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A stress-based formulation of the free material design problem with the trace constraint and single loading condition

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Abstract. The problem to find an optimal distribution of elastic moduli within a given plane domain to make the compliance minimal under the condition of a prescribed value of the integral of the trace of the elastic moduli tensor is called the free material design with the trace constraint. The present paper shows that this problem can be reduced to a new problem of minimization of the integral of the stress tensor norm over stresses being statically admissible. The eigenstates and Kelvin's moduli of the optimal Hooke tensor are determined by the stress state being the minimizer of this problem. This new problem can be directly treated numerically by using the Singular Value Decomposition (SVD) method to represent the statically admissible stress fields, along with any unconstrained optimization tool, e.g.: Conjugate Gradient (CG) or Variable Metric (VM) method in multidimensions.

Key words: free material design, anisotropic elasticity, compliance minimization.

1. Introduction

The structures made of a homogeneous material are non-optimal even if their shape is rationally designed or obtained by a certain optimization technique. The additional gain due to an optimal distribution of the material within a given domain can be essential, worth of incurring additional costs of replacing the homogeneous material by a non-homogeneous one with rationally designed material properties. A typical example of using composites is designing the plates and shells as laminated bodies in which the fiber directions are chosen optimally. The recent progress in the technology of composites makes it possible to control the microstructure in a continuous manner. This development is a driving force for the new methods of optimization taking into account variation of elastic moduli not only in thin-walled structures, but even in the spatial bodies.

The process of controlling the material anisotropy can be performed by designing the microstructure at the first step, or- temporarily- by omitting the microstructural analysis. The stream of papers based on the so-called relaxation by the homogenization method (see the book by Cherkaev [1] and the references therein) is concerned with just controlling and designing the microstructure, usually at finite number of levels (hence admitting a hierarchy of microstructures). The anisotropic and non-homogeneous properties are only the by-products of this modeling. However, one can take a risk to circumvent the previous method and attack the problem of designing the anisotropic properties directly, treating the process of designing the microstructures as a post-processing, see [2]. The latter stream of papers refers to the free material design (FMD). The model problem concerns the maximization of the stiffness or minimizing the compliance due to a single loading or by minimizing the linear combination of the compliances corresponding to various independently applied loadings, see e.g. Haslinger et al. [3] and [2, 4, 5].

In the papers on designing the microstructures, to make the problem well-posed, an isoperimetric condition is imposed on the amount of one material to be used or on the amount of two or more materials to be used. In the problems of designing of anisotropic properties a counterpart of the latter isoperimetric condition should be assumed, yet the mass density cannot now be the integrand of the condition, as this quantity does not enter the formulation. Instead, an isoperimetric condition is usually imposed on certain measures of the tensor $\mathbf{C}$ of elastic moduli. The condition of isotropy of the integrand $\Phi(\mathbf{C})$ of the isoperimetric condition means that $\Phi(\mathbf{C})$ is expressed in terms of the other function whose arguments are eigenvalues of the elasticity tensor $\mathbf{C}$, or in terms of Kelvin's moduli. The assumption of convexity of $\Phi(\mathbf{C})$ means that the aforementioned other function, of arguments being eigenvalues of $\mathbf{C}$ is also convex, see Yang [6]. The natural choice is the p-norm, applied in the present paper.

The version of the FMD method discussed here is stress-based. This formulation has made it possible to perform the optimization over the design variables analytically. A single load condition is dealt with. The FMD problem is reduced to the quasi-equilibrium problem

$$\min\left\{\int_{\Omega}\|\boldsymbol{\tau}\|\,dx\,|\,\boldsymbol{\tau}\text{ statically admissible in }\Omega\right\}\qquad\text{(P)}$$

the main feature of which is the integrand being of linear growth. A detailed formulation of this problem will be given in Sec. 5. Thus the minimizer $\boldsymbol{\tau}=\boldsymbol{\pi}$ of problem (P) is not characterized by the regularity properties of the elliptic prob-

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lem of elasticity in which the integrand of Castigliano's problem has a quadratic growth. To make the problem (P) well posed one should formulate appropriate assumptions within the notion of statical admissibility. We can draw upon the methods developed for the rigid-plastic bodies [7] and locking bodies, see [8–13]. The stresses are sought in the spaces of bounded measures on Ω and along its boundaries, imposing also the conditions on the divergence of stress and imposing, in a weak sense, the boundary conditions along the edges subjected to the tractions applied. To prove well-posedness of (P) one can use the mathematical analogy of this problem with the problem of equilibrium of the bodies with locking. These problems are of mathematical nature and are not discussed here; the reader is referred to [11–12].

Thus the solution $\boldsymbol{\pi}$ of problem (P) can be non-smooth and constant in some subdomains. Moreover, the stress $\boldsymbol{\pi}$ can be concentrated along some curves. The optimal tensor $\mathbf{C}$ is constructed by means of the solution $\boldsymbol{\pi}$; consequently, we take into account that it will vanish in some sub-domains. We see now, that an priori made assumption of the positive definiteness, imposed usually in the analysis of deformation of a body, becomes too restrictive in the FMD optimization problem. It is well seen during computations that the sequences of suboptimal solutions, each of which being characterized by positive definite Hooke tensors, tend inevitably to the optimum solution with Hooke's tensor being degenerated, optimally suited to the loading applied. Roughly speaking, no stiffness is required in the directions orthogonal to the non-zero stress components.

The problem (P) looks similar to Michell's problems of construction of the lightest fully stressed trusses transmitting given loads to a given contour of possible support. The stress-based Michell problem reads

$$\min \left\{ \int_{\Omega} \left(|\tau_I| + |\tau_{II}| \right) dx \,|\boldsymbol{\tau} \text{ statically admissible in } \Omega \right\}. \tag{P_M}$$

We note that the integrand in (P_M) is of linear growth, similarly as in (P). Thus one can expect that the solutions to both the problems are somehow similar.

An essential part of the present paper is the numerical treatment of the quasi-equilibrium problem (P) to which the FMD problem has been reduced. The statically admissible trial stress fields are interpolated within the finite element by the same shape functions as those used for the interpolation of the trial displacement fields. The unknowns describing the stress fields satisfy the set of equilibrium equations, being highly under-determinate. This set of equilibrium equations is solved by using the singular decomposition method SVD, but other numerical treatment have also been used, see Borkowski [14]. Its numerical treatment is implemented by using special tools of linear algebra, some of them available in Press et al. [15]. Consequently, the trial stress fields are expressed in terms of finite number of parameters on which no constraints are imposed. Let us remind that the redundants in the force method, known from structural analysis, are not subject to constraints, since equilibrium of the basic structure is satisfied for arbitrary values of the reduntants; such elementary analogies with the force method can be viewed as naive but seem helpful for better understanding of the method used. Thus the problem (P) is approximated by an unconditional minimization problem for which one of available optimization tools can be applied. Here the routines frprmn(...) and dfpmin(...) published in Press et al. [15] are used.

Having found the minimizer $\boldsymbol{\pi}$ of (P) one can determine optimal values of all components $C_{\alpha\beta\lambda\mu}$ of the Hooke tensor $\mathbf{C}$. The anisotropic and non-homogeneous plate thus constructed is the stiffest with respect to the given loading. In the examples shown the minimal compliance of the optimal anisotropic and non-homogeneous plate is more than 7 times smaller than the compliance of the isotropic and homogeneous plate satisfying the same isoperimetric condition on the trace of $\mathbf{C}$. This spectacular result reveals the gain we arrive at by using composites instead of conventional materials, at least in those cases when a single load is fixed. Extension of the results presented to the two-load case can be done, as announced in [16]. The multi-load case in 2D and 3D is still in progress.

