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Generalization of Vieta's formulae to the fractional polynomials, and generalizations the method of Graeffe-Lobachevsky

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Abstract. Two problems concerning polynomials are considered. For the first problem it is proved that the zeroes of the fractional polynomials of rational order fulfil relations similar to the Vieta's formulae for the polynomials.

In the second problem it is presented the iterative method of generalization of the Graeffe-Lobachevsky method to solution of the algebraic equations.

Key words: complex polynomials, fractional-order polynomials, fractional-order linear systems, roots of the fractional-order polynomials, formulae similar to Vieta's.

1. Introduction

We use the traditional notations

$N = \{1, 2, \ldots\}$ – the set of the natural numbers

R – set of the real numbers

C – set of the complex numbers

$P_n = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 :$
$a_i \in C, i = 0, 1, \ldots, n, a_n \neq 0,\ n \in N\}$ – set of the polynomials of n-th degree

$\deg(f)$ – degree of polynomial $f(x) \in P_n$.

For the function

$$\omega : C \longrightarrow C \quad \text{the set} \quad R(\omega) = \{z \in C :\ \omega(z) = 0\}$$

is called the set of the zeroes of the function $\omega(z)$.

The dynamics of the technical, or the physical process may be described by the differential equation with the fractional order derivatives.

The fractional derivatives have the long history – about 300 years. In the last years many interesting results were obtained concerning stability of the differential equations with the fractional derivatives. This type of equations have the application in technology, bionics, economics especially in the thermodynamics, automatics and many others. The properties and applications of such equations are presented in many publications: [1–4]. In the last decade many publications of this type of equations in automatics were published: [5–10].

The fractional polynomials are in close relation with these equations. The properties and algorithms of calculation for such polynomials are used in the research of systems dynamics with the fractional derivatives.

The function

$$f_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \\ = a_n(x - x_1)(x - x_2)\ldots(x - x_n), \tag{1}$$

where $a_i \in C$ $(i = 0, 1, 2, \ldots, n)$, $n \in N$ is called the polynomial or classic polynomial.

The function $g : C \longrightarrow C$ of the form

$$g(z) = b_k z^{\frac{\alpha_k}{\beta_k}} + b_{k-1} z^{\frac{\alpha_{k-1}}{\beta_{k-1}}} + \ldots + b_1 z^{\frac{\alpha_1}{\beta_1}} + b_0, \tag{2}$$

where $b_i \in C$, $\alpha_i \in N \cup \{0\}$, $\beta_i \in N$, α_i, β_i are relatively prime for $i = 1, 2, \ldots, k$ are called the fractional-order polynomial.

We assume in (2) that $\frac{\alpha_k}{\beta_k} > \frac{\alpha_{k-1}}{\beta_{k-1}} > \ldots > \frac{\alpha_1}{\beta_1}$ and $\beta_i = 1$ if $\alpha_i = 0$. For the sake of simplicity and without the loss of generality in the whole work we assume that $a_n = 1$ and $b_k = 1$. The numbers $b_k, b_{k-1}, \ldots, b_1, b_0$ are called the coefficients of the fractional polynomial (2). For the classic polynomials there are the well known Vieta's formulae. In this work the generalization of these formulae for the fractional polynomials is presented.

In the monographs [15–17] the method of Graeffe-Lobachevsky is presented for solving the equation

$$f_n(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \\ = (x - x_1)(x - x_2)\ldots(x - x_n) = 0,$$

where $a_i \in C$ $(i = 0, 1, \ldots, n-1)$, $n \in N$, $x_1, x_2, \ldots, x_n$ are the unknown roots.

The main idea of this method is based on the solution of the equation

$$\left(z - x_1^m\right)\left(z - x_2^m\right)\ldots\left(z - x_n^m\right) = \\ = z^n + A_{n-1}z^{n-1} + \ldots + A_1 z + A_0 = 0, \tag{3}$$

where $m = 2^k$, $(k = 1, 2, \ldots)$, and for the coefficients A_i $(i = 0, 1, \ldots, n-1)$ are calculated, going from the known a_i $(i = 0, 1, \ldots, n-1)$. In the simple case, when $|x_1| > \ldots > |x_n|$ from the Eq. (3) we obtain that

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$$x_i^m \approx -\frac{A_{n-i}}{A_{n+1-i}} \qquad (i=1,2,\ldots,n). \tag{4}$$

In the method of Graeffe-Lobachevsky for the Eq. (3) the number
$m = 2^k \in \{2,4,8,16,\ldots\}$. These values of the number m are determined by the way of calculation of the coefficients: $A_n, A_{n-1}, \ldots, A_0$ – the quadrature of the roots. This is the iterations method. If for example for $m = 32$ the iteration (4) does not fulfil the required accuracy, then in the next step of iteration put $m \geq 64$, and it is not possible to take a smaller m, for example $m = 40$.

