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PIERRE VAN HIELE: A DUTCH DIDACTICIAN OF MATHEMATICS WITH GLOBAL RENOWN

Summary: Pierre van Hiele, a Dutch secondary school mathematics teacher who developed textbooks and published quite extensively on the didactics of mathematics, received worldwide recognition for his theory of levels of mathematical thinking. We describe Van Hiele's life and work, in particular his 'level theory', and then take a closer look at his most important sources of inspiration: the doctoral research of his wife Dina van Hiele-Geldof, on whose experiments he based his level theory; the ideas of Tatiana Afanassjewa, who proposed a three-step approach to geometry education as early as the 1920s; and Hans Freudenthal, the supervisor of his thesis. Finally, we discuss some existing responses to Van Hiele's theory, both nationally and internationally.

Keywords: Dina van Hiele-Geldof, geometry education, Hans Freudenthal, Pierre van Hiele, Tatiana Afanassjewa

Introduction

Pierre van Hiele (Fig. 1) was a major personality in mathematics education in the second half of the 20th c., both in the Netherlands and internationally. In this paper, we outline the life and work of Van Hiele, as a mathematics teacher and author of textbooks, but above all as the creator of the theory of levels of mathematical thinking, known as 'level theory' for short. We consider the favorable

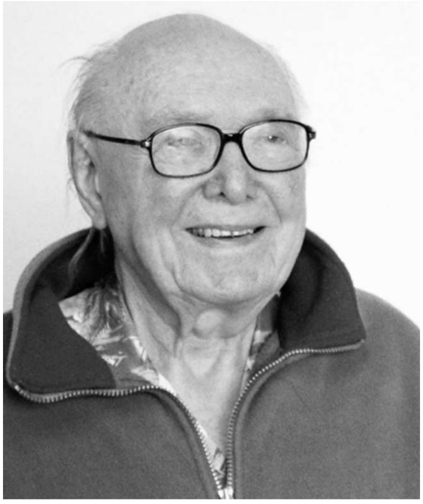

Fig. 1. Van Hiele during an interview with Gerard Alberts and Rainer Kaenders, 2005.

Credit: Courtesy of Gerard Alberts.

'milieu' in which his work originated; in particular, we discuss the influences on his work of Dina van Hiele-Geldof, his wife and 'co-researcher', of Tatiana Afanassjewa, and of Hans Freudenthal, his supervisor. We also discuss some national and international responses to his work.

Much has already been written about Van Hiele and his levels, but a comprehensive biographical study of the Van Hieles is still lacking. To describe Van Hiele's professional career, we have mainly relied on two extensive interviews: one conducted by Fred Goffree[1] and one by Gerard Alberts and Rainer Kaenders[2], as well as the many articles by Van Hiele and contemporaries in "Euclides", the journal of the *Nederlandse Vereniging van Wiskundeleraren* [Dutch Association of Mathematics Teachers]. Fundamental primary sources for the theory of levels of mathematical thinking are a first article on this subject by Van Hiele[3], his and his wife's dissertations[4], and several publications in which Van Hiele further clarifies this theory. In *Structure and Insight*[5], Van Hiele has presented a detailed explanation of his didactic views in English. Van Hiele's theory inspired many other researchers in the late 1970s and 1980s, especially in the US. As early as 1983,

[1] F. Goffree, *Pierre van Hiele, wiskundeleraar in Overveen, Bilthoven en Voorburg*, [in:] *Ik was wiskundeleraar*, SLO, Enschede 1985, p. 101–136.

[2] G. Alberts, R. Kaenders, *Interview Pierre van Hiele: Ik liet de kinderen wél iets leren*, "Nieuw Archief voor Wiskunde" 2005, vol. 6, no. 3, p. 247–251.

[3] P.M. van Hiele, *De niveaus in het denken, welke van belang zijn bij het onderwijs in de meetkunde in de eerste klasse van het V.H.M.O.* [The levels in thinking, which are important for teaching geometry in the first class of secondary school], "Paedagogische Studiën, Maandblad voor Onderwijs en Opvoeding" 1955, vol. 32, p. 289–297.

[4] P.M. van Hiele, *De problematiek van het inzicht, gedemonstreerd aan het inzicht van schoolkinderen in de meetkunde-leerstof* [The problems of insight, demonstrated by the insight of schoolchildren in geometry subject matter], J.M. Meulenhoff/J. Muusses/N.V. Erven P. Noordhoff/ N.V. Uitgeverij Nijgh & Van Ditmar/Spruyt, Van Mantgem, & De Does N.V., Amsterdam/Purmerend/Groningen/Den Haag/Leiden 1957; D. van Hiele-Geldof, *De didaktiek van de meetkunde in de eerste klas van het V.H.M.O.* [The didactics of geometry in the lowest class of secondary school], J.M. Meulenhoff/ J. Muusses/N.V. Erven P. Noordhoff/N.V. Uitgeverij Nijgh & Van Ditmar/Spruyt, Van Mantgem, & De Does N.V., Amsterdam/Purmerend/Groningen/Den Haag/Leiden 1957.

