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Relations between roots and coefficients of the transcendental equations

H. GÓRECKI* and S. BIAŁAS

[1] Faculty of Informatics, Higher School of Informatics, 17a Rzgowska St., 93-08 Łódź, Poland
[2] The School of Banking and Management, 4 Armii Krajowej St., 30-150 Kraków, Poland

Abstract. It is proved that there exist the relations between coefficients of the transcendental equations and the infinite number of their roots, similar to Vieta's formulae.

These relations may be obtained for the entire analytic functions using theorems of residues and argument principle. In particular the meromorphic functions will be considered.

Key words: transcendental equations, entire-meromorphic functions, roots, coefficients, residues, argument principle, quasipolynomials, principal term.

1. Introduction

The transcendental equations appear very frequently in many applications [1]. Stability problems, parameters optimization and optimal control are more difficult than in the system dynamics without time-delays [2].

2. Statement of the problem

It is well known that the number of roots of the algebraic equations is finite. According to this, Vieta's formulae have the finite number of the sum-products of their roots. In the transcendental equations the number of roots is infinite and the roots are going to infinity. This is the reason that only the sums of their inverse-roots may be finite for some type of the quasi-polynomials. Let us consider quasipolynomial

$$F(s) = A(s) + B(s)e^{-\tau s} \quad , \tag{1}$$

where

$$A(s) = \sum_{k=0}^{m} a_{m-k}s^k \quad , \tag{2}$$

$$B(s) = \sum_{k=0}^{n} b_{n-k}s^k \quad , \tag{3}$$

$$m < n, \qquad a_k, b_k, \tau \in R, \qquad \tau > 0 \ .$$

We assume that for assuring the asymptotic stability

$$F(s) = 0 \longrightarrow \mathrm{Re}\, s < 0 \ . \tag{4}$$

We use a notion of the principal term, closely connected with the stability problem. In order to find the principal term, the formula (1) is premultiplied by $e^{s\tau}$.

Definition 1. The principal term of quasipolynomial (1) after premultiplying it by $e^{s\tau}$ is the term $b_k s^k e^{s\tau}$ in which the argument of the power s and τ have the highest value for some $k = 0, 1, \ldots, n$.

Remark 1. There are some quasipolynomials which do not have a principal term, for example $F(s) = s^4e^{3s} + s^5e^{2s} + 1$.

Remark 2. In the case of multiple commensurable delays τ_k the quasipolynomial takes the form:

$$F(s, e^{s\tau}) = \sum_{i,j} b_{ij}s^i e^{-\tau_j s} = 0, \qquad i = 0, 1, 2, \ldots, n, \qquad j = 1, 2, \ldots, k. \tag{5}$$

Multiplying (5) by $e^{\tau_m s}$ with

$$\tau_m = \max_{i \le j \le k} \tau_j$$

we obtain an equivalent equation

$$\sum_{i,j} b_{ij}s^i e^{\lambda_j s} = 0,$$

$$i = 0, 1, 2, \ldots, n, \qquad j = 1, 2, \ldots, k.$$

Theorem 1 [3]. A quasipolynomial with no principal term has infinitely many roots with arbitrary large, positive real parts.

From this it results that the presence of principal term in a quasi-polynomial is a necessary condition for its stability (4).

3. Solution of the problem

In order to calculate the reciprocal roots sum the following theorem [4] can be used:

Theorem 2 [4]. Let $F(s)$ be a meromorphic function in a given closed area C and $\varphi(s)$ a meromorphic function in the same area provided that the poles of $\varphi(s)$ and the zeroes of the function $F(s)$ do not overlap. Then the integral $\oint_C$ on a closed curve C surrounding this area is equal to:

*e-mail: head@nova.ia.agh.edu.pl

$$\begin{aligned}\frac{1}{2\pi i}\oint\limits_{C}\frac{F^{(1)}(s)}{F(s)}\,\varphi(s)\,ds &= \\ = \sum\varphi(s_k)-\sum\varphi(p_j)+\sum\operatorname{Res}\frac{F^{(1)}(q_i)}{F(q_i)}\,\varphi(q_i).\end{aligned} \tag{6}$$

where s_k – zeroes of the function $F(s)$ inside this area, p_j – poles of the function $F(s)$ inside this area, q_i – poles of the function $\varphi(s)$ inside this area.