In this work we present the generalization of the Graeffe-Lobachevsky method. The recurrence formulae will be given for the calculation of the coefficients of the polynomial in the Eq. (3) for the arbitrary $m \in \{1,2,3,\ldots\}$, without the assumption that $m = 2^k$.

2. Elementary symmetric polynomials and Newton's formulae

For $t = (t_1, t_2, \ldots, t_n) \in C^n$ the functions:

$$\begin{aligned}\varphi_0(t_1,t_2,\ldots,t_n) &= 1,\\ \varphi_i(t_1,t_2,\ldots,t_n) &= \sum_{1\le j_1<\ldots<j_i\le n} t_{j_1}t_{j_2}\ldots t_{j_i}\\ &\qquad (i=1,2,\ldots,n)\end{aligned}$$

are the elementary symmetric polynomials.
For polynomial (1), $a_n = 1$, Vieta's formulae have the form:

$$\varphi_{n-i}(x_1,x_2,\ldots,x_n) = (-1)^{n-i}a_i \quad (i=1,2,\ldots,n). \tag{5}$$

The numbers

$$s_j = x_1^j + x_2^j + \ldots + x_n^j \quad (j=1,2,\ldots)$$

are called the Newton's sums for the polynomial $f_n(x)$, where $x_1, x_2, \ldots, x_n$ are the zeroes of the polynomial (1).

It is proved [18] that the sums s_j $(j=1,2,\ldots)$ fulfil the following relations:

$$\begin{aligned} s_0 &= n\\ s_1 + a_{n-1} &= 0\\ s_2 + a_{n-1}s_1 + 2a_{n-2} &= 0\\ s_3 + a_{n-1}s_2 + a_{n-2}s_1 + 3a_{n-3} &= 0\\ \ldots\ldots\ldots\ldots\ldots\ldots&\ldots\\ s_n + a_{n-1}s_{n-1} + \ldots + a_1s_1 + na_0 &= 0\\ s_{n+p} + a_{n-1}s_{n+p-1} + \ldots + a_0s_p &= 0 \quad (p=1,2,\ldots). \end{aligned} \tag{6}$$

When $a_0, a_1, \ldots, a_{n-1}$ are known we can consider the relations (6) as the system of the equations with unknowns $s_1, s_2, \ldots, s_{n+p}$ or for the known $s_1, s_2, \ldots, s_n$ as the system of equations with unknowns $a_0, a_1, \ldots, a_{n-1}$. In both cases the system (6) has exactly one solution.

In the whole work we assume that if $k > l$ then $\sum_{i=k}^{l} q_i = 0$.

From the relations (6) we obtain the following recurrence relation with unknowns $s_1, s_2, s_3, \ldots$:

$$s_j = \begin{cases} -\left[\sum_{i=1}^{j-1} a_{n-i}s_{j-i} + ja_{n-j}\right] & (j=1,2,\ldots,n)\\ -\left[\sum_{i=1}^{n} a_{n-i}s_{j-i}\right] & (j=n+1,n+2,\ldots)\ . \end{cases} \tag{7}$$

3. Generalization of the Graeffe-Lobachevsky method

Let

$$\begin{aligned} f_n(x) &= x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 =\\ &= (x-x_1)(x-x_2)\ldots(x-x_n)\end{aligned}$$

be the complex polynomial, where $n \in N$. For the given number $m \in N$ we present the method of determining the coefficients of polynomial

$$\begin{aligned} g_n(x) &= \left(x-x_1^m\right)\left(x-x_2^m\right)\ldots\left(x-x_n^m\right) =\\ &= x^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0.\end{aligned}$$