2. Castigliano formulation of the plane elasticity problem

Consider a 2D elasticity problem in the plane domain Ω; the body is fixed along the contour Γ_2. The Γ_1 segment of the contour is subject to the tractions of intensity $\mathbf{T}(s)$, s being the natural parameter of the contour line. The domain Ω is parameterized by the Cartesian coordinates (x_1, x_2) with the basis $(\mathbf{e}_1, \mathbf{e}_2)$; $x = (x_1, x_2)$is a point in Ω. The unknown displacement field $\mathbf{u} = (u_1, u_2)$ and the virtual displacement fields $\mathbf{v} = (v_1, v_2)$ are kinematically admissible if they vanish on Γ_2. In the preset paper we shall not discuss the regularity conditions, implicitly present in the definition of the space $V(\Omega)$ of the kinematically admissible displacement fields. The linear form on $V(\Omega)$

$$f(\mathbf{v}) = \int_{\Gamma_1} \mathbf{T} \cdot \mathbf{v} \, ds \tag{1}$$

is called the virtual work of the given loading. Let E_s^2 be the set of symmetric tensors of 2^{nd} rank. The strains constitute a tensor $\boldsymbol{\varepsilon} = (\varepsilon_{\alpha\beta}) \in E_s^2$ of components $\varepsilon_{\alpha\beta}(\mathbf{u})$ where $\varepsilon_{\alpha\beta}(.)$ is viewed as a symmetric part of the gradient

$$\varepsilon_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha}\right), \tag{2}$$

where $()_{,\alpha} = \partial/\partial x_\alpha$, $\alpha, \beta = 1, 2$. The stresses $\boldsymbol{\sigma} = (\sigma_{\alpha\beta}) \in E_s^2$ are linked with strains by the Hooke law

$$\boldsymbol{\sigma} = \mathbf{C}\varepsilon(\mathbf{u}) \quad , \quad \sigma_{\alpha\beta} = C_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}(\mathbf{u}), \tag{3}$$

where $\mathbf{C} = (C_{\alpha\beta\lambda\mu})$ is Hooke's tensor of known symmetries. Such tensors constitute a set E_s^4. Almost everywhere in Ω the components of tensor $\mathbf{C}$ satisfy the positivity condition

$$C_{\alpha\beta\lambda\mu}\varepsilon_{\alpha\beta}\varepsilon_{\lambda\mu} \geq c\varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta}, \; c > 0 \tag{4}$$

the summation convention over small Greek indices being applied. Moreover, we assume that $C_{\alpha\beta\lambda\mu} \in L^{\infty}(\Omega)$. Tensor fields $\mathbf{C}$ satisfying the conditions above constitute a set $H(\Omega)$ of admissible fields of Hooke's tensors. The test stress fields $\boldsymbol{\tau} = (\tau_{\alpha\beta})$ of E_s^2 class at each point x in Ω is said to be statically admissible if the following variational equation holds

$$\int_{\Omega} \boldsymbol{\tau} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx = f(\mathbf{v}) \qquad \forall \mathbf{v} \in V(\Omega), \tag{5}$$

where $\boldsymbol{\tau} \cdot \boldsymbol{\varepsilon} = \tau_{\alpha\beta}\varepsilon_{\alpha\beta}$ is a scalar product in E_s^2 which defines the norm: $\|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$. The set of the fields $\boldsymbol{\tau}$ satisfying (5) is denoted by $\Sigma(\Omega)$.

This is a linear-affine set, its full characterization is subtle, see [17].

If $\mathbf{u} \in V(\Omega)$ while $\boldsymbol{\sigma} = (\sigma_{\alpha\beta})$ given by (3) satisfies (5), then such a field $\mathbf{u}$ is a solution of the given equilibrium problem. Let us stress that $\mathbf{u}$ depends on the filed $\mathbf{C}$, which will be written as $\mathbf{u}(\mathbf{C})$. The quantity

$$\wp(\mathbf{C}) = f(\mathbf{u}(\mathbf{C})) \tag{6}$$

is called the compliance. It is emphasized that the compliance is determined by the field $\mathbf{C}$.

The Castigliano theorem provides the formula

$$\wp(\mathbf{C}) = \min\left\{ \int_{\Omega} \boldsymbol{\tau} \cdot \left(\mathbf{C}^{-1}\boldsymbol{\tau}\right) dx \,|\boldsymbol{\tau} \in \Sigma(\Omega) \right\} \tag{7}$$

see [18], [17]. Let us stress that this formula holds for the non-homogeneous distribution of the material properties. Thus the assumption $\mathbf{C} \in H(\Omega)$ is sufficient for (7) to hold.

3. Formulation of the FMD problem in its conventional setting

Having formulated the elasticity problem we can pass to the formulation of FMD. Assume that all components of tensor field $\mathbf{C}$ of class $H(\Omega)$ are design variables. Moreover, assume that these fields of moduli are subject to the resource constraint of the form

$$\langle \Phi(\mathbf{C}) \rangle = E_0, \qquad \langle f \rangle = |\Omega|^{-1} \int_{\Omega} f dx, \tag{8}$$

where $\langle . \rangle$ stands for the averaging over the body domain Ω; $\Phi(\mathbf{C})$ is a scalar function of arguments $C_{\alpha\beta\lambda\mu}$, while E_0 is a given elastic modulus which determines the averaged value of the measure $\Phi(\mathbf{C})$. The condition (8) can be interpreted as a constraint on the cost of the design, the integrand $\Phi(\mathbf{C})$ being viewed as a unit cost of the material. As stressed above, the solution $\mathbf{u}$ does depend on the distribution of $C_{\alpha\beta\lambda\mu}$ within Ω under the condition that the loading $\mathbf{T}$, the contours Γ_1, Γ_2 and the domain Ω are kept fixed. The problem

$$Y = \min\left\{ \wp(\mathbf{C}) \,|\mathbf{C} \in H(\Omega), \langle \Phi(\mathbf{C}) \rangle = E_0 \right\} \tag{9}$$

is usually called the free material design-FMD- (or free material optimization) problem. Its origin traces back to Refs. [19]. The meaning of the solutions of problems of type (9) depends upon the choice of the function $\Phi(\mathbf{C})$. It is usually required that this function is isotropic, see [20]. Here we shall assume additionally that this function is convex. By the theorem by Yang [7] such function can be expressed in terms of a convex function of the eigenvalues of $\mathbf{C}$. These eigenvalues $\lambda_K = \lambda_K(\mathbf{C})$, $K = 1, 2, 3$, are called Kelvin moduli, as proposed by Rychlewski [21], see [22]. In the present paper the function $\Phi(\mathbf{C})$ will be assumed as p-norm of the vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$

$$\Phi(\mathbf{C}) = \|\boldsymbol{\lambda}(\mathbf{C})\|_p, \qquad p \geq 1,$$
$$\|\boldsymbol{\lambda}(\mathbf{C})\|_p = \left[\sum_{K=1}^{d(d+1)/2} (\lambda_K)^p \right]^{1/p}, \tag{10}$$

where d is the dimension of Ω which is here equal to 2. Since $p-$norm is convex with respect to $\lambda_K = \lambda_K(\mathbf{C})$, $K = 1, 2, 3$, the function $\Phi(\mathbf{C})$ is convex either. Let us note that
a) p=1; $\Phi(\mathbf{C}) = tr\mathbf{C}$, $tr\mathbf{C} = C_{\alpha\beta\alpha\beta}$ and

$$\Phi(\mathbf{C}) = \lambda_1 + \lambda_2 + \lambda_3 \tag{11}$$

b) $p = 2$; $\Phi(\mathbf{C}) = (C_{\alpha\beta\lambda\mu}C_{\alpha\beta\lambda\mu})^{1/2}$ and

$$\Phi(\mathbf{C}) = \left((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\right)^{1/2}. \tag{12}$$

The FMD problem (9) with $\Phi(\mathbf{C})$ given by (10) for $p = 1$, $p = 2$ has been already discussed in Refs. [3]. [19]. If appropriately regularized (for $p = 1$) this problem is solvable, see Th. 2.8 in Ref. [3]. Insert now the formula (7) into (9) and interchange the sequence of operators *min*, thus obtaining

$$Y = \min\left\{ Z(\boldsymbol{\tau}) \,|\boldsymbol{\tau} \in \Sigma(\Omega) \right\} \tag{13}$$

with

$$Z(\boldsymbol{\tau}) = \min\left\{ \int_{\Omega} \tau \cdot \left(\mathbf{C}^{-1}\boldsymbol{\tau}\right) dx \,|\mathbf{C} \in H(\Omega), \langle \Phi(\mathbf{C}) \rangle = E_0 \right\}. \tag{14}$$

The problem (14) cannot be solved without having a deeper insight into the algebraic structure of tensor $\mathbf{C}$. This will be discussed in the sequel.