[5] P.M. van Hiele, *Structure and insight: A theory of mathematics education*, Academic Press, Orlando, Florida 1986.

Alan Hoffer wrote a substantial chapter on 'Van Hiele-based research'[6]; a more recent bibliographic orientation can be found in John Pegg's contribution to the *Encyclopedia of Mathematics Education*[7]. For more context and to place Van Hiele's work in the history of Dutch mathematics education, Harm Jan Smid's recent book *Theory and Practice*[8] and Freudenthal's biography[9] are useful reference works. In this paper, we also make a connection with the work of Tatiana Afanassjewa, whose life – and that of her husband Paul Ehrenfest – have recently been the subject of a double biography in Dutch[10] and an edited volume[11]. This connection and the way in which Van Hiele's work is framed within the 'milieu' of Dutch mathematics education in the last century is the most original contribution of the present study.

A short biography of Pierre van Hiele

Van Hiele was born on 4 May 1909 in Amsterdam. From 1927 to 1933, he studied mathematics at the University of Amsterdam. Gerrit Mannoury, Luitzen Brouwer, and the young Hans Freudenthal were the professors of mathematics who made the deepest impression on him[12]. In 1940, he married Dina (Dieke) Geldof (1911–1958), who had also studied mathematics in Amsterdam. They had two daughters.

During the recession of the 1930s, Van Hiele had several small teaching jobs, mainly as a personal tutor. In 1938, he started as a mathematics and chemistry teacher at the Montessori Lyceum in Overveen. Thereafter, he taught mathematics in Bilthoven and Voorburg[13]. Van Hiele would remain a teacher for his entire professional career. He stated very explicitly that he was 'possessed' by teaching: 'I wished to teach children'[14]. First, together with his wife, later with a larger

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[6] A. Hoffer, *Van Hiele-based research*, [in:] *Acquisition of mathematics concepts and processes*, ed. by R.A. Lesh, M. Landau, Academic Press, New York 1983, p. 205–227.

[7] J. Pegg, *The van Hiele theory*, [in:] *Encyclopedia of mathematics education*, ed. by S. Lerman, Springer, Dordrecht 2014, p. 613–615.

[8] H.J. Smid, *Theory and practice: A history of two centuries of Dutch mathematics education*, Springer, Cham 2022.

[9] S. La Bastide-van Gemert, *All positive action starts with criticism. Hans Freudenthal and the didactics of mathematics*, Springer, New York 2015.

[10] M. van der Heijden, *Denken is verrukkelijk: het leven van Tatiana Afanassjewa en Paul Ehrenfest*, Prometheus, Amsterdam 2021.

[11] *The legacy of Tatjana Afanassjewa: Philosophical insights from the work of an original physicist and mathematician*, ed. by J. Uffink, G. Valente, C. Werndl, L. Zuchowski, Springer, Cham 2021.

[12] G. Alberts, R. Kaenders, *Interview Pierre van Hiele*, p. 247.

[13] F. Goffree, *Pierre van Hiele, wiskundeleraar in Overveen, Bilthoven en Voorburg*, p. 101–103.

[14] G. Alberts, R. Kaenders, *Interview Pierre van Hiele*, p. 247. This and all other quotations were translated by the authors.

group of teachers, he wrote mathematics textbooks for secondary education. In 1957, Van Hiele and his wife earned a PhD at Utrecht University (see the next section).

From the late 1940s, Van Hiele and his wife were active members of the *Wiskunde Werkgroep* [Mathematics Working Group] (see below). In the 1950s/1960s, Van Hiele taught didactics of mathematics at one of the Dutch institutes for prospective teachers. From 1962 to 1982, he was one of the editors of "Euclides". Van Hiele died on 1 November 2010, at the age of 101. In "Euclides", Harrie Broekman wrote an obituary:

> Throughout his working life, Pierre was inspired by observing learners; but beyond that, describing and thinking through those observations. In doing so, he was very emphatic about the principle that the learning process should connect to the intuitive, informal mathematics that students have learned through their own experiments.[15]

Broekman concluded his obituary with:

> All those who worked with Pierre will remember him as a persisting researcher with strong opinions and – especially – a man of great passion. What he had liked to see happening was a mathematics curriculum based on vectors – a unification of the visual/geometrical and the arithmetical/algebraic. Whenever he had the opportunity he promoted that idea but, to his disappointment, it never found enough support.
>
> Although not any longer among us, Van Hiele's ideas are still as intriguing as they were 50 years ago and will be quoted often.[16]

The Mathematics Working Group

The Mathematics Working Group was part of the Dutch branch of the New Education Fellowship, an internationally operating network of progressive and experimental education, founded by several pedagogues in 1919. The working group started in 1936 and lasted until 1974. The group was instrumental in shaping Dutch mathematics education in general, but also for the professional development of the Van Hieles. In this group, ideas about the didactics of mathematics emerged, or, as Van Hiele himself stated it: 'Lovely time, where didactics was created'[17]. Mrs. Ehrenfest-Afanassjewa was (probably) one of the first members of this working group. Freudenthal was also an early member; in 1950 he became the chair of the group. The group organized not only monthly meetings but also a series of 'weekend conferences'. Fig. 2 shows a picture of the participants of the

[15] H. Broekman, *In memoriam: Pierre Marie van Hiele, 4 mei 1909 – 1 november 2010*, "Euclides" 2010, vol. 86, no. 3, p. 107.