We denote

$$S_i = s_{m*i} = x_1^{m*i} + x_2^{m*i} + \ldots + x_n^{m*i}, \quad (i=1,2,\ldots,n),$$

where $*$ denotes multiplying.
We prove

Theorem 1. If

$$\begin{aligned} f_n(x) &= x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 =\\ &= (x-x_1)(x-x_2)\ldots(x-x_n)\end{aligned}$$

is the complex polynomial, $m \in N$, then the coefficients of polynomial

$$\begin{aligned} g_n(x) &= (x-x_1^m)(x-x_2^m)\ldots(x-x_n^m) =\\ &= x^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0\end{aligned}$$

are determined by the following recurrence relations

$$A_{n-k} = -\frac{1}{k}\left[\sum_{i=1}^{k-1} A_{n-i}S_{k-i} + S_k\right] \tag{8}$$
$$(k=1,2,\ldots,n),$$

where $S_i = s_{m*i}$ $(i=1,2,\ldots,n)$ can be obtained from the recurrence formulae (7).

Proof. The relations (6) can be applied to the polynomial $g_n(x)$ and we obtain:

$$\begin{aligned} S_0 &= n,\\ S_1 + A_{n-1} &= 0,\\ S_2 + A_{n-1}S_1 + 2A_{n-2} &= 0,\\ \ldots\ldots\ldots\ldots\ldots&\ldots,\\ S_n + A_{n-1}S_{n-1} + \ldots + A_1S_1 + nA_0 &= 0.\end{aligned}$$

From these relations the recurrence formulae (8) result.

From Theorem 1 we have an algorythm of determining the coefficients of the polynomial $g_n(x)$ for arbitrary $m \in N$.

In the Graeffe-Lobachevsky method, the parameter m could have only the value $m = 2^k$ $(k=1,2,\ldots)$.

Example 1. For the complex polynomial

$$f_3(x) = x^3 + a_2x^2 + a_1x + a_0 = \\ = (x-x_1)(x-x_2)(x-x_3)$$

and $m = 2$ from the recurrence relations (7), (8) we obtain:

$$\begin{aligned}
& s_1 = -a_2 \\
S_1 = \quad & s_2 = a_2^2 - 2a_1 \\
& s_3 = -a_2^3 + 3a_1a_2 - 3a_0 \\
S_2 = \quad & s_4 = a_2^4 - 4a_1a_2^2 + 4a_0a_2 + 2a_1^2 \\
& s_5 = -a_2^5 + 5a_1a_2^3 - 5a_0a_2^2 - 5a_1^2a_2 + 5a_0a_1 \\
S_3 = \quad & s_6 = a_2^6 - 6a_1a_2^4 + 6a_0a_2^3 + 9a_1^2a_2^2 - 12a_0a_1a_2 \\
& \qquad -2a_1^3 + 3a_0^2 \\
& A_2 = -a_2^2 + 2a_1 \\
& A_1 = a_1^2 - 2a_0a_2 \\
& A_0 = -a_0^2.
\end{aligned}$$

Therefore, the polynomial

$$g_3(x) = \left(x-x_1^2\right)\left(x-x_2^2\right)\left(x-x_3^2\right) = \\ = x^3 + A_2x^2 + A_1x + A_0 = \\ = x^3 + \left[-a_2^2 + 2a_1\right]x^2 + \left[a_1^2 - 2a_0a_2\right]x + \left[-a_0^2\right].$$

For polynomial $f_3(x)$ and $m = 3$ we obtain:

$$\begin{aligned}
S_1 = \quad & s_3 = -a_2^3 + 3a_1a_2 - 3a_0 \\
S_2 = \quad & s_6 = a_2^6 - 6a_1a_2^4 + 6a_0a_2^3 + 9a_1^2a_2^2 - 12a_0a_1a_2 - \\
& -2a_1^3 + 3a_0^2 \\
S_3 = \quad & s_9 = -a_2^9 + 9a_1a_2^7 - 9a_0a_2^6 - 27a_1^2a_2^5 + \\
& + 45a_0a_1a_2^4 + 30a_1^3a_2^3 - 18a_0^2a_2^3 - 54a_0a_1^2a_2^2 + \\
& + 27a_0^2a_1a_2 - 9a_1^4a_2 + 9a_0a_1^3 - 3a_0^3 \\
& A_2 = a_2^3 - 3a_1a_2 + 3a_0 \\
& A_1 = a_1^3 + 3a_0^2 - 3a_0a_1a_2 \\
& A_0 = a_0^3
\end{aligned}$$