4. A version of the free material design with prescribed Kelvin moduli

Prior to formulate the optimal design problem we are interested in let us write down the spectral decomposition of the Hooke tensor $\mathbf{C}$

$$\mathbf{C} = \lambda_1\mathbf{P}_1 + \lambda_2\mathbf{P}_2 + \lambda_3\mathbf{P}_3, \tag{15}$$

where

$$P_1 = \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1, \qquad P_2 = \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_2, \qquad P_3 = \boldsymbol{\omega}_3 \otimes \boldsymbol{\omega}_3 \tag{16}$$

and $\boldsymbol{\omega}_K \in E_s^2$, $K = 1, 2, 3$. The quantities λ_K are Kelvin moduli, see [21–22]. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_0$ and $\lambda_0 > 0$. Tensors $\boldsymbol{\omega}_K$ are called eigenstates and satisfy the local orthonormality conditions

$$\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}, \qquad K, L = 1, 2, 3. \tag{17}$$

Let us compute the integrand in (7) by using representation (15)

$$\boldsymbol{\tau}\cdot\left(\mathbf{C}^{-1}\boldsymbol{\tau}\right)=\frac{1}{\lambda_1}\left(\boldsymbol{\omega}_1\cdot\tau\right)^2+\frac{1}{\lambda_2}\left(\boldsymbol{\omega}_2\cdot\boldsymbol{\tau}\right)^2+\frac{1}{\lambda_3}\left(\boldsymbol{\omega}_3\cdot\boldsymbol{\tau}\right)^2. \tag{18}$$

By using the spectral representation of the unit tensor in E_s^4

$$\mathbf{I}=\left(I_{\alpha\beta\lambda\mu}\right),\qquad I_{\alpha\beta\lambda\mu}=\frac{1}{2}\left(\delta_{\alpha\lambda}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\lambda}\right) \tag{19}$$

or

$$\mathbf{I}=\boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_1+\boldsymbol{\omega}_2\otimes\boldsymbol{\omega}_2+\boldsymbol{\omega}_3\otimes\boldsymbol{\omega}_3 \tag{20}$$

we eliminate $\boldsymbol{\omega}_1$ in (18) to find

$$\boldsymbol{\tau}\cdot\left(\mathbf{C}^{-1}\boldsymbol{\tau}\right)=\frac{1}{\lambda_1}\left\|\boldsymbol{\tau}\right\|^2+\nu_2\left(\boldsymbol{\omega}_2\cdot\boldsymbol{\tau}\right)^2+\nu_3\left(\boldsymbol{\omega}_3\cdot\boldsymbol{\tau}\right)^2, \tag{21}$$

where $\left\|\boldsymbol{\tau}\right\|^2=\boldsymbol{\tau}\cdot\boldsymbol{\tau}$ and

$$\nu_2=\frac{1}{\lambda_2}-\frac{1}{\lambda_1},\qquad \nu_3=\frac{1}{\lambda_3}-\frac{1}{\lambda_1}, \tag{22}$$

where $\nu_3>\nu_2>0$. We re-write (7) as

$$\wp\left(\boldsymbol{\lambda},\boldsymbol{\omega}_2,\boldsymbol{\omega}_3\right)=\min\left\{\int_\Omega\left[\frac{1}{\lambda_1}\left\|\boldsymbol{\tau}\right\|^2+\underline{\nu_2\left(\boldsymbol{\omega}_2\cdot\boldsymbol{\tau}\right)^2+\nu_3\left(\boldsymbol{\omega}_3\cdot\boldsymbol{\tau}\right)^2}\right]\cdot dx\;\middle|\boldsymbol{\tau}\in\Sigma\left(\Omega\right)\right\}. \tag{23}$$

Assume that the Kelvin moduli are fixed within Ω while the eigenstates $\boldsymbol{\omega}_K$, $K=2,3$ are design variables in the optimum design problem:

find the eigenstate fields $\boldsymbol{\omega}_2,\boldsymbol{\omega}_3$ within Ω satisfying (17) pointwise and making the compliance (23) minimal. (P_1)

Let us write this problem more formally. Let $Q\left(\Omega\right)$ be the set of pairs $\mathbf{a}$, $\mathbf{b}$ of tensor fields on Ω such that $\mathbf{a}\cdot\mathbf{b}=0$, $\mathbf{a}\cdot\mathbf{a}=1$, $\mathbf{b}\cdot\mathbf{b}=1$. Like in (23) the compliance $\wp$ is treated as a functional depending on the vector $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$ (which is fixed here) and depending on $(\boldsymbol{\omega}_2,\boldsymbol{\omega}_3)\in Q\left(\Omega\right)$. Problem (P_1) is rewritten as

$$I\left(\boldsymbol{\lambda}\right)=\min\left\{\wp\left(\boldsymbol{\lambda},\boldsymbol{\omega}_2,\boldsymbol{\omega}_3\right)\left|\left(\boldsymbol{\omega}_2,\boldsymbol{\omega}_3\right)\in Q\left(\Omega\right)\right.\right\},\quad (P_1)$$

where $\wp\left(\boldsymbol{\lambda},\boldsymbol{\omega}_2,\omega_3\right)$ is given by (23). Since the terms underlined in (23) are non-negative, the minimum is attained on tensors $\boldsymbol{\omega}_K$, $K=2,3$ such that

$$\boldsymbol{\omega}_2\cdot\boldsymbol{\tau}=0,\qquad \boldsymbol{\omega}_3\cdot\boldsymbol{\tau}=0. \tag{24}$$

If one assumes

$$\boldsymbol{\omega}_1=\frac{1}{\left\|\boldsymbol{\tau}\right\|}\boldsymbol{\tau} \tag{25}$$

and $\boldsymbol{\omega}_2$ is assumed such that

$$\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_2=0,\qquad \boldsymbol{\omega}_2\cdot\boldsymbol{\omega}_2=1 \tag{26}$$

and if we fix $\boldsymbol{\omega}_3$ such that

$$\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_3=0,\qquad \boldsymbol{\omega}_2\cdot\boldsymbol{\omega}_3=0,\qquad \boldsymbol{\omega}_3\cdot\boldsymbol{\omega}_3=1 \tag{27}$$

then the fields $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ chosen this way are minimizers of (P_1). Then $I\left(\boldsymbol{\lambda}\right)=I_1\left(\boldsymbol{\lambda}\right)$ and

$$I_1\left(\boldsymbol{\lambda}\right)=\left\{\int_\Omega\frac{1}{\lambda_1}\left\|\boldsymbol{\tau}\right\|^2dx\,\middle|\boldsymbol{\tau}\in\Sigma\left(\Omega\right)\right\} \tag{28}$$

Let $\boldsymbol{\tau}=\widetilde{\boldsymbol{\sigma}}$ be the minimizer of I_1. Then the formulae

$$\begin{aligned}&\boldsymbol{\omega}_1=\frac{1}{\left\|\widetilde{\boldsymbol{\sigma}}\right\|}\widetilde{\boldsymbol{\sigma}},\qquad &&\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_2=0,\\ &\boldsymbol{\omega}_2\cdot\boldsymbol{\omega}_2=1,\qquad &&\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_3=0,\\ &\boldsymbol{\omega}_2\cdot\boldsymbol{\omega}_3=0,\qquad &&\boldsymbol{\omega}_3\cdot\boldsymbol{\omega}_3=1\end{aligned} \tag{29}$$

determine the eigenstates $\boldsymbol{\omega}_K$, $K=1,2,3$. The optimal layout of the moduli is given by (15), (16). These formulae determine the moduli $C_{\alpha\beta\lambda\mu}$ up to one parameter which fixes juxtaposition of $\boldsymbol{\omega}_2$ with respect to $\boldsymbol{\omega}_1$.

5. Releasing the Kelvin moduli

5.1. Formulation of the free material design problem in the stress-based setting. Let us assume that $\Phi\left(\mathbf{C}\right)$ is given by (10). Let us re-write the problem (9) as follows

$$Y_p=\min\left\{\wp\left(\boldsymbol{\lambda},\boldsymbol{\omega}_2,\boldsymbol{\omega}_3\right)\left|\left(\boldsymbol{\omega}_2,\boldsymbol{\omega}_3\right)\in Q\left(\Omega\right),\right.\right.\\ \left.\left.\boldsymbol{\lambda}_K\in L^\infty\left(\Omega\right),\lambda_K\geq\lambda_0,\left\langle\left\|\boldsymbol{\lambda}\right\|_p\right\rangle=E_0\right\}. \tag{30}$$

We have assumed that $\lambda_K>\lambda_0$, where $\lambda_0>0$ is a given modulus. Assume, that $\lambda_0<\Lambda/(3\left|\Omega\right|)$. Minimization in (30) over $(\boldsymbol{\omega}_2,\,\boldsymbol{\omega}_3)$ has been done in Sec. 4. Thus the problem above reduces to

$$Y_p=\min\left\{I_1\left(\boldsymbol{\lambda}\right)\left|\boldsymbol{\lambda}_K\in L^\infty\left(\Omega\right),\lambda_K\geq\lambda_0,\left\langle\left\|\boldsymbol{\lambda}\right\|_p\right\rangle=E_0\right.\right\}, \tag{31}$$

where $I_1\left(\boldsymbol{\lambda}\right)$ is defined by (28). The operation of *min* over $\boldsymbol{\tau}$ can be put before operation *min* over $\boldsymbol{\lambda}$ thus obtaining

$$Y_p=\min\left\{J_p\left(\boldsymbol{\tau}\right)\left|\boldsymbol{\tau}\in\Sigma\left(\Omega\right)\right.\right\}, \tag{32}$$

where

$$J_p\left(\boldsymbol{\tau}\right)=\min\left\{\int_\Omega\frac{1}{\lambda_1}\left\|\boldsymbol{\tau}\right\|^2dx\,\middle|\boldsymbol{\lambda}_K\in L^\infty\left(\Omega\right),\right.\\ \left.\lambda_K\geq\lambda_0,\int_\Omega\left\|\boldsymbol{\lambda}\right\|_p dx=\Lambda\right\} \tag{33}$$

and $\Lambda=\left|\Omega\right|E_0$.

