

# Positive linear systems with different fractional orders

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**Abstract.** A new class of positive linear systems with different fractional orders is introduced. A solution to the set of linear differential equations with different fractional orders is derived. Necessary and sufficient conditions for the positivity of the fractional systems are established. It is shown that the linear electrical circuits composed of resistors, supercondensators, coils and voltage (current) sources are positive systems with different fractional orders.

**Key words:** positive, fractional order, linear, solution electrical circuit.

## 1. Introduction

A dynamical system is called positive if and only if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in monographs [1, 2]. Variety of models having positive linear behavior can be found in engineering, management sciences, economics, social sciences, biology and medicine, etc.. Mathematical fundamentals of the fractional calculus are given in the monographs [3–5]. The positive fractional linear systems have been introduced in [6, 7]. Stability of fractional linear 1D discrete-time and continuous-time systems has been investigated in the papers [8–11] and of 2D fractional positive linear systems in [12]. The notion of practical stability of positive fractional discrete-time linear systems has been introduced in [13] and the relationship between fractional order continuous-time systems configuration and its dynamics has been analyzed in [3]. Some recent interesting results in fractional systems theory and its applications can be found in [14, 15].

In this paper a new class of positive linear systems with different fractional orders will be introduced. Using the Laplace transform method the solution to the set of linear differential equations with different fractional orders will be derived and necessary and sufficient conditions for the positivity of the fractional systems will be established.

The paper is organized as follows. In Sec. 2 the set of linear differential equations with different fractional orders is introduced and its solutions is derived using Laplace transform method. Necessary and sufficient conditions for the positivity of the fractional linear systems are established in Sec. 3. In Sec. 4 it is shown that the linear electrical circuits are positive systems with different fractional orders. Concluding remarks are given in Sec. 5.

The following notation will be used in this paper. The set of real  $n \times m$  matrices will be denoted by  $\mathfrak{R}^{n \times m}$  and the set of  $n \times m$  real matrices with nonnegative entries will be

denoted  $\mathfrak{R}_+^{n \times m}$  ( $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$ ). The  $n \times n$  identity matrix will be denoted by  $I_n$ .

## 2. Linear differential equations with different fractional orders and their solutions

In this paper the following Caputo definition of the fractional derivative will be used

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \quad (1)$$

where  $n-1 < \alpha < n$ ,  $n \in N = \{1, 2, \dots\}$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (2)$$

is the gamma Euler function and

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}. \quad (3)$$

It is well known [5, 7] that the Laplace transform ( $L$ ) of (1) is given by the formula

$$L \left[ \frac{d^\alpha f(t)}{dt^\alpha} \right] = \int_0^\infty \frac{d^\alpha f(t)}{dt^\alpha} e^{-st} dt = s^\alpha F(s) - \sum_{k=1}^n s^{\alpha-k} f^{(k-1)}(0+), \quad (4)$$

where  $F(s) = L[f(t)]$  and  $n-1 < \alpha < n$ ,  $n \in N$ .

Consider a fractional linear system described by the equation

$$\begin{bmatrix} \frac{d^\alpha x_1}{dt^\alpha} \\ \frac{d^\beta x_2}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad (5)$$

$$p-1 < \alpha < p; \quad q-1 < \beta < q; \quad p, q \in N$$

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where  $x_1 \in \mathbb{R}^{n_1}$  and  $x_2 \in \mathbb{R}^{n_2}$  are the state vectors,  $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ ,  $B_i \in \mathbb{R}^{n_i \times m}$ ;  $i, j = 1, 2$ , and  $u \in \mathbb{R}^m$  is the input vector.

Initial conditions for (5) have the form

$$x_1(0) = x_{10} \quad \text{and} \quad x_2(0) = x_{20}. \tag{6}$$

**Theorem 1.** The solution of the equation (5) for  $0 < \alpha < 1$ ;  $0 < \beta < 1$  with initial conditions (6) has the form

$$x(t) = \Phi_0(t)x_0 + \int_0^t [\Phi_1(t - \tau)B_{10} + \Phi_2(t - \tau)B_{01}] u(\tau) d\tau \tag{7}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$T_{kl} = \begin{cases} I_n & \text{for } k = l = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for } k = 0, l = 1 \\ T_{10}T_{k-1,l} + T_{01}T_{k,l-1} & \text{for } k + l > 0 \end{cases} \tag{8}$$

$$\Phi_0(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}, \tag{9a}$$

$$\Phi_1(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k+1)\alpha+l\beta]}, \tag{9b}$$