For our purposes we consider the function (1)

$$F(s) = A(s) + B(s)e^{-\tau s}$$

which does not have poles, but instead it has an infinite number of zeroes s_k. It is a holomorphic function, which is the specific case of the meromorphic function. Its derivative

$$F^{(1)}(s) = A^{(1)}(s) + \left(B^{(1)}(s) - \tau B(s)\right)e^{-\tau s}. \tag{7}$$

In our problem we shall consider the function

$$\varphi(s) = \frac{1}{s}, \tag{8}$$

since we want to find $\sum\limits_{k=1}^{\infty}\frac{1}{s_k}$, where s_k are the zeroes of the function $F(s)$. The function $\varphi(s)$ has one pole $q_i = 0$, which does not overlap with any zero of the function $F(s)$. With the exception of this point it is a holomorphic function. Taking into consideration the Eqs. (6)–(8) and (2), (3) in the formula (6) we have to calculate the integrals:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{s_k} &= \frac{1}{2\pi i}\oint\limits_{C_R\to\infty}\frac{F^{(1)}(s)}{F(s)}\frac{ds}{s} - \frac{1}{2\pi i}\oint\limits_{C_r\to 0}\frac{F^{(1)}(s)}{F(s)}\frac{ds}{s} = \\ &= \frac{1}{2\pi i}\int\limits_{C_R\to\infty}\frac{\sum\limits_{k=1}^{m}ka_{m-k}s^{k-1}+M}{\sum\limits_{k=0}^{m}a_{m-k}s^k+\sum\limits_{k=0}^{n}b_{n-k}s^ke^{-s\tau}}\frac{ds}{s} - \\ &- \frac{1}{2\pi i}\oint\limits_{C_r\to 0}\frac{\sum\limits_{k=1}^{m}ka_{m-k}s^{k-1}+M}{\sum\limits_{k=0}^{m}a_{m-k}s^k+\sum\limits_{k=0}^{n}b_{n-k}s^ke^{-s\tau}}\frac{ds}{s},\end{aligned} \tag{9}$$

where $M = \left(\sum\limits_{k=1}^{n}kb_{n-k}s^{k-1} - \tau\sum\limits_{k=0}^{n}b_{n-k}s^k\right)e^{-s\tau}$.

Because

$$\int\limits_{C_R\to\infty}\frac{F'(s)}{F(s)}\frac{ds}{s} = 0 \qquad \text{for} \quad \operatorname{Re}s > 0\ .$$

The integration area has been presented in the Fig. 1[1].

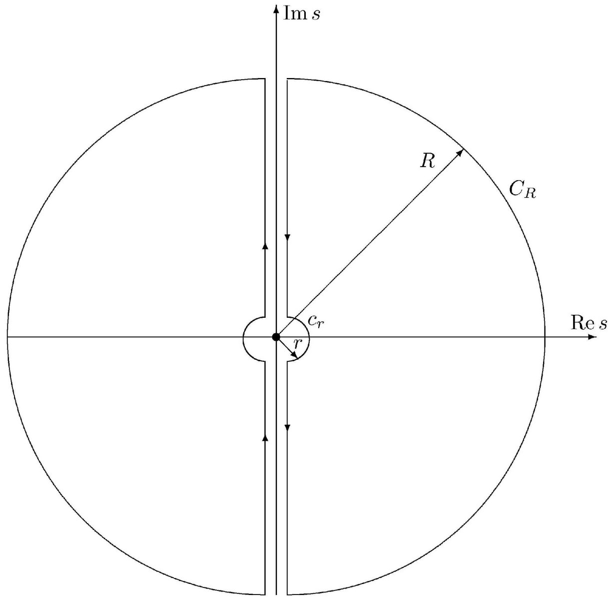

Fig. 2. Integration path C or contour of integration

The first integral in (9) depends only on the rising of the function $b_0s^ne^{-s\tau}$ for $R\to\infty$. So from (9) we have that