5.2. Case of $p=1$: the free material design with the trace condition. Consider the case of $p=1$, which means$\Phi\left(\boldsymbol{\lambda}(\mathbf{C})\right)=tr\,\mathbf{C}$, $tr\,\mathbf{C}=C_{\alpha\beta\alpha\beta}$. The functional (33) assumes the form

$$J_1\left(\boldsymbol{\tau}\right)=\min\left\{\int_\Omega\frac{1}{\lambda_1}\left\|\boldsymbol{\tau}\right\|^2dx\,\middle|\lambda_K\geq\lambda_0,\right.\\ \left.\int_\Omega\left(\lambda_1+\lambda_2+\lambda_3\right)dx=\Lambda\right\}. \tag{34}$$

The minimum will be attained if λ_1 runs over the largest possible set or in the case when λ_2, λ_3 assume the minimal value λ_0 in the whole domain Ω. Thus

$$J_1(\boldsymbol{\tau}) = \min \left\{ \int_\Omega \frac{1}{\lambda_1} \|\boldsymbol{\tau}\|^2 dx \,|\lambda_0 - \lambda_1 \leq 0, \right. \\ \left. \int_\Omega \lambda_1 dx = \Lambda_0 \right\}, \tag{35}$$

where

$$\Lambda_0 = \Lambda - 2\lambda_0 |\Omega|. \tag{36}$$

The lagrangian has the form

$$L = \int_\Omega \frac{1}{\lambda_1} \|\boldsymbol{\tau}\|^2 dx + \mu \left(\int_\Omega \lambda_1 dx - \Lambda_0 \right) \\ + \int_\Omega \gamma(x) (\lambda_0 - \lambda_1(x)) dx. \tag{37}$$

The stationarity conditions and the Karush-Kuhn-Tucker (K-K-T) conditions imply

$$-\frac{1}{(\lambda_1(x))^2} \|\boldsymbol{\tau}(x)\|^2 + \mu - \gamma(x) = 0, \\ \gamma(x) \geq 0, \\ (\lambda_0 - \lambda_1(x)) \gamma(x) = 0, \\ \lambda_0 - \lambda_1(x) \leq 0, \\ \int_\Omega \lambda_1(x) dx = \Lambda_0, \qquad \mu \in R. \tag{38}$$

The domain Ω is divided into

$$\Omega_1 = \{x \in \Omega \,|\lambda_1(x) > \lambda_0\}, \quad \Omega_0 = \{x \in \Omega \,|\lambda_1(x) = \lambda_0\}.$$

Note that $\gamma(x) = 0$ if $x \in \Omega_1$, hence for such x

$$\lambda_1(x) = \frac{\|\boldsymbol{\tau}(x)\|}{\sqrt{\mu}}. \tag{39}$$

Let us re-write the isoperimetric condition in the form

$$\int_{\Omega_1} \lambda_1 dx + \int_{\Omega_0} \lambda_1 dx = \Lambda_0, \tag{40}$$

hence, by (38, 39), we have

$$\frac{1}{\sqrt{\mu}} \int_{\Omega_1} \|\boldsymbol{\tau}\| dx + \lambda_0 |\Omega_0| = \Lambda_0, \tag{41}$$

which gives

$$\frac{1}{\sqrt{\mu}} = \frac{\Lambda_1}{\int_{\Omega_1} \|\boldsymbol{\tau}\| dx}, \tag{42}$$

where

$$\Lambda_1 = \Lambda_0 - \lambda_0 |\Omega_0|. \tag{43}$$

Note that $|\Omega_0| \leq |\Omega|$, hence $\Lambda_1 \geq \Lambda - 3\lambda_0 |\Omega|$ and by the condition $\lambda_0 < \Lambda/(3|\Omega|)$we conclude that $\Lambda_1 \geq 0$. Thus we have

$$\lambda_1(x) = \frac{\Lambda_1}{\int_{\Omega_1} \|\boldsymbol{\tau}\| dx} \|\boldsymbol{\tau}\|. \tag{44}$$

The distribution of the main Kelvin modulus is

$$\lambda_1(x) = \begin{cases} \lambda_0 & \text{for } x \in \Omega_0 \\ \dfrac{\Lambda_1}{\int_{\Omega_1} \|\boldsymbol{\tau}\| dx} \|\boldsymbol{\tau}\| & \text{for } x \in \Omega_1 \end{cases}. \tag{45}$$

The functional (35) can be written as

$$J_1(\boldsymbol{\tau}) = \frac{1}{\lambda_0} \int_{\Omega_0(\boldsymbol{\tau})} \|\boldsymbol{\tau}\|^2 dx + \frac{1}{\Lambda_1} \left(\int_{\Omega_1(\boldsymbol{\tau})} \|\boldsymbol{\tau}\| dx \right)^2, \tag{46}$$

where it is stressed that the domains $\Omega_0(\boldsymbol{\tau})$ and $\Omega_1(\boldsymbol{\tau})$ do depend on $\boldsymbol{\tau}$. We come back to Y_1

$$Y_1 = \min \{J_1(\boldsymbol{\tau}) \,|\boldsymbol{\tau} \in \Sigma(\Omega)\}, \tag{47}$$

where $J_1(\boldsymbol{\tau})$ is given by (46). The formula becomes effective if one admits $\lambda_0 = 0$ in (34). Let us write (35) for the case of $\lambda_0 = 0$

$$J_1(\boldsymbol{\tau}) = \min \left\{ \int_\Omega \frac{1}{\lambda_1} \|\boldsymbol{\tau}\|^2 dx \,\middle|\, -\lambda_1 \leq 0, \int_\Omega \lambda_1 dx = \Lambda \right\}. \tag{48}$$

The lagrangian has the form

$$L = \int_\Omega \frac{1}{\lambda_1} \|\boldsymbol{\tau}\|^2 dx + \mu \left(\int_\Omega \lambda_1 dx - \Lambda \right), \tag{49}$$

while the condition $\lambda_1 \geq 0$ can be omitted. Indeed, the K-K-T conditions and stationarity conditions assume the form

$$-\frac{1}{(\lambda_1(x))^2} \|\boldsymbol{\tau}(x)\|^2 + \mu = 0, \\ \int_\Omega \lambda_1(x) dx = \Lambda, \qquad \mu > 0. \tag{50}$$

Thus

$$\lambda_1(x) = \frac{1}{\sqrt{\mu}} \|\boldsymbol{\tau}(x)\|. \tag{51}$$

Substitution into the isoperimetric condition leads to the formula for

$$\frac{1}{\sqrt{\mu}} = \frac{\Lambda}{\int_\Omega \|\boldsymbol{\tau}\| dx} \tag{52}$$

and substitution into (51) results in

$$\lambda_1(x) = \frac{\Lambda}{\int_\Omega \|\boldsymbol{\tau}\| dx} \|\boldsymbol{\tau}(x)\|. \tag{53}$$

One can compute the target function

$$J_1(\boldsymbol{\tau}) = \frac{1}{\Lambda}\left(\int_\Omega \|\boldsymbol{\tau}\|\, dx\right)^2. \tag{54}$$

Substitution into (47) gives

$$Y_1 = \frac{1}{\Lambda}\min\left\{\left(\int_\Omega \|\boldsymbol{\tau}\|\, dx\right)^2 \,|\, \boldsymbol{\tau} \in \Sigma(\Omega)\right\} \tag{55}$$

which is equivalent to

$$Y_1 = \frac{1}{\Lambda}\left(\min\left\{\int_\Omega \|\boldsymbol{\tau}\|\, dx \,|\, \boldsymbol{\tau} \in \Sigma(\Omega)\right\}\right)^2. \tag{56}$$