[16] Ibidem, p. 107.

[17] P.M. van Hiele, *Freudenthal en de didactiek der wiskunde*, "Euclides" 1975, vol. 51, no. 1, p. 8–10.

Fig. 2. The Mathematics Working Group during the conference of 13–14 November 1948; Freudenthal in the middle of the first row (with bow tie), Dina van Hiele-Geldof in the third row on the right (woman with head partially hidden).
Credit: Courtesy of Martinus van Hoorn.

conference of 13–14 November 1948. Van Hiele and Mrs. Ehrenfest are not on the picture; however, Van Hiele is mentioned in the list of participants.

At the 1948 conference, Van Hiele presented the beginning of the work on the new curriculum as what he called 'an attempt to establish guidelines for a didactics of mathematics'[18]. He discussed the following key aspects: 'the goal of the mathematics education', 'the knowledge the students already possess', and 'how this knowledge can be expanded effectively'. In 1953, the Mathematics Working Group proposed a new curriculum for mathematics in secondary schools[19]. The proposal was prepared in five commissions (algebra, geometry, analytical geometry, trigonometry and descriptive geometry). Van Hiele was a member of the commissions for algebra, geometry and trigonometry, while Van Hiele-Geldof

[18] P.M. van Hiele, *Een poging om de richtlijnen op te stellen voor een didactiek der wiskunde* [An attempt to establish guidelines for the didactics of mathematics], "Euclides" 1948, vol. 24, no. 3, p. 122–133.

[19] Wiskunde Werkgroep, *Het wiskundeprogramma voor het V.H.M.O. Een ontwerp van de Wiskunde Werkgroep van de W.V.O.* [The mathematics program for secondary education. A proposal of the Mathematics Working Group of the W.V.O.], "Euclides" 1953, vol. 28, no. 5, p. 206–226.

Fig. 3. Working books of geometry and algebra by Van Hiele and Van Hiele-Geldof.
Credit: Collection Bert Zwaneveld.

was only a member of the commission for trigonometry[20]. In 1955, Van Hiele analyzed the motivation for the proposed curriculum[21]. In his motivation, he considered the proposed curriculum to be a major step forward and expressed the hope that it would be implemented. However, he also felt it necessary to warn of some dangers arising from certain didactic views in the proposed curriculum. A little later, Wimecos, one of the two precursors of the Dutch Association of Mathematics Teachers, published a slightly different proposal[22]. In 1958, a new curriculum based on both proposals was implemented.

Van Hiele and Van Hiele-Geldof as textbook authors

The Van Hieles began writing mathematics textbooks in the mid-1940s. *Werkboek der meetkunde: van figuren naar begrippen* [Working book of geometry: from figures to concepts] and *Werkboek der algebra I* [Working book of alge-

[20] F. Goffree, *Pierre van Hiele, wiskundeleraar in Overveen, Bilthoven en Voorburg*, p. 122.

[21] P.M. van Hiele, *De motivering in het rapport van de leerplancommissie-1954 van Wimecos* [Motivation in the report of the 1954 curriculum commission of Wimecos (Wi=mathematics, me=mechanics, cos=cosmography)], "Euclides" 1955, vol. 31, no. 2, p. 126–131.

[22] Wimecos, *Rapport van de leerplan-commissie-1954 van Wimecos inzake het opstellen van een ontwerp-leerplan en een ontwerp-eindexamenprogramma voor wiskunde voor de H.B.S-B.*, "Euclides" 1954, vol. 30, no. 4, p. 149–176.

bra I] (Fig. 3) became well-known first-year mathematics courses for VHMO[23] (= secondary education). The use of the word 'working' in the titles of the books was chosen in accordance with the authors' objective that students should learn mathematics by doing. More generally, textbooks were a way for them to express their didactic ideas in a concrete form.

In 1958, following the reform of the curriculum, the Van Hieles rewrote their textbooks without changing the titles, now for all types of secondary education. After a new curriculum reform in 1970, Van Hiele led a team of mathematics teachers and educationalists that revised all textbooks, but now without his wife, who had since passed away. In the 1958 and 1970 versions of the textbooks, Van Hiele applied his level theory, although traces of it can already be found in earlier editions. We provide two quotations on how the textbooks were received. About the algebra book, Rudolf Troelstra wrote: 'A book with special didactical qualities, which places quite high demands on the student, as well as on the teacher'[24]. And Jakob Groenman, a reviewer of the geometry book, wrote: 'The order is unorthodox. However, it may well be that the student, who is supposed to have never worked with the subject, does not find this as difficult as the teacher, who cannot dismiss his knowledge. And if that is indeed the case, then this can be considered a plus'[25].