and

$$g_3(x) = \left(x-x_1^3\right)\left(x-x_2^3\right)\left(x-x_3^3\right) = \\ = x^3 + A_2x^2 + A_1x + A_0 = \\ = x^3 + \left[a_2^3 - 3a_1a_2 + 3a_0\right]x^2 + \\ + \left[a_1^3 + 3a_0^2 - 3a_0a_1a_2\right]x + a_0^3.$$

4. Vieta's formulae for the fractional polynomials

We consider a fractional polynomial

$$g(z) = z^{\frac{\alpha_k}{\beta_k}} + b_{k-1}z^{\frac{\alpha_{k-1}}{\beta_{k-1}}} + \ldots + b_1z^{\frac{\alpha_1}{\beta_1}} + b_0$$

where $b_i \in C$, $\alpha_i \in N \cup \{0\}$, $\beta_i \in N$, $\frac{\alpha_i}{\beta_i}$ are relatively prime for $i = 1, 2, \ldots, k$. We assume that $\frac{\alpha_i}{\beta_i} > \frac{\alpha_{i-1}}{\beta_{i-1}}$ $(i = 2, 3, \ldots, k)$, $\beta_i = 1$ if $\alpha_i = 0$.

Let $m \in N$ be the least common denominator of the fractions: $\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \ldots, \frac{\alpha_k}{\beta_k}$. In this case we can write $g(z)$ in the form:

$$g(z) = z^{\frac{l_k}{m}} + b_{k-1}z^{\frac{l_{k-1}}{m}} + \ldots + b_1z^{\frac{l_1}{m}} + b_0,$$

where $l_i \in N \cup \{0\}$, $(i = 1, 2, \ldots, k)$, $l_k > l_{k-1} > \ldots > l_1$ and we assume that $l_1 \geq 1$.

It is evident that for $m > 1$ $g(z)$ is a multivalued function of variable $z \in C$. If we denote $n = l_k$, $c_0 = b_0$ then the polynomial $g(z)$ have the following form:

$$g(z) = z^{\frac{n}{m}} + c_{n-1}z^{\frac{n-1}{m}} + \ldots + c_1z^{\frac{1}{m}} + c_0, \tag{9}$$

and

$$c_i = \begin{cases} 0 & \text{if} \quad i \notin \{l_k, l_{k-1}, \ldots, l_1\} \\ b_j & \text{if} \quad i \in \{l_k, l_{k-1}, \ldots, l_1\} \quad \text{and} \quad i = l_j \,. \end{cases}$$

Polynomial

$$h(y) = y^n + c_{n-1}y^{n-1} + \ldots + c_1y + c_0 \tag{10}$$

is associated with the fractional polynomial $g(z)$.

The number $n = l_k$ is the degree of the fractional polynomial $g(z)$ and we write $n = f\deg(g)$.

Example 2. For the fractional polynomial

$$g(z) = z^{\frac{3}{2}} + 4z + 5 = z^{\frac{3}{2}} + 4z^{\frac{2}{2}} + 5$$

we have $m = 2$, $fdeg(g) = 3$, $h(y) = y^3 + 4y^2 + 5$.

Let $y_1, y_2, \ldots, y_n$ are the zeroes of the polynomial (10), then

$$h(y) = y^n + c_{n-1}y^{n-1} + \ldots + c_1y + c_0 = \\ = (y-y_1)(y-y_2)\ldots(y-y_n),$$

$$R(h) = \{y_1, y_2, \ldots, y_n\}.$$

From these denotation it results

Conclusion 1. If

$$h(y) = y^n + c_{n-1}y^{n-1} + \ldots + c_1y + c_0 = \\ = (y-y_1)(y-y_2)\ldots(y-y_n),$$

then the numers

$$z_i = y_i^m \quad (i = 1, 2, \ldots, n)$$

are the zeroes of the fractional polynomial

$$g(z) = z^{\frac{n}{m}} + c_{n-1}z^{\frac{n-1}{m}} + \ldots + c_1z^{\frac{1}{m}} + c_0,$$

or

$$R(g) = \{y_1^m, y_2^m, \ldots, y_n^m\} = \{z_1, z_2, \ldots, z_n\}$$

is the set of zeroes of the function $g(z)$, where $f\deg(g) = n$.