The minimizer $\boldsymbol{\tau} = \boldsymbol{\pi}$ of the problem

$$\min\left\{\int_\Omega \|\boldsymbol{\tau}\|\, dx \,|\, \boldsymbol{\tau} \in \Sigma(\Omega)\right\} \qquad (P) \tag{57}$$

determines the optimal distribution of the modulus λ_1 according to (53)

$$\lambda_1(x) = \frac{\Lambda}{\int_\Omega \|\boldsymbol{\pi}\|\, dx}\|\boldsymbol{\pi}(x)\| \tag{58}$$

and $\lambda_2 = \lambda_3 = 0$. Thus the optimal material is degenerated. Let us re-write (58) as follows

$$\lambda_1(x) = E(\boldsymbol{\pi}(x)) \tag{59}$$

with

$$E(\boldsymbol{\pi}) = E_0\frac{\|\boldsymbol{\pi}\|}{\langle\|\boldsymbol{\pi}\|\rangle} \tag{60}$$

the averaging $\langle . \rangle$ being defined by (8). Let us introduce the stress tensor of unit norm

$$\widehat{\boldsymbol{\pi}}(x) = \frac{1}{\|\boldsymbol{\pi}\|}\boldsymbol{\pi}(x) \tag{61}$$

and let $\boldsymbol{\omega}_2(x), \boldsymbol{\omega}_3(x) \in E_s^2$, $x \in \Omega$, satisfy the conditions (29) or

$$\begin{gathered}\widehat{\boldsymbol{\pi}}\cdot\boldsymbol{\omega}_2 = 0, \qquad \widehat{\boldsymbol{\pi}}\cdot\boldsymbol{\omega}_3 = 0,\\ \boldsymbol{\omega}_2\cdot\boldsymbol{\omega}_3 = 0, \qquad \|\boldsymbol{\omega}_2\| = 1,\\ \|\boldsymbol{\omega}_3\| = 1.\end{gathered} \tag{62}$$

The components of the tensor $\mathbf{C}$ are expressed by

$$C_{\alpha\beta\lambda\mu} = E(\boldsymbol{\pi})\,\hat{\pi}_{\alpha\beta}\hat{\pi}_{\lambda\mu} + 0\cdot(\omega_{2\alpha\beta}\omega_{2\lambda\mu} + \omega_{3\alpha\beta}\omega_{3\lambda\mu}) \tag{63}$$

hence

$$\sigma_{\alpha\beta} = E(\boldsymbol{\pi})\,\widehat{\boldsymbol{\pi}}_{\alpha\beta}\widehat{\boldsymbol{\pi}}_{\lambda\mu}\varepsilon_{\lambda\mu} \tag{64}$$

or

$$\boldsymbol{\sigma} = E(\boldsymbol{\pi})\,(\widehat{\boldsymbol{\pi}}\cdot\boldsymbol{\varepsilon})\widehat{\boldsymbol{\pi}}. \tag{65}$$

Remark. The tensor fields $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ are colinear. The stress trajectories of the initial problem with optimally chosen elastic moduli follow the trajectories of the pseudo-stresses of problem (P).

The solution to problem (30) for $\lambda_0 = 0$ and $p = 1$ has the following non-unique form
λ_1 – given by (59), $\lambda_2 = \lambda_3 = 0$
$\boldsymbol{\omega}_1 = \widehat{\boldsymbol{\pi}}$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ chosen such that the triplet $(\widehat{\boldsymbol{\pi}}, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ is orthonormal.

The field $\boldsymbol{\pi}$ is the minimizer of the problem (P) or (57).

The nonuniqueness of the choice of $(\boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ can be interpreted as the consequence of the formulation (30), where only one loading condition is assumed. If two loading conditions are assumed, and if the target function is the weighted sum of two compliances both $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ are determined by optimization, the smallest Kelvin modulus vanishes, while the triplet $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$ is usually uniquely determined, see [16, 23]. If three loading conditions are assumed, all three Kelvin moduli are uniquely determined.

Let us compute the trace of $\mathbf{C}$

$$tr\mathbf{C} = C_{\alpha\beta\alpha\beta} = E(\boldsymbol{\pi}) \tag{66}$$

and the average of the trace

$$\langle tr\mathbf{C}\rangle = E_0 = \Lambda\,|\Omega|^{-1} \tag{67}$$

which confirms that the condition (8) is fulfilled. Let us compute the Frobenius norm of $\mathbf{C}$

$$\|\mathbf{C}\|_2 = (C_{\alpha\beta\lambda\mu}C_{\lambda\mu\alpha\beta})^{1/2} = E(\boldsymbol{\pi}). \tag{68}$$

We note: $tr\mathbf{C} = \|\mathbf{C}\|_2 = E(\boldsymbol{\pi})$ and this equality suggests that the solutions to problem (30) for $p = 1$, $p = 2$ are identical if $\lambda_0 = 0$. This issue will be discussed later.

The problem (30) for $p = 1$ has been discussed for the first time in [19] in the form

$$\widetilde{Y}_1 = \min\left\{\wp(\mathbf{C}) \,\middle|\, \mathbf{C} \in E_s^4, \mathbf{C} \text{ being positive semi – definite}, \int_\Omega tr\ \mathbf{C}\, dx = \Lambda\right\} \tag{69}$$

which means that here $\lambda_0 = 0$. The equivalence of (30) for p=1 and (69) follows from the equality:

$$tr\mathbf{C} = C_{\alpha\beta\alpha\beta} = \lambda_1 + \lambda_2 + \lambda_3 = \|\boldsymbol{\lambda}\|_1. \tag{70}$$

The assumption $\lambda_0 = 0$ implies $\lambda_2 = \lambda_3 = 0$. Hence the optimal material is degenerated.

5.3. Case of $p > 1$. We come back to the problem (33). To enable λ_1 run over possibly large set one should choose $\lambda_2 = \lambda_3 = \lambda_0$. Thus the only unknown is λ_1 to solve the problem

$$J_p(\boldsymbol{\tau}) = \min\left\{\int_\Omega \frac{1}{\lambda_1}\|\boldsymbol{\tau}\|^2\, dx \,|\, \lambda_0 - \lambda_1 \leq 0, \int_\Omega [(\lambda_1)^p + 2(\lambda_0)^p]^{1/p}\, dx = \Lambda\right\}. \tag{71}$$

Consider the lagrangian

$$L = \int_{\Omega} \frac{1}{\lambda_1} \|\boldsymbol{\tau}\|^2 \, dx + \mu \left(\int_{\Omega} \left[(\lambda_1)^p + 2 (\lambda_0)^p \right]^{1/p} dx - \Lambda \right) + \int_{\Omega} \gamma(x) (\lambda_0 - \lambda_1(x)) \, dx. \tag{72}$$

There, where $\gamma = 0$ the stationarity condition of L with respect to small changes $\delta\lambda_1$ assumes the form

$$\frac{(\lambda_1)^{p+1}}{\left[(\lambda_1)^p + 2 (\lambda_0)^p \right]^{1-\frac{1}{p}}} = \frac{\|\boldsymbol{\tau}\|^2}{\mu}. \tag{73}$$

We shall give up computing λ_1 since this is analytically cumbersome in the case of $\lambda_0 \neq 0$. On the other hand the assumption $\lambda_0 = 0$ reduces (73) to (51), which is p independent. Indeed, in the case of $\lambda_0 = 0$ the problem (71) becomes p independent and the solution for p=1 is here valid. The case of p=2 was already discussed in [19].

The results above extend to the 3D case. If Ω is a 3D domain and $x = (x_1, x_2, x_3)$ then representation (15) can be extended to the case of $d = 3$. The conditions (17) hold for $K, L = 1, \ldots, 6$. The results (28), (29) are valid with $\Sigma(\Omega)$ referring to the 3D setting. The results (64), (65) also extend to the 3D case.