Pierre van Hiele as an author on didactics of mathematics

After defending his PhD thesis, Van Hiele wrote many articles on the teaching and learning of mathematics, mainly in "Euclides", in which he almost always referred to his level theory. And often there was a further specification of it. Van Hiele also published several books. We mention:

- In 1973: *Begrip en inzicht: werkboek van de wiskundedidactiek* [Understanding and insight: working book of mathematics didactics] with 26 articles on many aspects of mathematics didactics, including his level theory; some articles are adaptations of previously published articles.
- In 1975: *Mogelijkheden van het wiskundeonderwijs* [Possibilities of mathematics education] with 10 articles more or less comparable to articles in *Begrip en inzicht*; this booklet deals primarily with mathematics education for students continuing their education in vocational schools.
- In 1981: *Structuur* [Structure], in which he tried to answer, with the help of many self-made pictures, questions such as:
  - What is the relationship between structure and insight?

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[23] *Voorbereidend Hoger en Middelbaar Onderwijs* [Preparatory higher and secondary education].

[24] R. Troelstra, *Review of Werkboek der algebra, I and II*, "Euclides" 1962, vol. 38, no. 1, p. 28.

[25] J. Groenman, *Review of Van figuren naar begrippen I and II*, "Euclides" 1964, vol. 39, no. 9, p. 284.

- Is mathematics, as used in other sciences, just a model or does it have more value? This question is related to a topic he already addressed in his PhD thesis: what is the summative/formative value of mathematics?
- Is it possible to describe a concept such as causality more clearly, so that the major disagreements about it diminish?
- Is the approach to processes, such as the learning process, with a description of the phases in the process, a necessary approach or is it just a 'verbal game'?
- Frequently it occurs that in an investigation someone cannot follow the line of thoughts. Is this caused by a lack of intelligence of the reader or is the author (or teacher) not careful enough in the design of the arguments?

• In 1986: *Structure and insight*[26], a revised translation of his 1981 book, which is still widely used in American higher education.

Theory of levels of mathematical thinking

Van Hiele's and Van Hiele-Geldof's PhD theses

The theory of levels of mathematical thinking developed by Van Hiele was based on classrooms observation protocols compiled by Van Hiele-Geldof. Van Hiele gave two main reasons for developing his level theory[27]:

• Students did not understand what their teacher was saying/explaining;
• Students stayed on the level where they could use the learned algorithms as shown by the teacher, but were unable to handle more complex mathematical situations.

The essence of the level theory is apparent in both Van Hiele's and Van Hiele-Geldof's PhD theses[28], which they defended on 4 July 1957 at Utrecht University (Fig. 4). Their research was supervised by Freudenthal (for Van Hiele) and Martinus Langeveld, developmental psychologist (for Van Hiele-Geldof). Van Hiele's central research question was: what is insight (in geometry) and what influence can the teacher have on it in a learning situation? Van Hiele-Geldof's research is, in a sense, a concretization of Van Hiele's ideas. Her central research question was: is a didactic approach possible in which the child's initial visual thinking (in geometry) develops into abstract thinking? Their companion doctoral

26 P.M. van Hiele, *Structure and insight: A theory of mathematics education*.

27 P.M. van Hiele, *De wortels van de minder gewenste habitus van de leerlingen in de lagere klassen van het middelbaar onderwijs en de mogelijkheden tot verbetering hiervan*, [in:] *Voordrachten gehouden op het tiende conferentie-weekeinde van de Wiskunde-werkgroep van de W.V.O.*, Conferentieoord "De Grasheuvel", Amersfoort, 19–20 oktober 1957 (Publicatie uit de Wiskunde-werkgroep van de Werkgemeenschap voor Vernieuwing van Opvoeding en Onderwijs, no. 7), J. Muusses, Purmerend 1958.

28 P.M. van Hiele, *De problematiek van het inzicht*; D. van Hiele-Geldof, *De didaktiek van de meetkunde in de eerste klas van het V.H.M.O*.

work resulted in Van Hiele's theory of levels of mathematical thinking. Initially, Van Hiele distinguished three levels (levels 1–3 described below), later four (including a 'ground' level) or five[29]. For readers unfamiliar with this theory, we outline the five-level version, exemplified in the domain of (initial) teaching of (Euclidean) geometry at the secondary level, where Van Hiele developed his theory. Papers by other researchers referring to Van Hiele's theory typically present the same five levels[30].

Fig. 4. Van Hiele (right) and Van Hiele-Geldof at the defense of their PhD theses, 1957.
Source: F. Goffree, *Pierre van Hiele, wiskundeleraar in Overveen, Bilthoven en Voorburg*, [in:] *Ik was wiskundeleraar*, SLO, Enschede 1985, p. 109.

[29] G. Alberts, R. Kaenders, *Interview Pierre van Hiele*, p. 249–250.
[30] See, e.g., J. Pegg, *The van Hiele theory*, p. 613–615; S. Watan, Sugiman, *The Van Hiele theory and realistic mathematics education: As teachers' instruction for teaching geometry*, "AIP Conference Proceedings" 2018, vol. 2014, no. 1, p. 020075-1–020075-7.