$$\frac{1}{2\pi i}\int\limits_{C_R\to\infty}\frac{\sum\limits_{k=1}^{m}ka_{m-k}s^{k-1}+M}{\sum\limits_{k=0}^{m}a_{m-k}s^k+\sum\limits_{k=0}^{n}b_{n-k}s^ke^{-s\tau}}\frac{ds}{s} = -\frac{\tau}{2}, \tag{10}$$

where $M = \left(\sum\limits_{k=1}^{n}kb_{n-k}s^{k-1} - \tau\sum\limits_{k=0}^{n}b_{n-k}s^k\right)e^{-s\tau}$, for $\operatorname{Re}s < 0$. The second integral in (9) is equal:

$$\begin{aligned}&\frac{1}{2\pi i}\oint\limits_{C_r\to 0}\frac{\sum\limits_{k=1}^{m}ka_{m-k}s^{k-1}+M}{\sum\limits_{k=0}^{m}a_{m-k}s^k+\sum\limits_{k=0}^{n}b_{n-k}s^ke^{-s\tau}}\frac{ds}{s} = \\ &= \operatorname*{Res}_{s=0}\frac{1}{s}\,\frac{\sum\limits_{k=1}^{m}ka_{m-k}s^{k-1}+M}{\sum\limits_{k=0}^{m}a_{m-k}s^k+\sum\limits_{k=0}^{n}b_{n-k}s^ke^{-s\tau}} = \\ &= \frac{a_{m-1}+b_{n-1}-\tau b_n}{a_m+b_n},\end{aligned} \tag{11}$$

where $M = \left(\sum\limits_{k=1}^{n}kb_{n-k}s^{k-1} - \tau\sum\limits_{k=0}^{n}b_{n-k}s^k\right)e^{-s\tau}$.

From (9), (10) and (11) we have finally that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{s_k} &= -\frac{\tau}{2} - \frac{a_{m-1}+b_{n-1}-\tau b_n}{a_m+b_n} = \\ &= -\frac{(a_m-b_n)\tau+2(a_{n-1}+b_{n-1})}{2(a_m+b_n)}.\end{aligned}$$

[1]By Piotr Grabowski's courtesy.

Theorem 3 (Basic result). The relation between coefficients and the roots of the quasipolynomial equations of the type (1), (2), (3) is given by the following formula:

$$\sum_{k=1}^{\infty} \frac{1}{s_k} = \frac{1}{2} \left[\frac{F^{(1)}(s)}{F(s)} \right]_{s=\infty} - \left[\frac{F^{(1)}(s)}{F(s)} \right]_{s=0}. \tag{12}$$

In case when it is necessary to calculate $\sum_{k=1}^{\infty} \frac{1}{s_k^p}$, $p = 1, 2, \ldots$ an analogous procedure leads to the formula:

$$\sum_{k=1}^{\infty} \frac{1}{s_k^p} = \frac{1}{2(p-1)!} \left[\frac{F^{(1)}(s)}{F(s)} \right]_{s=\infty}^{(p-1)} - $$
$$- \frac{1}{(p-1)!} \left[\frac{F^{(1)}(s)}{F(s)} \right]_{s=0}^{(p-1)}.$$

4. Conclusions

The method proposed may be used for much more general functional equations, for example partial differential equations or other, under the restriction that the transmittance of the system is an analytic entire function [4].