6. The stress-based numerical approach to solving the FMD problem with trace constraint

6.1. Approximation of problem (P) .The problem (P) or (57) will be solved numerically with using the new developed scheme of construction of statically admissible stress fields defined element-wise along with the optimizer solving the minimization problem. The numerical approach starts from division of the domain Ω into 4-node, quadrilateral, isoparametric finite elements with bilinear shape functions interpolating four stress fields τ_{11}, τ_{22}, τ_{12}, τ_{21}

$$\begin{aligned} \tau_{11}(\xi,\eta) &= N_0(\xi,\eta)\tau_0 + N_1(\xi,\eta)\tau_3 + N_2(\xi,\eta)\tau_6 \\ &\quad + N_3(\xi,\eta)\tau_9, \\ \tau_{22}(\xi,\eta) &= N_0(\xi,\eta)\tau_1 + N_1(\xi,\eta)\tau_4 + N_2(\xi,\eta)\tau_7 \\ &\quad + N_3(\xi,\eta)\tau_{10}, \\ \tau_{12}(\xi,\eta) &= N_0(\xi,\eta)\tau_2 + N_1(\xi,\eta)\tau_5 + N_2(\xi,\eta)\tau_8 \\ &\quad + N_3(\xi,\eta)\tau_{11}, \\ \tau_{21}(\xi,\eta) &= \tau_{12}(\xi,\eta) \end{aligned} \tag{74}$$

within an e-th element Ω_e , where ξ, η parameterize master element $\varpi = [-1,1] \times [-1,1]$ and τ_{3i+0}, τ_{3i+1}, τ_{3i+2} $(i = 0,1,2,3)$ are the unknown nodal stresses τ_{11} ,τ_{22}, τ_{12} in the nodes 0,1,2,3 respectively (see Fig. 1) (we simplify notation and omit upper index e writing e.g. τ_j instead more correctly τ_j^e).

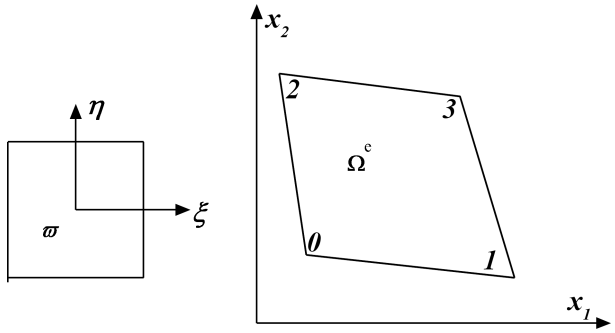

Fig. 1. Mapping of the master element ϖ onto e-th real finite element Ω^e

The shape functions N_0, N_1, N_2, N_3 are given by the formulae

$$N_i(\xi,\eta) = \frac{\left[1 - (-1)^{i \bmod 2}\,\xi\right]\left[1 - (-1)^{i \operatorname{div} 2}\,\eta\right]}{4} \quad (i = 0,1,2,3), \tag{75}$$

where **mod** operator computes the integral part of the result of dividing its first operand by its second, and **div** operator returns the integer remainder of the result of dividing its first operand by its second.

The virtual displacement field $\mathbf{v} = (v_1, v_2)$ in variational Eq. (5) is interpolated within an e-th element Ω_e similarly as the stress field

$$v_k(\xi,\eta) = \sum_{i=0}^{3} N_i(\xi,\eta)\, p_{2i+k-1} \quad (k = 1,2), \tag{76}$$

where p_{2i+0}, p_{2i+1} $(i = 0,1,2,3)$ are the unknown nodal virtual displacements v_1, v_2 at the nodes 0,1,2,3 respectively. The components of the gradient $(\nabla\mathbf{v})_{\alpha\beta}$ are computed by the following formulae

$$\begin{bmatrix} (\nabla\mathbf{v})_{\alpha 1}(x) \\ (\nabla\mathbf{v})_{\alpha 2}(x) \end{bmatrix} = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} \begin{bmatrix} v_{\alpha,\xi} \\ v_{\alpha,\eta} \end{bmatrix} \quad (\alpha = 1,2), \tag{77}$$

where $j_{\alpha\beta} = j_{\alpha\beta}(\xi,\eta)$ are the components of the inverse of the jacobian matrix defined by the partial derivatives $\dfrac{\partial x_\alpha}{\partial \xi}$, $\dfrac{\partial x_\alpha}{\partial \eta}$ $(\alpha = 1,2)$ of the transformation $x = (x_1, x_2)$ defining the shape of the body Ω. Substitution of (74)–(77) into the variational Eq. (5) results in the set of linear equations representing the equilibrium conditions

$$\mathbf{B}\,\boldsymbol{\Xi} = \boldsymbol{\Theta}, \tag{78}$$

where $\mathbf{B} \in M_{M\times n}$ is the rectangular $M \times n$ statics matrix, $\boldsymbol{\Theta} \in R^M$, $\boldsymbol{\Xi} \in R^n$ are vectors of nodal forces (only partially known !) and unknown nodal parameters τ_j defining the stress fields, respectively. Number M of rows and number n of columns (unknown nodal parameters τ_j, $j = 1, ..., n$ in global notation) in the matrix $\mathbf{B}$ are equal (in 2D case) to $2 \times N$ and $3 \times N$ respectively, where N denotes the number of all nodes in the global finite element mesh. Similarly as in the Force Method (FM) (see [24]) we perform the partition of the rectangular matrix $\mathbf{B}$ and vector $\boldsymbol{\Theta}$ into two matrices: upper $\mathbf{B}^u \in M_{m\times n}$, lower $\mathbf{B}^l \in M_{(M-m)\times n}$ and two vectors

$\mathbf{\Theta}^u \in R^m, \mathbf{\Theta}^l \in R^{M-m}$, respectively. The first m indices of the rows in upper matrix $\mathbf{B}^u$ correspond to the indices defining the global, unknown, free degrees of freedom and the remaining $M-m$ indices of the rows in lower matrix $\mathbf{B}^l$ correspond to the indices of the global, known, constrained degrees of freedom (boundary conditions). All components of the vector $\mathbf{\Theta}^u$ are known and the vector $\mathbf{\Theta}^l$ of the unknown boundary reactions can be calculated from the relation $\mathbf{\Theta}^l = \mathbf{B}^l \mathbf{\Xi}$ upon finding the vector $\mathbf{\Xi}$ from the system of rectangular linear equations $\mathbf{B}^u\ \mathbf{\Xi} = \mathbf{\Theta}^u$. In the Force Method, the number $M-m$ defines the degree of statical indeterminacy of the bar structure and of course could be equal 0. In the problem considered the number $M-m$ is always positive. The set of all solutions of the equations

$$\mathbf{B}^u\ \mathbf{\Xi} = \mathbf{\Theta}^u \tag{79}$$

can be expressed as

$$\begin{aligned} \Im = \Big\{ \mathbf{\Xi} = \mathbf{\Xi}\left(\alpha_1, ..., \alpha_s\right) \in R^n \,|\, \mathbf{\Xi} \\ = \mathbf{\Xi}^* + \sum_{k=1}^{s} \alpha_k \mathbf{h}_k,\ \alpha_k \in R \Big\}, \end{aligned} \tag{80}$$

where

$$\mathbf{h}_k = \left[h_{1\,k}, h_{2\,k}, ..., h_{n\,k}\right]^T, \quad k = 1, 2, ..., s \tag{81}$$

are the vectors that span the s-dimensional kernel of the matrix $\mathbf{B}^u$ and $\mathbf{\Xi}^* = \left[\tau_1^*, \tau_2^*, ..., \tau_n^*\right]^T \in R^n$ is the arbitrary, fundamental solution of the set of linear equations $\mathbf{B}^u\ \mathbf{\Xi} = \mathbf{\Theta}^u$.

The source codes implementing the decomposition (80) are provided by Borkowski [14], the Appendix, the code in FORTRAN. The representation (80) can be also performed with using the SVD decomposition and just this method has been chosen in the present paper. The short description of the SVD algorithm is given in the Appendix.