$$\Phi_2(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]}, \tag{9c}$$

**Proof.** Using the Laplace transforms

$$X_i(s) = L[x_i(t)], \quad i = 1, 2 \quad \text{and} \quad U(s) = L[u(t)]$$

we may write the equations (5) for  $0 < \alpha < 1$ ;  $0 < \beta < 1$  in the form

$$\begin{bmatrix} s^\alpha X_1(s) \\ s^\beta X_2(s) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} s^{\alpha-1}x_{10} \\ s^{\beta-1}x_{20} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s) \tag{10}$$

since by (4) for  $0 < \alpha < 1$ ;  $0 < \beta < 1$

$$\mathcal{L} \left[ \frac{d^\alpha x_1}{dt^\alpha} \right] = s^\alpha X_1(s) - s^{\alpha-1}x_{10},$$

$$\mathcal{L} \left[ \frac{d^\beta x_2}{dt^\beta} \right] = s^\beta X_2(s) - s^{\beta-1}x_{20}.$$

From (10) we have

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} I_{n_1}s^\alpha - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}s^\beta - A_{22} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} s^{\alpha-1}x_{10} \\ s^{\beta-1}x_{20} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s) \right\}. \tag{11}$$

Using (8) it is easy to verify that

$$\begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}s^{-\beta} & I_{n_2} - A_{22}s^{-\beta} \end{bmatrix} \left[ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} s^{-(k\alpha+l\beta)} \right] = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{12}$$

where the matrices  $T_{kl}$  are defined by (8).

Using (12) we obtain

$$\begin{aligned} & \begin{bmatrix} I_{n_1}s^\alpha - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}s^\beta - A_{22} \end{bmatrix}^{-1} = \\ & = \left\{ \begin{bmatrix} I_{n_1}s^\alpha & 0 \\ 0 & I_{n_2}s^\beta \end{bmatrix} \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}s^{-\beta} & I_{n_2} - A_{22}s^{-\beta} \end{bmatrix} \right\}^{-1} \\ & = \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}s^{-\beta} & I_{n_2} - A_{22}s^{-\beta} \end{bmatrix}^{-1} \\ & = \begin{bmatrix} I_{n_1}s^{-\alpha} & 0 \\ 0 & I_{n_2}s^{-\beta} \end{bmatrix} = \\ & = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} s^{-(k\alpha+l\beta)} \begin{bmatrix} I_{n_1}s^{-\alpha} & 0 \\ 0 & I_{n_2}s^{-\beta} \end{bmatrix}. \end{aligned} \tag{13}$$

Substitution of (13) into (11) yields

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \left[ x_0 s^{-(k\alpha+l\beta+1)} + (B_{10}s^{-[(k+1)\alpha+l\beta]} + B_{01}s^{-[k\alpha+(l+1)\beta]}) U(s) \right]. \tag{14}$$

Applying the inverse Laplace transform ( $\mathcal{L}^{-1}$ ) and the convolution theorem to (14) we obtain

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \mathcal{L}^{-1} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \mathcal{L}^{-1} \\ & \left[ x_0 s^{-(k\alpha+l\beta+1)} + (B_{10}s^{-[(k+1)\alpha+l\beta]} + B_{01}s^{-[k\alpha+(l+1)\beta]}) U(s) \right] \\ &= \Phi_0(t)x_0 + \int_0^t [\Phi_1(t - \tau)B_{10} + \Phi_2(t - \tau)B_{01}] u(\tau) d\tau \end{aligned} \tag{15}$$

since  $\mathcal{L}^{-1} \left[ \frac{1}{s^{\alpha+1}} \right] = \frac{t^\alpha}{\Gamma(\alpha+1)}$ .

**Remark.** Note that if  $\alpha = \beta$  then from (9a) we have [6, 7]

$$\Phi_0|_{\alpha=\beta}(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \quad (16a)$$

From comparison of (9a) and (16a) and using (8) it is easy to show that

$$\sum_{i=0}^k \sum_{j=0}^k T_{ij} \frac{t^{i\alpha+j\beta}}{\Gamma(i\alpha + j\beta + 1)} \Big|_{\alpha=\beta} = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \quad (16b)$$

### 3. Positive fractional systems

**Definition 1.** The fractional system (5) is called positive if  $x_1 \in \mathbb{R}_+^{n_1}$  and  $x_2 \in \mathbb{R}_+^{n_2}$ ,  $t \geq 0$  for any initial conditions  $x_{10} \in \mathbb{R}_+^{n_1}$ ,  $x_{20} \in \mathbb{R}_+^{n_2}$  and all input vectors  $u \in \mathbb{R}_+^m$ ,  $t \geq 0$ .