Level 0. This 'ground' level (concrete or intuitive level) is purely visual. Students recognize geometrical figures based on their global shape and compare the shapes based on a given figure or everyday objects. Properties of geometrical figures are not recognized, but the student can, for example, distinguish a square from a rectangle (but at that level, a square is not seen as a rhombus, particularly when the diagonals are in vertical and horizontal position).

Level 1. At this descriptive level, a geometric object is recognized by its properties. For example, a rhombus has four equal sides, the opposite angles are equal, the diagonals divide each other into two equal parts and are perpendicular to each other, but these properties are seen independent of each other. A square is recognized as a rhombus, but the property 'being a square' does not imply 'being a rhombus'. Deduction is not yet available.

Level 2. At this 'informal deductive' level, students no longer see the properties of figures as independent. They recognize that a property precedes or follows from other properties. For example, the student understands that the equality of two angles in a triangle implies the equality of two sides, and vice versa. Relationships between different shapes are understood based on their properties: squares are rhombuses *because* they have equal sides.

Level 3. At this 'theoretical deductive' level, students understand the place of deduction and the role of definitions, axioms, theorems. They develop logic proofs for properties instead of memorizing them. They are able to distinguish between necessary and sufficient conditions, and between a theorem and its converse.

Level 4. At this 'formal deductive' level, thinking focusses on the mathematical system itself. Axioms are no longer regarded as 'absolute truths', but can be manipulated. Students can compare axiom systems and explore different geometries based on different axiom systems. Evidently, this level of thinking is rarely achieved by secondary school students.

In principle, Van Hiele's levels are hierarchical, but that does not mean that a particular level must be fully mastered before moving on to the next level. Sometimes skills that belong to concepts or structures of a certain level tend to function at a lower level. Van Hiele calls this phenomenon 'level reduction' and argues that this is sometimes necessary. It gives the Van Hiele levels a more cyclical than hierarchical character.

Although the above description of the Van Hiele levels is content-specific (geometry), they can be viewed more broadly as stages of cognitive development that have much in common with those of Piaget (see below). In Van Hiele's words, 'the levels are situated not in the subject matter but in the thinking of man'[31]. Van Hiele himself occasionally refers to the development of numerical and algebraic thinking. For this reason, we allow ourselves to speak of levels of *mathematical* thinking. In

[31] P.M. van Hiele, *Structure and insight: A theory of mathematics education*, p. 41.

fact, most disciplines have different levels of thinking: the visual level, the descriptive level and the theoretical level, each with its own network of relations and its own judgments of truth[32], but the Van Hiele theory is most convincingly developed and best known in the domain where it originated: geometry.

Van Hiele associates progress through the levels, and in particular the attainment of the theoretical level, with the development of *insight*, which is also his ultimate goal. The level theory is not about teaching geometric skills or solving problems in geometry, but is first and foremost intended to help students develop insight into the structure of (Euclidean) geometry (or more generally: a particular mathematical domain). Students on the path to insight are referred to by Van Hiele as 'structuring students', in contrast to 'algorithmic students', who can be quite successful in mathematics at school by learning to master algorithms.

Van Hiele acknowledged that he had been influenced by Piaget's work in the field of developmental psychology. However, he considered Piaget's findings to be primarily of theoretical interest and of little relevance to education[33]. There are also clear differences between Piaget's and Van Hiele's 'level theories'. Piaget's levels are more or less fixed stages, and progress is basically the result of maturation or natural development, while Van Hiele's levels are not age-related, and advancement from a lower to a higher level is the result of a learning process that a teacher can facilitate, for example, by providing appropriate materials or situations. In this sense, Van Hiele's theory is more of a pedagogical theory, while Piaget's is a psychological theory.

In addition to Piaget, the doctoral research of Van Hiele's wife Dina (Dieke), and in particular the lessons she had analyzed, were an important source of inspiration for Van Hiele. They provided the empirical evidence for the level theory. 'But then I saw Dieke's work […] And suddenly, the levels became clear. […] Her work helped me overcome my doubts […] and enough courage to bring the levels to the fore'[34]. In *Understanding and insight*, Van Hiele identified five phases in a learning process (inquiry/information, directed orientation, explication, free orientation and integration) that allowed students to develop from one level to the next[35]. These phases also appeared through the doctoral work of his wife. Van Hiele-Geldof's contribution was also recognized by her contemporaries. In "Euclides", Hermen Jacobs, secretary of the Mathematics Working Group, wrote a detailed review of the Van Hieles' theses[36]. In this review, he praised

[32] P.M. van Hiele, *Similarities and differences between the theory of learning and teaching of Skemp and the Van Hiele levels of thinking*, [in:] *Intelligence, learning and understanding: A tribute to Richard Skemp*, ed. by D.O. Tall, M.O.J. Thomas, Post Pressed, Flaxton 2002, p. 27–47.

[33] P.M. van Hiele, *De problematiek van het inzicht*, p. 32.

[34] F. Goffree, *Pierre van Hiele, wiskundeleraar in Overveen, Bilthoven en Voorburg*, p. 114–115.

[35] See also J. Pegg, *The van Hiele theory*, p. 613–615.