In each e-th finite element Ω^e, the stress components (74) depend not only on $(\xi, \eta) \in \varpi$ and local nodal parameters $\tau_j^*\ \ (j = 0, 1, ..., 11)$ but additionally on global parameters $\alpha_1, ..., \alpha_s$ defining the linear combinations of the particular local components $h_{j\,k}\ \ (j = 1, ..., n,\ k = 1, ..., s)$. In other words, upon constructing the solution (found only once!) of the linear, rectangular algebraic system $\mathbf{B}^u\ \mathbf{\Xi} = \mathbf{\Theta}^u$, one obtains a very simple approximation of the statically admissible set of the stress fields $\Sigma(\Omega)$ determined by s global parameters $\alpha_i \in R$

$$\begin{aligned} \Sigma\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) = \Big\{ \boldsymbol{\tau} = \boldsymbol{\tau}\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) \\ = \begin{bmatrix} \tau_{11}\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) & \tau_{12}\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) \\ \tau_{21}\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) & \tau_{22}\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) \end{bmatrix} \in M_{2\times 2} \Big\}, \end{aligned}$$

where $\boldsymbol{\zeta} = (\xi, \eta) \in \varpi$ and

$$\begin{aligned} \tau_{11} &= N_0\ \tau_0^* + N_1\ \tau_3^* + N_2\ \tau_6^* + N_3\ \tau_9^* \\ &\quad + \sum_{k=1}^{s} \alpha_k\ N_0\ h_{I_0\,k} + \sum_{k=1}^{s} \alpha_k\ N_1\ h_{I_3\,k} \\ &\quad + \sum_{k=1}^{s} \alpha_k\ N_2\ h_{I_6\,k} + \sum_{k=1}^{s} \alpha_k\ N_3\ h_{I_9\,k}, \\ \tau_{22} &= N_0\ \tau_1^* + N_1\ \tau_4^* + N_2\ \tau_7^* + N_3\ \tau_{10}^* \\ &\quad + \sum_{k=1}^{s} \alpha_k\ N_0\ h_{I_1\,k} + \sum_{k=1}^{s} \alpha_k\ N_1\ h_{I_4\,k} \\ &\quad + \sum_{k=1}^{s} \alpha_k\ N_2\ h_{I_7\,k} + \sum_{k=1}^{s} \alpha_k\ N_3\ h_{I_{10}\,k}, \\ \tau_{12} &= N_0\ \tau_2^* + N_1\ \tau_5^* + N_2\ \tau_8^* + N_3\ \tau_{11}^* \\ &\quad + \sum_{k=1}^{s} \alpha_k\ N_0\ h_{I_2\,k} + \sum_{k=1}^{s} \alpha_k\ N_1\ h_{I_5\,k} \\ &\quad + \sum_{k=1}^{s} \alpha_k\ N_2\ h_{I_8\,k} + \sum_{k=1}^{s} \alpha_k\ N_3\ h_{I_{11}\,k}, \\ \tau_{21} &= \tau_{12}. \end{aligned} \tag{82}$$

Note that j in τ_j^* runs over $\{0,1,\ldots,11\}$, while j in $h_{j\,k}$runs over $\{1,2,\ldots,n\}$.

The (constrained) problem (P), see (57) is now reformulated to the algebraic (unconstrained) problem P^α

$$\begin{aligned} &\min_{(\alpha_1,...,\alpha_s)\in R^s} \left\{F^\alpha\right\}, \quad F^\alpha\left(\alpha_1, ..., \alpha_s\right) \\ &= \sum_e \sum_{\boldsymbol{\zeta}^Q \in \varpi} w\left(\boldsymbol{\zeta}^Q\right) \left\| \boldsymbol{\tau}^e\left(\boldsymbol{\zeta}^Q, \alpha_1, ..., \alpha_s\right) \right\| \\ &\qquad \cdot \left| \det \nabla x\left(\boldsymbol{\zeta}^Q\right) \right| \quad (P^\alpha). \end{aligned} \tag{83}$$

where $\boldsymbol{\zeta}^Q = \left(\xi^Q, \eta^Q\right) \in \varpi$ and $w = w\left(\boldsymbol{\zeta}^Q\right)$ are Gauss integration points and weights, respectively.

In each arbitrary, but fixed point $\boldsymbol{\zeta} = (\xi, \eta) \in \varpi$, the gradient

$$df = df\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) = \left[\begin{array}{ccc} \dfrac{\partial f}{\partial \alpha_1} & ... & \dfrac{\partial f}{\partial \alpha_s} \end{array} \right]^T \in R^s \tag{84}$$

of the function

$$f = f\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) = \left\| \boldsymbol{\tau}\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) \right\| \tag{85}$$

is defined by the following formula

$$\begin{aligned} &df\left(\boldsymbol{\zeta}, \alpha_1, ..., \alpha_s\right) \\ &= \frac{1}{\|\boldsymbol{\tau}\|} \left[\begin{array}{ccc} \boldsymbol{\tau} \cdot \dfrac{\partial \boldsymbol{\tau}}{\partial \alpha_1} & ... & \boldsymbol{\tau} \cdot \dfrac{\partial \boldsymbol{\tau}}{\partial \alpha_s} \end{array} \right]^T \in R^s, \end{aligned} \tag{86}$$

where see (82)

$$\begin{aligned} \frac{\partial \tau_{11}}{\partial \alpha_k} &= N_0\ h_{I_0\,k} + N_1\ h_{I_3\,k} + N_2\ h_{I_6\,k} + N_3\ h_{I_9\,k}, \\ \frac{\partial \tau_{22}}{\partial \alpha_k} &= N_0\ h_{I_1\,k} + N_1\ h_{I_4\,k} + N_2\ h_{I_7\,k} + N_3\ h_{I_{10}\,k}, \\ \frac{\partial \tau_{12}}{\partial \alpha_k} &= N_0\ h_{I_2\,k} + N_1\ h_{I_5\,k} + N_2\ h_{I_8\,k} + N_3\ h_{I_{11}\,k}. \end{aligned} \tag{87}$$

The formulae (85)–(87) make it possible to calculate the components of the gradient of the objective function $F^{\alpha}(\alpha_1, ..., \alpha_s)$ in (83).

6.2. The algorithm. The computational procedure consists of the following steps

Step 1. Set the rectangular system of linear algebraic Eqs. (78) $\mathbf{B}\,\boldsymbol{\Xi} = \boldsymbol{\Theta}$, according with FEM.

Step 2. Apply FM to separate upper sub-matrix $\mathbf{B}^u$ and upper sub-vector $\mathbf{Q}^u$ corresponding to the unknown degrees of freedom of the FEM.

Step 3. Define the set (80), i.e. find the solutions $\boldsymbol{\Xi} = \boldsymbol{\Xi}^* + \sum_{k=1}^{s} \alpha_k \, \mathbf{h}_k$ of the rectangular system of linear algebraic equations $\mathbf{B}^u \, \boldsymbol{\Xi} = \boldsymbol{\Theta}^u$.

Step 4. Apply any algorithm of the nonlinear mathematical programming to find the solution $[\alpha_1, ..., \alpha_s]^T \in R^s$ of the minimization problem

$$\min_{(\alpha_1,...,\alpha_s)\in R^s} \left\{ \sum_{e} \sum_{\zeta^Q \in \varpi} w\left(\zeta^Q\right) f^e\left(\zeta^Q, \alpha_1, ..., \alpha_s\right) \left|\det \nabla x\left(\zeta^Q\right)\right| \right\},$$

where the function $f^e = f^e\left(\zeta^Q, \alpha_1, ..., \alpha_s\right)$ and its gradient df^e are defined in (85) and (86), respectively.

Step 5. According with the formulae (82), find the all components $\pi_{11}, \pi_{22}, \pi_{12}$ of the optimal stress tensor $\boldsymbol{\pi}$ defined by the optimal solution $\alpha_1, ..., \alpha_s$ found in step 4.

Step 6. According with the formulae (60) and (63) find the distribution of $\lambda_1 = E(\boldsymbol{\pi})$ and optimal moduli $C_{\alpha\beta\gamma\delta}$.

We have implemented two various, gradient-oriented numerical routines in C language: frprmn(...) and dfpmin(...) (see [15], p. 423 and 428) for obtaining the optimal solution $\alpha_1, ..., \alpha_s$. These routines implement the Fletcher-Reeves (FR), Polak-Ribiere (PR) or Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithms of non-linear mathematical programming. We always call routines frprmn(...) or dfpmin(...) for the initial values of design parameters $\alpha_1, ..., \alpha_s$ randomly generated within the arbitrary range $[-z, +z]$, $z > 0$ or for all $\alpha_i = 0$, $i = 1, ..., s$. The average number of the main iteration loops for the assumed tolerance $tol = 1.0 \cdot 10^{-5}$ was equal from ∼50 to ∼150. The optimal value of the objective function was independent of the initial values of α_i , $i = 1, ..., s$ and was always the same or nearly the same after the both routines finished its job. The time of executing of the optimization procedures was essentially shorter in comparison with the time of finding the solution with SVD decomposition of the FEM static problem (for the presented below two examples: seconds, e.g. ∼ 60-120 and minutes, e.g. ∼60-120 on VAIO laptop, CPU 2.53 GHz). The non-gradient algorithm powell(...) [15] gave the same numerical results of optimum point $(\alpha_1, ..., \alpha_s)$ but the time of executing of the program was for the relatively small number of the unknowns unacceptable long (above few hours and more).

