Consider the system (9) with the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}. \tag{42}$$

The matrices of the desired positive asymptotically stable system are

$$\overline{A} = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0 \end{bmatrix}. \tag{43}$$

Using Procedure 1 and (42), (43) we obtain

Step 1. From (42) and (28) we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}. \tag{44}$$

Step 2. Using (43) and (29) we obtain

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.3 & 0.2 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{45}$$

Step 3. Using (39) and (44), (45) we obtain

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.3 & 0.2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.7 & 0.2 & 1 \\ 1.7 & -0.2 & -1 \\ 0.9 & -0.1 & -1 \end{bmatrix}. \tag{46}$$

The matrix (46) is nonsingular.

Now we shall consider the following problem.

For given matrices A, B, C, D such that

$$\operatorname{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = n + p \tag{47}$$

and the desired transfer matrix

$$\overline{T}(z) = \overline{C} \left[I_n z - \overline{A} \right]^{-1} \overline{B} + \overline{D} \tag{48}$$

of the positive stable system with desired stable poles $z_1, \ldots, z_n$ and stable desired zeros $z_{01}, \ldots, z_{0n}$, find (compute) the matrix N satisfying (39). To solve the problem the following procedure can be used.

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Procedure 2.

Step 1. Knowing the matrices A, B, C, D find the matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{49}$$

Step 2. Knowing the given poles $z_1, \ldots, z_n$ and the given zeros $z_{01}, \ldots, z_{0n}$ of the transfer matrix (48) find the matrix

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix}. \tag{50}$$

Step 3. Using (39) compute the desired matrix

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix}. \tag{51}$$

Example 3. Compute matrix N knowing the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}. \tag{52}$$

of the unstable system (9) and the desired stable poles $z_1 = 0.2$, $z_2 = 0.4$ and zero $z_{01} = -0.3$ of the transfer function

$$\overline{T}(z) = \frac{z + 0.3}{z^2 + 0.6z + 0.08}. \tag{53}$$

Using Procedure 2 we obtain the following.

Step 1. Using (49) and (52) we obtain

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{54}$$

Step 2. The matrices $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ with the desired stable poles $z_1 = 0.2$, $z_2 = 0.4$ and stable zero $z_{01} = -0.3$ have the forms

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -0.08 & -0.6 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 0.3 & 1 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0 \end{bmatrix} \tag{55}$$

since

$$\overline{T}(z) = \overline{C}\left[I_2 z - \overline{A}\right]^{-1} \overline{B} + \overline{D} = \begin{bmatrix} 0.3 & 1 \end{bmatrix} \begin{bmatrix} z & -1 \\ 0.08 & z + 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$
$$= \frac{z + 0.3}{z^2 + 0.6z + 0.08}. \tag{56}$$

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Therefore, we have

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.08 & -0.6 & 1 \\ 0.3 & 1 & 0 \end{bmatrix}. \tag{57}$$

Step 3. Using (51), (54) and (55) we obtain

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -0.08 & -0.6 & 1 \\ 0.3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.3 & 1 & 0 \\ -0.3 & 0 & 0 \\ 0.52 & -0.6 & 1 \end{bmatrix}. \tag{58}$$

The matrix (58) is nonsingular.

5. Reduction of the discrete-time linear systems to the systems with nilpotent state matrices

To simplify the notation, we assume $m = p = 1$ and $D = \begin{bmatrix} 0 \end{bmatrix}$. The desired matrices A, B, C have the canonical forms

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \Re_+^{n\times n}, \qquad \overline{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re_+^{n\times 1},$$
$$\overline{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in \Re_+^{1\times n}. \tag{59}$$

Remark 1. *It is well known that if the matrix* $\overline{A}$ *has the form (59) then* $\overline{A}^k x_0 = 0$ *for* $k = n, n{+}1, \dots$ *and any nonzero* $x_0 \in \Re_+^n$.

For the given matrices A, B, C and (59) compute the nonsingular matrix $N \in \Re^{(n+1)\times(n+1)}$ satisfying the equation

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} N. \tag{60}$$

It is assumed that

$$\operatorname{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + 1. \tag{61}$$

From (59) it follows that

$$\operatorname{rank} \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = n + 1. \tag{62}$$

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By Theorem 4 the matrix equation (60) has unique nonsingular solution since

$$\operatorname{rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} \tag{63}$$

and we have the following theorem.

Theorem 13. *The equation (60) has a unique nonsingular solution*

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} \tag{64}$$

if and only if the condition (63) is satisfied.

Knowing the matrices A, B, C and (59) matrix N can be computed using of the following procedure.

Procedure 3.
Step 1. Knowing the matrices A, B, C compute the matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \Re^{(n+1)\times(n+1)}. \tag{65}$$

Step 2. For given matrices (59) compute the matrix

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} \in \Re^{(n+1)\times(n+1)}. \tag{66}$$

Step 3. Using (64) compute the matrix N.

Example 4. For given matrices (42) and the desired matrices

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0 \end{bmatrix}. \tag{67}$$

compute matrix N

In this case the condition (63) for the matrices (42) and (67) is satisfied, since

$$\operatorname{rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 3 \tag{68}$$

and

$$\operatorname{rank}\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 3. \tag{69}$$

Using Procedure 3 and (42), (67) we obtain the following.

Step 1. Using (65) and (42) we obtain the nonsingular matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{70}$$

Step 2. Using (66) and (67) we obtain the nonsingular matrix

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{71}$$

Step 3. From (65), (70) and (71) we have

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}. \tag{72}$$

The matrix (72) is nonsingular.

The considerations can be easily extended to the case $m > 1$, $p > 1$ and nonzero matrix $D \in \Re_+^{p\times m}$.

6. Concluding remarks

A new approach to design of the positive linear continuous-time and discrete-time systems with desired poles and zeros of their transfer matrices has been proposed. Conditions have been established under which the transfer matrices of the positive systems have the desired stable poles and zeros. Procedures for computation of the matrices of the systems with desired poles and zeros of the transfer matrices has been proposed and illustrated by simple numerical examples. The reduction of the discrete-time linear systems to the systems with nilpotent state matrices has been also analyzed. The proposed approach can be easily implemented in practice. The approach can be extended to the linear continuous-time and discrete-time fractional orders linear systems.

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