Let  $M_n$  be the set of  $n \times n$  Metzler matrices, i.e. real matrices with nonnegative off-diagonal entries.

**Theorem 2.** The fractional system (5) for  $0 < \alpha < 1$ ;  $0 < \beta < 1$  is positive if and only if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_n, \quad (17a)$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n \times m}, \quad (n = n_1 + n_2) \quad (17b)$$

**Proof.** First we shall show that

$$\Phi_k(t) \in \mathbb{R}_+^{n \times n} \quad \text{for } k = 0, 1, 2 \quad \text{and } t \geq 0 \quad (18)$$

only if (17a) holds.

From the expansions (9) we have

$$\begin{aligned} \Phi_0(t) &= \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t}{\Gamma(\alpha + 1)} + \\ &+ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t}{\Gamma(\beta + 1)} + \dots \end{aligned} \quad (19a)$$

$$\begin{aligned} \Phi_1(t) &= \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \\ &+ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} + \dots \end{aligned} \quad (19b)$$

$$\begin{aligned} \Phi_2(t) &= \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} \frac{t^{\beta-1}}{\Gamma(\beta)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} + \\ &+ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{2\beta-1}}{\Gamma(2\beta)} + \dots \end{aligned} \quad (19c)$$

From (19) it follows that  $\Phi_k(t) \in \mathbb{R}_+^{n \times n}$ ,  $k = 0, 1, 2$  for small value of  $t > 0$  only if the condition (17a) is satisfied.

Using (16) in a similar way as in [8, 10] it can be shown that if (17) holds then

$$\Phi_0(t) \in \mathbb{R}_+^{n \times n} \quad t \geq 0 \quad (20a)$$

and

$$\Phi_1(t)B_{10} + \Phi_2(t)B_{01} \in \mathbb{R}_+^{n \times n} \quad t \geq 0. \quad (20b)$$

In this case from (7) we have  $x(t) \in \mathbb{R}_+^n$ ,  $t \geq 0$  since by definition  $x_0 \in \mathbb{R}_+^n$  and  $u(t) \in \mathbb{R}_+^m$ ,  $t \geq 0$ . The remaining part of the proof is similar as in [6, 7].

These considerations can be extended for the set of  $p$  matrix differential equations with different fractional orders.

### 4. Fractional linear electrical circuits

Consider linear electrical circuits composed of resistors, supercondensators (ultracapacitors), coils and voltage (current) sources. As the state variables (the components of the state vector  $x$ ) the voltage across the supercondensators and the currents in the coils are usually chosen. It is well-known [14, 16] that the current  $i(t)$  in supercondensator with its voltage  $u_C(t)$  is related by the formula

$$i_C(t) = C \frac{d^\alpha u_C(t)}{dt^\alpha} \quad \text{for } 0 < \alpha < 1, \quad (21)$$

where  $C$  is the capacity of the supercondensator.

Similarly, the voltage  $u_L(t)$  on the coil with its current  $i_L(t)$  is related by the formula

$$u_L(t) = L \frac{d^\beta i_L(t)}{dt^\beta} \quad \text{for } 0 < \beta < 1, \quad (22)$$

where  $L$  is the inductance of the coil.

Using the relations (21), (22) and the Kirchoff's laws we may write for the fractional linear circuits the following state equation

$$\begin{bmatrix} \frac{d^\alpha x_C}{dt^\alpha} \\ \frac{d^\beta x_L}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e \quad (23)$$

where the components of  $x_C \in \mathbb{R}^{n_1}$  are voltages across the supercondensators, the components of  $x_L \in \mathbb{R}^{n_2}$  are currents in coils and the components of  $e \in \mathbb{R}^m$  are the voltages of the circuit.

**Example 1.** Consider the linear electrical circuit shown on Fig. 1 with known resistances  $R_1, R_2, R_3$ , capacitances  $C_1, C_2$ , inductances  $L_1, L_2$  and sources voltages  $e_1, e_2$ .