5. Simple example

Let

$$F(s) = a + bs + e^{-s\tau}, \qquad \tau > 0.$$

After premultiplying it by $e^{s\tau}$ we have that the principal term is $bse^{s\tau}$. The first derivative is

$$F^{(1)}(s) = b - \tau e^{-s\tau}.$$

We have that

$$\left. \frac{F^{(1)}(s)}{F(s)} \right|_{s=0} = \frac{b-\tau}{a+1}$$

and

$$\frac{1}{2} \left. \frac{F^{(1)}(s)}{F(s)} \right|_{s=\infty} = \frac{1}{2} \left. \frac{b - \tau e^{-s\tau}}{a + bs + e^{-s\tau}} \right|_{s=\infty} = -\frac{\tau}{2}\ .$$

Finally

$$\sum_{k=1}^{\infty} \frac{1}{s_k} = \frac{1}{2} \left. \frac{F^{(1)}(s)}{F(s)} \right|_{s=\infty} - \left. \frac{F^{(1)}(s)}{F(s)} \right|_{s=0} =$$
$$= -\frac{\tau}{2} - \frac{b-\tau}{a+1} = \frac{(1-a)\tau - 2b}{2(a+1)}\ .$$

6. Numerical example

Let us consider an numerical example for which we can apply another method which gives verification of the results.

The following neutral equation is given

$$x(t) = Kx(t-1).$$

The characteristic equation is

$$1 - Ke^{-s} = 0. \tag{13}$$

The roots of the equation (13) are

$$s_m = \ln K \pm j2\pi m, \qquad m = 0, 1, \ldots. \tag{14}$$

The sum for calculation is

$$\sum_{m=1}^{\infty} \frac{1}{s_m} = \left[\frac{1}{\ln K} + \sum_{m=1}^{\infty} \frac{2 \ln K}{\ln^2 K + 4\pi^2 m^2} \right]. \tag{15}$$

In [5] there is formulae 4, which will be useful for our calculation

$$\sum_{m=-\infty}^{\infty} \frac{1}{m^2 + x^2} = \frac{1}{x^2} + \sum_{m=1}^{\infty} \frac{2}{m^2 + x^2} = \frac{\pi}{x} \coth(\pi x). \tag{16}$$

After substitution into (16)

$$x = \ln \frac{K}{2\pi}.$$

We obtain that

$$\frac{2\pi^2}{\ln K} \coth \frac{\ln K}{2} = \frac{4\pi^2}{\ln^2 K} + \sum_{m=1}^{\infty} \frac{2}{m^2 + \frac{\ln^2 K}{4\pi^2}} =$$
$$= \frac{4\pi^2}{\ln^2 K} + \sum_{m=1}^{\infty} \frac{8\pi^2}{4m^2\pi^2 + \ln^2 K}.$$

After division through $\frac{4\pi^2}{\ln K}$ we obtain finally that

$$\frac{1}{2} \coth \frac{\ln K}{2} = \frac{1}{\ln K} + \sum_{m=1}^{\infty} \frac{2 \ln K}{\ln^2 K + 4\pi^2 m^2}. \tag{17}$$

But

$$\coth \frac{\ln K}{2} = \frac{\left(e^{\ln K}\right)^{1/2} + \left(e^{-\ln K}\right)^{1/2}}{\left(e^{\ln K}\right)^{1/2} - \left(e^{-\ln K}\right)^{1/2}} =$$
$$= \frac{K^{1/2} + K^{-1/2}}{K^{1/2} - K^{-1/2}} = \frac{K+1}{K-1}$$

and from (15) and (17) we have

$$\sum_{m=1}^{\infty} \frac{1}{s_m} = \frac{1}{2} \coth \frac{\ln K}{2} = \frac{1}{2} \frac{K+1}{K-1} = -\frac{1}{2} \frac{K+1}{1-K}, \tag{18}$$
$$\text{for} \qquad 0 < K < 1.$$