If y_i is r-multiple zero of polynomial (10) then $z_i = y_i^m$ is r-multiple zero of the fractional polynomials $g(z)$.

Conclusion 2. If $f\deg(g) = n$, then the equation $g(z) = 0$ has exactly n zeroes and r-multiple zero we take r-times.

If

$$f_n(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = \\ = (x-x_1)(x-x_2)\ldots(x-x_n) \in P_n,$$

then the set of the roots $\{x_1, x_2, \ldots, x_n\}$ explicite determine the coefficients: $a_0, a_1, \ldots, a_{n-1}$.

This property is not true for the fractional polynomials, as illustrated by the following

Example 3. For the fractional polynomials

$$g_1(z) = z^{\frac{1}{2}} + 1; \quad g_2(z) = z^{\frac{1}{2}} - 1$$

we have

$$R(g_1) = R(g_2) = \{1\}, \quad \text{and} \quad g_1(z) \not\equiv g_2(z).$$

In the next of the work we say that the fractional polynomial (9) is given, if $n, m, c_{n-1}, \ldots, c_0$ are given.
From Vieta's formulae it is evident that for the calssic polynomial

$$f_n(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \\ = (x - x_1)(x - x_2)\ldots(x - x_n)$$

the values of the elementary symmetric functions

$$\varphi_1(x_1, x_2, \ldots, x_n),$$
$$\varphi_2(x_1, x_2, \ldots, x_n),$$
$$\ldots$$
$$\varphi_n(x_1, x_2, \ldots, x_n)$$

we can calculate from the simple algebraic relations which depend on: $n, a_{n-1}, a_{n-2}, \ldots, a_0$. We prove that if $z_1, z_2, \ldots, z_n$ are zeroes of the fractional polynomial

$$g(z) = z^{\frac{n}{m}} + c_{n-1}z^{\frac{n-1}{m}} + \ldots + c_1 z^{\frac{1}{m}} + c_0,$$

then the values of the functions

$$\varphi_1(z_1, z_2, \ldots, z_n),$$
$$\varphi_2(z_1, z_2, \ldots, z_n),$$
$$\ldots$$
$$\varphi_n(z_1, z_2, \ldots, z_n),$$

we can calculate from the algebraic relations which depend on: $n, m, c_{n-1}, c_{n-2}, c_0$.
From the Conclusion 1 it is evident that:
if

$$g(z) = z^{\frac{n}{m}} + c_{n-1}z^{\frac{n-1}{m}} + \ldots + c_1 z^{\frac{1}{m}} + c_0,$$

$$h(y) = y^n + c_{n-1}y^{n-1} + \ldots + c_1 y + c_0 = \\ = (y - y_1)(y - y_2)\ldots(y - y_n),$$

then $z_i = y_i^m$, $(i = 1, 2, \ldots, n)$ are the zeroes of the fractional polynomial $g(z)$.
Polynomial

$$F(z) = (z - z_1)(z - z_2)\ldots(z - z_n) = \\ = \left(z - y_1^m\right)\left(z - y_2^m\right)\ldots\left(z - y_n^m\right) = \\ = z^n + \widetilde{A}_{n-1}z^{n-1} + \ldots + \widetilde{A}_1 z + \widetilde{A}_0, \tag{11}$$

is associated with the fractional polynomial $g(z)$.

We prove now the theorem, which represents the method of calculation of the values of elementary symmetric functions for the fractional polynomials.

Theorem 2 [generalization of Vieta's formulae]. If

$$g(z) = z^{\frac{n}{m}} + c_{n-1}z^{\frac{n-1}{m}} + \ldots + c_1 z^{\frac{1}{m}} + c_0,$$

is the fractional polynomial of degree $n \geq 1$ the zeroes of which are the numbers: $z_1, z_2, \ldots, z_n$, then for the elementary symmetric functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ the following relations are true:

$$\varphi_{n-i}(z_1, z_2, \ldots, z_n) = (-1)^{n-i}\widetilde{A}_i, \\ (i = 0, 1, \ldots, n), \tag{12}$$

where $\widetilde{A}_n = 1$, $\widetilde{A}_{n-1}, \ldots, \widetilde{A}_0$ are the coefficients of the polynomial (11).