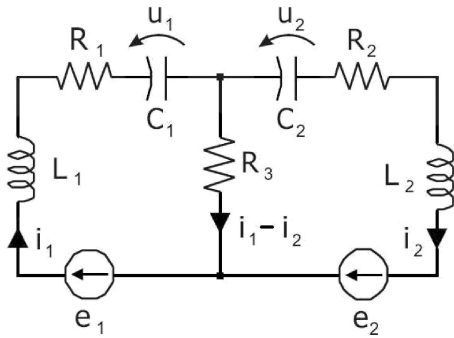


Fig. 1. Electrical circuit

Using relations (21), (22) and the Kirchoff's laws we may write for the circuit the following equations.

$$i_1 = C_1 \frac{d^\alpha u_1}{dt^\alpha}, \quad i_2 = C_2 \frac{d^\alpha u_2}{dt^\alpha}$$

$$e_1 = (R_1 + R_3)i_1 + L_1 \frac{d^\beta i_1}{dt^\beta} + u_1 - R_3 i_2 \quad (24)$$

$$e_2 = (R_2 + R_3)i_2 + L_2 \frac{d^\beta i_2}{dt^\beta} + u_2 - R_3 i_1.$$

The Eq. (24) can be written in the form

$$\begin{bmatrix} \frac{d^\alpha u_1}{dt^\alpha} \\ \frac{d^\alpha u_2}{dt^\alpha} \\ \frac{d^\beta i_1}{dt^\beta} \\ \frac{d^\beta i_2}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (25)$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L_1} & 0 & -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ 0 & -\frac{1}{L_2} & \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad (26)$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}.$$

From (26) it follows that the fractional electrical circuit is not positive since the matrix  $A$  has some negative off-diagonal entries.

If the fractional linear circuit is not positive but the matrix  $B$  has nonnegative entries (see for example the circuit in Fig. 1) then using the state-feedback

$$e = K \begin{bmatrix} x_C \\ x_L \end{bmatrix} \quad (27)$$

we may usually choose the gain matrix  $K \in R^{m \times n}$  so that the closed-loop system matrix (obtained by substitution of (27) into (23))

$$A_c = A + BK \quad (28)$$

is a Metzler matrix.

**Theorem 3.** Let  $A$  be not a Metzler matrix but  $B \in R_+^{n \times m}$ . Then there exists a gain matrix  $K$  such that the closed-loop system matrix  $A_c \in M_n$  if and only if

$$\text{rank}[B, A_c - A] = \text{rank} B \quad (29)$$

**Proof.** By Kronecker-Cappely theorem the equation

$$BK = A_c - A \quad (30)$$

has a solution  $K$  for any given  $B$  and  $A_c - A$  if and only if the conditions (29) is satisfied.

**Example 2.** (continuation of Example 1).

Let

$$A_c = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ \frac{a_1}{L_1} & 0 & -\frac{R_1 + R_3}{L_1} & \frac{a_3}{L_1} \\ 0 & \frac{a_2}{L_2} & \frac{a_4}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix} \quad (31)$$

for  $a_k \geq 0 \quad k = 1, 2, 3, 4$ .

In this case the condition (29) is satisfied since

$$\text{rank}[B, A_c - A] = \text{rank} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{L_1} & 0 & \frac{a_1 + 1}{L_1} & 0 & 0 & \frac{a_3 - R_3}{L_1} \\ 0 & \frac{1}{L_2} & 0 & \frac{a_2 + 1}{L_2} & \frac{a_4 - R_3}{L_2} & 0 \end{bmatrix} =$$

$$= \text{rank} \begin{bmatrix} 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} = 2. \quad (32)$$

The equation (30) has the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_1 + 1}{L_1} & 0 & 0 & \frac{a_3 - R_3}{L_1} \\ 0 & \frac{a_2 + 1}{L_2} & \frac{a_4 - R_3}{L_2} & 0 \end{bmatrix} \quad (33)$$

and its solution is

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} a_1 + 1 & 0 & 0 & a_3 - R_3 \\ 0 & a_2 + 1 & a_4 - R_3 & 0 \end{bmatrix} \quad (34)$$

The matrix (34) has nonnegative entries if  $a_k \geq 0$  for  $k = 1, 2$  and  $a_k \geq R_2$  for  $k = 3, 4$ .

On the following two examples of fractional linear circuits we shall show that it is not always possible to choose the gain matrix  $K$  so that the two conditions are satisfied:

1. the closed-loop system matrix  $A_c \in M_n$ ,
2. the closed-loop system is asymptotically stable.

**Example 3.** Consider the fractional linear circuit shown in Fig. 2 with given resistance  $R$ , capacitance  $C$ , inductance  $L$  and source of voltage  $e$ .