[36] H.J. Jr. Jacobs, *De dissertaties van de Van Hiele's*, "Euclides" 1959, vol. 34, no. 8, p. 246–253.

Van Hiele-Geldof's work because it was based on protocols of many lessons. These protocols make it possible to understand the method, the way in which pupils think, and the important didactic conclusions. According to Jacobs, they form the basis of level theory, and he even considered them more important than the theoretical considerations. Unfortunately, Van Hiele-Geldof was unable to continue her work: she died in 1958, the year after she defended her thesis. In an obituary, Piet Vredenduin wrote:

> Dieke van Hiele was one of the pioneers in modernizing mathematics education. She devoted all her efforts to achieving a responsible method of teaching mathematics to students in a way that the subject-matter was adapted to the abilities of the child. As could be expected, she chose experiences as a starting point; yet her aim was acquiring true mathematical understanding. Together with Van Hiele, she conducted in-depth theoretical en practical investigations in this field. Their PhD theses are, in a sense, the crown of their work.[37]

Reactions to the level theory

In the interview with Goffree[38], Van Hiele referred to the international recognition of the level theory. He told the following anecdote: 'At the 1980 ICME conference held in Berkeley, California, was someone who lectured on Van Hiele's theory. When he – it was a Canadian – saw that I was there too, he immediately asked if I would stand up. Now nothing could go wrong, he added, Van Hiele would tell if he had understood the theory correctly'[39].

Izaak Wirszup (University of Chicago) played an essential role in popularizing Van Hiele's ideas in the US. He became acquainted with Van Hiele's ideas through Soviet literature[40], which indicates that these ideas had already crossed the Iron Curtain in the 1960s. After publishing an initial paper on the subject, entitled 'Breakthroughs in the Psychology of Learning and Teaching Geometry'[41], Wirszup presented Van Hiele's theory to the National Council of Teachers of Mathematics. The theory then gained widespread recognition in the US (and beyond) and was used in various studies. Van Hiele mentioned Alan Hoffer (Palo Alto), Zalman Usiskin (Chicago), Joanne Mayberry (Athens), Dorothy Geddes (New York), Sharon L. Senk (Syracuse), and Dieter Lunkenbein (Sherbrooke). A more recent example is Whitney George's (Wisconsin) merging of the

[37] P.G.J. Vredenduin, *In memoriam Mevrouw Dr. D. van Hiele-Geldof*, "Euclides" 1958, vol. 34, no. 1, p. 1.

[38] F. Goffree, *Pierre van Hiele, wiskundeleraar in Overveen, Bilthoven en Voorburg*, p. 101–136.

[39] Ibidem, p. 129.

[40] D.L. Roberts, *Interview with Izaak Wirszup*, "International Journal for the History of Mathematics Education" 2010, vol. 5, no. 1, p. 53–74.

[41] I. Wirszup, *Breakthroughs in the psychology of learning and teaching geometry*, [in:] *Space and geometry: Papers from a Research Workshop*, ed. by J. Martin, ERIC/SMEAC, Columbus, Ohio 1976, p. 75–97.

Van Hiele model with Piaget's model of spatial reasoning to present a coherent framework for describing the development of geometric and topological intuition and understanding in young learners[42].

International recognition was also apparent when Van Hiele was invited to write a chapter in a tribute to Richard Skemp[43]. Van Hiele discussed the similarities and differences between both their theories. Van Hiele shared Skemp's opinion that 'instrumental' thinking cannot be considered as real thinking and that intuitive thinking is the base of all rational thinking. Skemp would object to an instrumental action such as differentiating $1/x^3$ by remembering $1/x^3 = x^{-3}$, and then applying the rule that the derivative of x^n is nx^{n-1}, whereas Van Hiele would be content with the above calculation if he was sure that the student once in his studies had comprehended a series of proofs demonstrating that the formula for the differentiation of powers is true for all real exponents[44].

The 'milieu' in which Van Hiele's ideas originated: Afanassjewa as a predecessor?

At the time of the Van Hieles (and before), geometry was central to Dutch debates on mathematics education[45]. Textbooks offered a (more or less) rigorous deductive approach to Euclidean geometry, which was believed to contribute to 'learning to think'. The figurehead of the (very) strict approach to geometry education was Eduard Jan Dijksterhuis, mathematician and historian of science. However, this approach was not appreciated equally by all mathematics teachers. A 'didactic alternative' was proposed by Tatiana Afanassjewa (Fig. 5). She outlined a geometry course in three parts: first an intuitive introduction, followed by a more systematic treatment, and finally a rigorous deductive exposition.

Tatiana Afanassjewa (1876–1964)

As early as 1924, Afanassjewa published a pamphlet on geometry education[46]. Her premise is that every scientific discipline starts from intuitive ideas. After becoming increasingly sure of the relevance of these intuitive ideas, people working in that discipline will try to formulate these ideas abstractly, in a more logical way. The development of mathematical thinking also follows this pattern. Reflecting on the role of intuition and logic, especially in Euclidean geometry, she

[42] W. George, *Bringing van Hiele and Piaget together: A case for topology in early mathematics learning*, "Journal of Humanistic Mathematics" 2017, vol. 7, no. 1, p. 105–116.