The coefficients $\widetilde{A}_k$, $(k = 0, 1, \ldots, n-1)$ can be calculated from the recurrence relations

$$\widetilde{A}_{n-k} = -\frac{1}{k}\left[\sum_{i=1}^{k-1}\widetilde{A}_{n-i}\widetilde{S}_{k-i} + \widetilde{S}_k\right], \\ (k = 1, 2, \ldots, n), \tag{13}$$

where $\widetilde{S}_k = \widetilde{s}_{m*k}$, and $\widetilde{s}_j = y_1^j + y_2^j + \ldots + y_n^j$, $(j = 1, 2, \ldots, m*n)$ and

$$\widetilde{s}_j = \begin{cases} -\left[\sum_{i=1}^{j-1} c_{n-i}\widetilde{s}_{j-i} + jc_{n-j}\right], & (j = 1, 2, \ldots, n) \\ -\left[\sum_{i=1}^{n} c_{n-i}\widetilde{s}_{j-i}\right], & (j = n+1, n+2, \ldots) \end{cases}. \tag{14}$$

Proof. The formulae (12) result from Vieta's relations for the polynomial (11). The relations (13), (14) can be obtained from Theorem 1 applied to the polynomial $h(y)$ and polynomial (11). In the particular case, when $m = 1$ the fractional polynomial $g(z)$ is identical with the classic polynomial, $F(z) \equiv g(z)$ and the relations (12) are Vieta's formulae (5).

Example 4. Let us consider the fractional polynomial

$$g(z) = z^{\frac{3}{2}} + 4z + 5 = z^{\frac{3}{2}} + 4z^{\frac{2}{2}} + 5 = \\ = z^{\frac{3}{2}} + c_2 z^{\frac{2}{2}} + c_1 z^{\frac{1}{2}} + c_0.$$

In this example we have:

$$m = 2, \quad fdeg(g) = 3, \quad c_2 = 4, \quad c_1 = 0, \quad c_0 = 5.$$

The associated polynomial with $g(z)$ has the following form:

$$h(y) = y^3 + 4y^2 + 5 = (y - y_1)(y - y_2)(y - y_3)$$

where y_1, y_2, y_3 are not known.
From Conclusion 2 we have that

$$z_1 = y_1^2, \quad z_2 = y_2^2, \quad z_3 = y_2^2$$

are the zeroes of the fractional polynomial $g(z)$. From the relations (13) and (14) applied to the polynomial $g(y)$ we obtain:

$$\widetilde{s}_1 = -4, \quad \widetilde{s}_2 = 16, \quad \widetilde{s}_3 = -79, \quad \widetilde{s}_4 = 336, \\ \widetilde{s}_5 = -1424, \quad \widetilde{s}_6 = 6091,$$

$$\widetilde{S}_1 = \widetilde{s}_2 = 16, \quad \widetilde{S}_2 = \widetilde{s}_4 = 336, \quad \widetilde{S}_3 = \widetilde{s}_6 = 6091 \ ,$$

$$\widetilde{A}_2 = -16, \quad \widetilde{A}_1 = -40, \quad \widetilde{A}_0 = -25 \ .$$

The polynomial (11) for this fractional polynomial $g(z)$ has the form:

$$F(z) = z^3 - 16z^2 - 40z - 25 = (z - z_1)(z - z_2)(z - z_3).$$

Taking this and Theorem 2 into account we obtain:

$$\varphi_1(z_2, z_2, z_2) = z_1 + z_2 + z_3 = -\widetilde{A}_2 = 16,$$

$$\varphi_2(z_2, z_2, z_2) = z_1 z_2 + z_1 z_3 + z_2 z_3 = \widetilde{A}_1 = -40,$$

$$\varphi_3(z_2, z_2, z_2) = z_1 z_2 z_3 = -\widetilde{A}_0 = 25,$$

where z_1, z_2, z_3 are zeroes of the fractional polynomial $g(z)$.

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