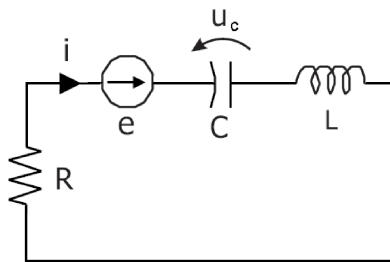


Fig. 2. Simple electrical circuit

Using (21), (22) and the second Kirchhoff's law we obtain for the circuit the state equation

$$\begin{bmatrix} \frac{d^\alpha u_C}{dt^\alpha} \\ \frac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_C \\ i \end{bmatrix} + Be, \quad 0 < \alpha < 1; \quad 0 < \beta < 1, \quad (35)$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \quad (36)$$

From (36) it follows that  $A$  is not a Metzler matrix but  $B \in R_+^2$ . It is easy to see that the condition (29) is satisfied for

$$A_c = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{a}{L} & \frac{b - R}{L} \end{bmatrix} \quad (37)$$

and from (30) we obtain

$$\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 0 \\ \frac{a + 1}{L} & \frac{b}{L} \end{bmatrix} \quad (38)$$

and

$$K = [k_1 \quad k_2] = [a + 1 \quad b] \quad (39)$$

Note that the characteristic polynomial of the matrix (37)

$$\det \begin{bmatrix} I_{n_1} s^\alpha - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} s^\beta - A_{22} \end{bmatrix} = \begin{vmatrix} s^\alpha & -\frac{1}{C} \\ -\frac{a}{L} & s^\beta + \frac{R - b}{L} \end{vmatrix} = s^{\alpha + \beta} + \frac{R - b}{L} s^\alpha - \frac{a}{LC} \quad (40)$$

has one negative coefficient and the closed-loop circuit is unstable for  $a \geq 0$  and any  $b$ .

**Example 4.** Consider the fractional linear system shown in Fig. 3 with given resistances  $R_1, R_2$ , capacitance  $C$ , inductance  $L$  and source of voltage  $e$ .

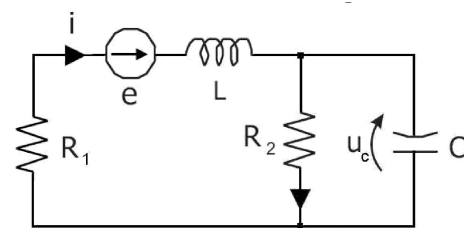


Fig. 3. Electrical circuit

Using the relations (21), (22) and the second Kirchhoff's law we obtain for the circuit the state equation

$$\begin{bmatrix} \frac{d^\alpha u_C}{dt^\alpha} \\ \frac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_C \\ i \end{bmatrix} + Be, \quad (41)$$

where

$$A = \begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \quad (42)$$

The matrix  $A$  is not a Metzler matrix but  $B \in R_+^2$ . It is easy to check that the condition (29) is satisfied for

$$A_c = \begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{C} \\ \frac{a}{L} & \frac{b - R_1}{L} \end{bmatrix}, \quad a, b \geq 0 \quad (43)$$

and from (30) we obtain

$$\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 0 \\ \frac{a+1}{L} & \frac{b}{L} \end{bmatrix} \quad (44)$$

and

$$K = [k_1 \quad k_2] = [a+1 \quad b]. \quad (45)$$

In this case the characteristic polynomial of the matrix (43) has the form

$$p(s) = \begin{vmatrix} s^\alpha + \frac{1}{R_2 C} & -\frac{1}{C} \\ -\frac{a}{L} & s^\beta + \frac{R_1 - b}{L} \end{vmatrix} = \quad (46)$$

$$= s^{\alpha+\beta} + \frac{R_1 - b}{L} s^\alpha + \frac{1}{R_2 C} s^\beta + \frac{R_1 - aR_2 - b}{R_2 CL}$$

and it is possible to choose the values of parameters  $a, b$  so that the closed-loop system is asymptotically stable [17].

The following question arises. What are the necessary and sufficient conditions under which there exists a gain matrix  $K$  such that the closed-loop system is positive and asymptotically stable? This problem will be the topic of the subsequent paper.

## 5. Concluding remarks

A new class of positive linear systems with different fractional orders has been introduced. Solution to the set of linear differential equations with different fractional orders has been derived using Laplace transform method (Theorem 1). It has been shown that the fractional linear systems are positive if and only if the system matrix  $A$  is a Metzler matrix and the matrix  $B$  has nonnegative entries (Theorem 2). It has been also shown that linear electrical circuits are positive systems with different fractional orders. If the systems matrix  $A$  is not Metzler matrix but the matrix  $B$  has nonnegative entries then there exists a gain matrix of the state-feedbacks such that the closed-loop system matrix is a Metzler matrix if and only if the condition (29) is satisfied (Theorem 3). The consideration

have been illustrated by examples of linear electrical circuits. An open problem has been formulated.

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