[43] P.M. van Hiele, *Similarities and differences*, p. 27–47.

[44] Ibidem, p. 43–44.

[45] E. de Moor, S. Kemme, *Meetkundeonderwijs op gymnasium en hbs 1900–1968*, "Nieuw Archief voor Wiskunde" 2012, vol. 13, no. 2, p. 102–109.

[46] T. Ehrenfest-Afanassjewa, *Wat kan en moet het meetkundeonderwijs aan een niet-wiskundige geven?*, J.B. Wolters, Groningen 1924.

Fig. 5. Tatiana Afanassjewa around 1910.

Source: H.J. Smid, *Formative years: Hans Freudenthal in prewar Amsterdam*, [in:] *International Study Group on the Relations between History and Pedagogy of Mathematics: Proceedings of the 2016 ICME Satellite Meeting*, ed. by L. Radford, F. Furinghetti, T. Hausberger, IREM de Montpellier, Montpellier 2016, p. 74.

concluded that the syllogisms of logic are not an instrument of thought: 'Can another, to whom we present our syllogistic reasoning, follow us, unless he himself *thinks*... in a non-syllogistic way?' And she added that this applies not only to syllogisms, but to any pure one-dimensional chain of thought. We quote from Chapter II 'What is logical?' of Afanassjewa's pamphlet.

> Understanding a subject does not happen at those moments when one abstracts from it and focuses on the formal-logical relations between the propositions by which it is described: at such moments, of course, one thinks, but now no longer about that subject, but about something else – about those formal relations. One no longer relates to the subject itself as a 'thinking' but as a 'calculating' person. Similarly, operating on algebraic formulas in physics is not thinking about the physical relations themselves, as represented in those formulas.
>
> I reserve the word thinking only for: the processing of intuitive material by consciousness, but I suppose that is a matter of terminology. But this touches the essence of the matter, that in a scientific investigation I distinguish *two* things: 'thinking' and 'calculating' […] and therefore I do not refer to them both with the same word 'thinking'.[47]

And also in the same chapter:

> *The actual work of logic occurs at those moments when intuition is brought to consciousness.*[48]

Afanassjewa then argues that in geometry, since Euclid, two completely different branches of science have been mixed: the *science of space* and the *axiomatics of geometry*. The intuitive material of the science of space provides the *spatial imaginative faculty*. Its logical processing consists in searching for the most essential spatial relations, formulating and establishing them. The intuitive material of axiomatics are all those propositions that constitute for us the science of space. The logical operations on them consist of determining which are the most essential from which the other propositions can be logically derived, and in proving their (formal) logical independence. In the science of space, to prove means to understand that a proposition is correct. In axiomatics, proving means deriving the proposition

[47] Ibidem, p. 4–6.
[48] Ibidem, p. 6, italics in original.

from the axioms. She argues that in teaching, the intuitive approach of the science of space precedes the abstract approach of axiomatics.

Afanassjewa elaborated her ideas in a geometry course in three parts: first an intuitive part, aimed at the imaginative faculty – the propaedeutic course –, followed by a systematic course, in which theorems are proved only if they are not evident to one of the students; the theorems are accepted as a kind of axioms; proving happens in a dialogue between teacher and students. The third part consists of an axiomatic revision in which the provisional axioms are proved. In 1931, Afanassjewa concretized her ideas of such a propaedeutic course in a booklet titled *Uebungensammlung zu einer Geometrischen Propädeuse*[49].

The title of this section is a question: is Afanassjewa a (sort of) predecessor of Van Hiele with respect to his level theory? Afanassjewa distinguished between intuitive and abstract/logic, thinking and calculating. The aspects 'intuitive' and 'abstract/logic' are prominently present in Van Hiele's level theory. The comparable keywords are 'intuitive/concrete', maybe 'visual' versus 'logical/theoretical', and the 'structure-oriented' student versus the 'algorithmic-oriented' student. Surely, Van Hiele's theory is much more comprehensive and detailed. Moreover, it is underpinned by the classroom observations of Van Hiele-Geldof. But in our opinion, core elements of the level theory are already present in Afanassjewa's ideas. Therefore, we think that the question of whether Afanassjewa is a predecessor of Van Hiele deserves an affirmative answer.

It is fair to add that Van Hiele mentioned Afanassjewa only in the chapter of his PhD thesis on the formative value of mathematics, where he described a discussion between Afanassjewa and Freudenthal about this (supposed) formative value. In short, Freudenthal was absolutely not convinced that 'sound reasoning' – outside mathematics – was best learned within mathematics, while Afanassjewa was convinced of the opposite. The discussion between Afanassjewa and Freudenthal was published in 1951 in a joint brochure entitled: *Kan het Wiskundeonderwijs tot de Opvoeding van het Denkvermogen Bijdragen?*[50]. From Van Hiele's PhD thesis and a contribution to "Euclides"[51], together with his wife, it is clear that the Van Hieles followed Afanassjewa. The difference in views between Afanassjewa and Freudenthal, according to them, has its origin in their different views on mathematics:

> Mrs. Ehrenfest can only come to the conclusion that mathematics can have great formative value by giving a broader meaning to the term 'mathematics' than is usual.