From (14) we see that the stability limited is for $K = 1$. From (18) we have that if

$$K = \frac{1}{2}, \qquad \sum_{m=1}^{\infty} \frac{1}{s_m} = \frac{1}{2} \frac{\frac{1}{2} + 1}{\frac{1}{2} - 1} = -\frac{3}{2}$$

$$K = \frac{1}{3}, \qquad \sum_{m=1}^{\infty} \frac{1}{s_m} = \frac{1}{2} \frac{\frac{1}{3} + 1}{\frac{1}{3} - 1} = -1$$

$$K = \frac{1}{4}, \qquad \sum_{m=1}^{\infty} \frac{1}{s_m} = \frac{1}{2} \frac{\frac{1}{4} + 1}{\frac{1}{4} - 1} = -\frac{5}{6}$$

which agree with our method because

$$\sum_{m=1}^{\infty} \frac{1}{s_m} = \frac{1}{2} \left[\frac{F^{(1)}(s)}{F(s)} \right]_{s \to -\infty} - \left[\frac{F^{(1)}(s)}{F(s)} \right]_{s=0}.$$

In our case $F(s) = 1 - Ke^{-s}$, $F^{(1)}(s) = Ke^{-s}$ and

$$\sum_{m=1}^{\infty} \frac{1}{s_m} = \frac{1}{2}\left[\frac{Ke^{-s}}{1-Ke^{-s}}\right]_{s\to-\infty} - \left[\frac{F^{(1)}(s)}{F(s)}\right]_{s=0} =$$

$$= -\frac{1}{2} - \frac{K}{1-K} = -\frac{1+K}{2(1-K)}.$$

7. Remarks

Remark 3. It is worth to note that for the stable systems $s < 0$ the term $\left[\frac{F^{(1)}(s)}{F(s)}\right]_{s\to\infty} = 0$ because e^{-s} for $s > 0$ and tending to infinity is equal zero.

The same expression $\left[\frac{F^{(1)}(s)}{F(s)}\right]_{s\to\infty}$ for $s < 0$ is going to the $-\frac{\tau}{2}$.

The formula (12) is not depended on the symmetrising efect as in [6].

Remark 4. Generalization for the equations with many delays: Let us consider quasipolynomial

$$F(s) = A_0(s) + A_1(s)e^{-\tau_1 s} + \ldots + A_n(s)e^{-\tau_n s},$$

where $A_i(s)$ are the polynomials of s with real coefficients, and

$$\deg A_0(s) > \deg A_1(s) > \ldots > \deg A_n(s),$$

$$0 < \tau_1 < \tau_2 < \ldots < \tau_n.$$

From the general formulae (12) we obtain

$$\sum_{k=1}^{\infty} \frac{1}{s_k} = -\frac{1}{2}\tau_n - \frac{A_0^{(1)}(s) + \sum_{i=1}^{n}\left[A_i^{(1)}(0) - \tau_i A_i(0)\right]}{A_0(0) + \sum_{i=1}^{n} A_i(0)}.$$

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REFERENCES

[1] H. Górecki, *Analysis and Synthesis of Control Systems with delay*, WNT, Warszawa, 1971, (in Polish).

[2] H. Górecki and L. Popek, ”Parametric optimization problem for control systems with time-delay”, *9th World Congress of IFAC* IX, CD-ROM (1984).

[3] A.C. Pontrjagin, ”O nuljach niekotorych prostych transcendentnych funkcyj”, *Izw. ANSSSR* 6, 115 (1942), (in Russian).

[4] S. Saks and A. Zygmund, *Analytical Functions*, Publishing House – Czytelnik, Warszawa, 1948, (in Polish).

[5] L.M. Ryżyk and I.S. Gradsztejn, *Integral, Sum, Series and Product Table*, PWN, Warszawa, 1964, (in Polish).

[6] P. Grabowski, “How one can apply the Cartwright-Levinson theorem to quasi-polynomials”, *Private Communication*, (2010).