[49] T. Ehrenfest-Afanassjewa, *Uebungensammlung zu einer Geometrischen Propädeuse*, Martinus Nijhoff, Den Haag 1931.

[50] Can mathematics education contribute to the development of the intellectual capacity?

[51] P.M. van Hiele, D. van Hiele-Geldof, *Een fenomenologische inleiding tot de meetkunde*, "Euclides" 1957, vol. 33, no. 2, p. 33–46.

> If someone wants mathematics education to have great formative value, then this education should be primarily concerned with teaching how concrete (originally non-mathematical) problems can be transformed into mathematics.[52]

About Afanassjewa's influence on her work, Van Hiele-Geldof wrote that the idea of starting geometry with plane tilings was suggested 'years earlier at the house of Mrs. Ehrenfest-Afanassjewa'[53]. In the aforementioned interview with Alberts and Kaenders, Van Hiele, when asked whether he had already become acquainted with Mrs. Ehrenfest-Afanassjewa during his studies, replied as follows: 'Yes, I knew her well for a long time. Through Freudenthal; he told me that Mrs. Ehrenfest was an important person I should contact'[54]. It is not clear from that interview why Freudenthal advised this to Van Hiele. Freudenthal, according to Van Hiele, was not engaged in didactics at the time: 'About didactics, we actually did not even know that you could study it. That was unknown at that time'[55].

Following the passing of Afanassjewa, Van Hiele and Gerrit Krooshof, the later editor-in-chief of "Euclides", wrote an In memoriam[56]. The authors praised two qualities: her profound modesty and the indefatigable will to keep thinking and speaking about the didactics of mathematics, in particular geometry. She considered geometry to be part of physics. After the launch of the Mathematics Working Group, the fourth meeting was already held at her home. She actively participated in the discussions about a new conception of a mathematics/geometry curriculum for secondary schools. Her ideal for geometry was a thin textbook with only the necessary theorems, and as assignments only those that contributed to the theory.

Hans Freudenthal (1905–1990)

Freudenthal knew the Van Hieles since they were studying mathematics in Amsterdam. The pedagogical interests of the 'research mathematician' Freudenthal may have been sparked in the 1930s by the work of Afanassjewa, whom he held in high regard. It was Freudenthal who advised Van Hiele to contact her and learn more about her ideas.

However, although Freudenthal was Van Hiele's thesis supervisor, his influence on that scholarly work seems rather modest. Freudenthal's initial idea for a PhD thesis in didactics – a very detailed study of the learning process of

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52 P.M. van Hiele, D. van Hiele-Geldof, *De vormende waarde der wiskunde*, "Euclides" 1957, vol. 32, no. 2, p. 281.

53 D. van Hiele-Geldof, *De didaktiek van de meetkunde in de eerste klas van het V.H.M.O.*, p. 9.

54 G. Alberts, R. Kaenders, *Interview Pierre van Hiele*, p. 247.

55 Ibidem, p. 247.

56 P.M. van Hiele, G. Krooshof, *T. Ehrenfest-Afanassjewa*, "Euclides" 1957, vol. 39, no. 9, p. 257–259.

a single student throughout his or her entire school career – was rejected by Van Hiele because it was simply not feasible. Their personal relationship was not always smooth. In the interview with Alberts and Kaenders, Van Hiele called Freudenthal 'bossy'. At a certain point, Van Hiele even considered requesting a different supervisor, but they eventually found a *modus vivendi*. In terms of mutual influence, Sacha La Bastide-van Gemert has argued that the work of Van Hiele (and that of his wife) had a fundamental influence on Freudenthal (rather than vice versa): key ideas in Freudenthal's later work, such as 'mathematization' and 'guided re-invention', are already present in that work (although they are not referred to as such)[57].

To conclude

We discussed Van Hiele's work in mathematics education, in particular his theory of levels of mathematical thinking. We have argued that and how Van Hiele was influenced by Piaget, his wife Dina, Afanassjewa, and (to a limited extent) Freudenthal.

Van Hiele's level theory found an international resonance. We know that the theory already crossed the Iron Curtain in the 1960s, but it became particularly popular in the US (and elsewhere) in the late 1970s and 1980s. In the 1970s, after a brief period of New Math, interest in deductive Euclidean geometry began to wane in the Netherlands and new goals for geometry education were set[58]. In the Netherlands, this led to the development of the model of Realistic Mathematics Education. In other parts of the world, constructivist trends in mathematics education emerged from the 1980s. The publication in 1989 of the *Standards for School Mathematics* was a well-known milestone in the US[59]. It is likely that the change in the goals of geometry education (and mathematics education in general) led to a decline in interest in Van Hiele's theory from the 1990s, although there are still individual researchers today who draw inspiration from this theory.

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[57] S. La Bastide-van Gemert, *All positive action starts with criticism*, p. 179–204.

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