Example 1. The first example concerns a rectangular plate of length $L_x = 4.0\,[m]$ and height $L_y = 2.0\,[m]$. The finite element mesh is defined by $n_x \times n_y = 60 \times 30 = 1800$ quadrilateral modules (see Fig. 2). The total number of nodes $N = (n_x + 1)(n_y + 1) = 1891$, which gives the total number of the columns and rows in matrix $\mathbf{B}$ equal to $n = 3N = 5673$ and $M = 2N = 3782$, respectively (n and M are also equal to the total number of the unknown nodal, stress parameters τ_j and total number of the degrees of freedom, respectively). The 9- and 3- points Hammer – Stroud rules of the Gauss integration for the two-dimensional cube and one- dimensional segment are adopted, respectively.

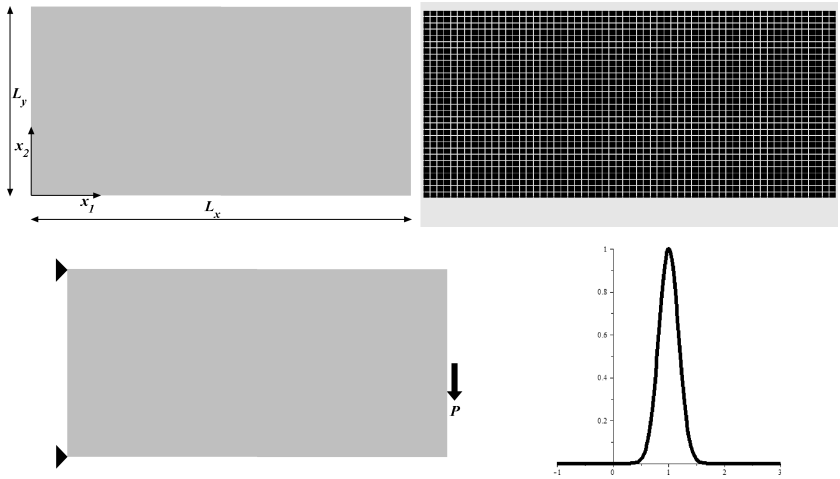

Fig. 2. The body Ω – rectangular plate $L_x \times L_y$, FEM mesh $n_x \times n_y$, boundary conditions, and loading P defined by weight function traction

The plate is supported at two non-sliding supports on the left edge (lower and upper node) and subject to a vertical load at the right edge (tangent to the vertical edge), see Fig. 2. The two point supports are constructed by the three fully clamped nodes in horizontal and three fully clamped nodes in vertical direction in the local neighborhood of the both left nodes: lower and upper, respectively. The traction T is modeled by the weight function (see Fig. 2)

$$\forall y \in [0, L_y] \quad T = T(y) = T_{\max}\, e^{-\left(\frac{y-y_0}{w}\right)^2}, \tag{88}$$

where $T_{\max} = 1.0\,[N/m^2]$, $y_0 = 1.0\,[m]$, $w = 0.25\,[m]$. The vertical resultant of the traction loading equals $P = 0.44\,T_{\max}$. The optimal distributions of the modulus$\lambda_1(x) = E(\boldsymbol{\pi}(x))$ and of the components C_{1111}, C_{1122}, C_{1112}, C_{2222}, C_{2212}, C_{1212} are shown in Fig. 3 (referred to E_0) using the contours graphic output in Voxler Graphical System. The initial and optimal values of the objective function (for all initial $\alpha_i = 0$) were 6.6[Nm] and 5.6[Nm] after 133 iterations, respectively.

The optimal value of the compliance function can be easily calculated from the formula (56). In our case, we have

$$Y_1 = \frac{1}{\Lambda}\left(\min\left\{\int_\Omega \|\boldsymbol{\tau}\|\,dx \mid \boldsymbol{\tau} \in \Sigma(\Omega)\right\}\right)^2 = \frac{\left(\int_\Omega \|\boldsymbol{\pi}\|\,dx\right)^2}{E_0\,|\Omega|} = \frac{(5.633115)^2}{E_0\;8.0} \cong \frac{3.9665}{E_0}. \tag{89}$$

This value can be alternatively calculated from the formula

$$\int_{\Gamma_1} \mathbf{T}\cdot\mathbf{u}\,ds, \tag{90}$$

where $\mathbf{u} \in V(\Omega)$ is the solution of Eq. (5), i.e.

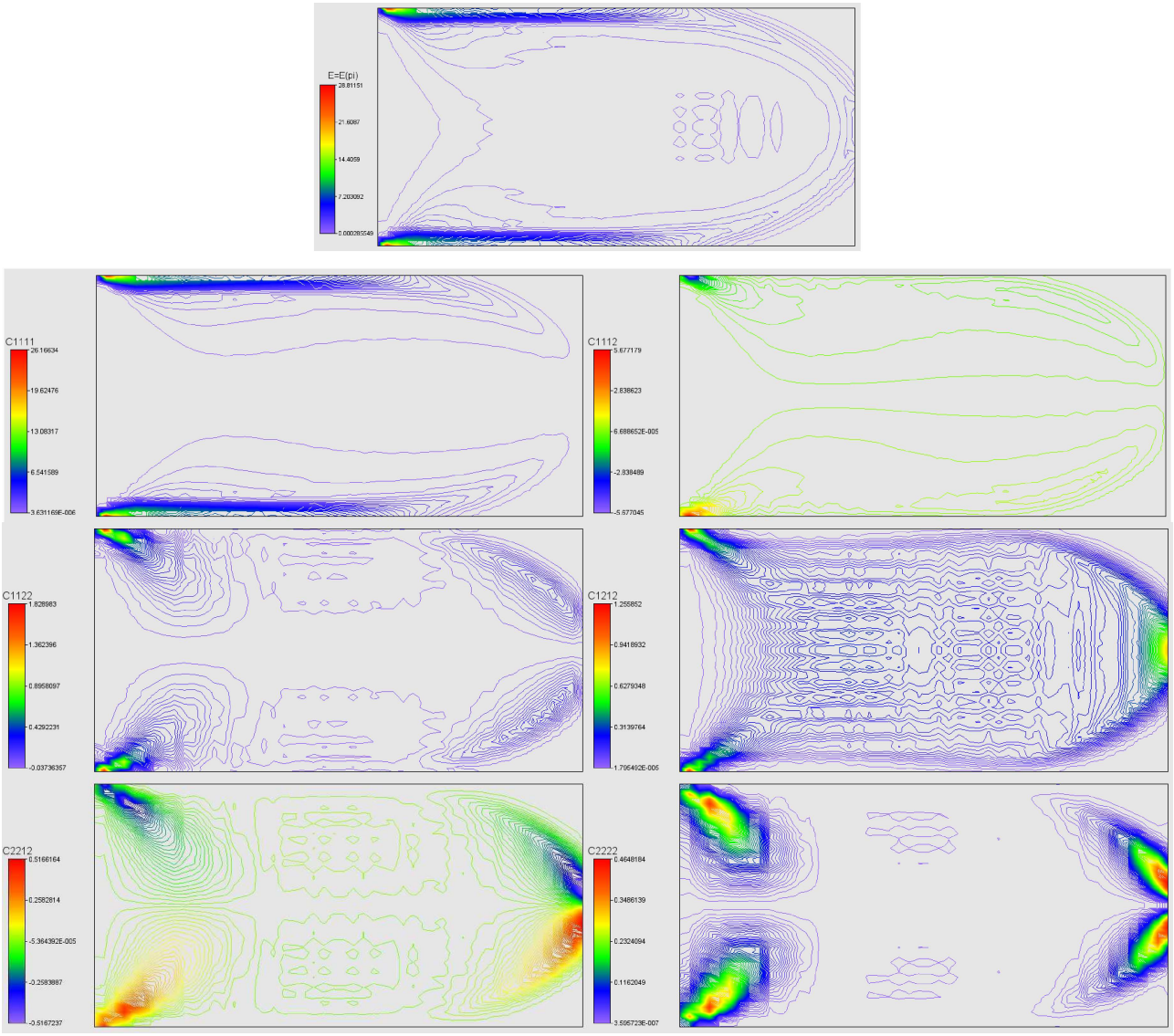

Fig. 3. Solution of Example 1. Distribution of the optimal $\lambda_1(\boldsymbol{\pi})/E_0$, denoted symbolically as E(pi), and optimal moduli $C\alpha\beta\gamma\delta = C_{\alpha\beta\gamma\delta}/E_0$

$$\forall \mathbf{v} \in V(\Omega) \quad \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx = \int_{\Gamma_1} \mathbf{T} \cdot \mathbf{v}\, ds, \qquad \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{C}(\boldsymbol{\pi})\boldsymbol{\varepsilon}(\mathbf{u}) \tag{91}$$

for the optimal Hooke tensor $\mathbf{C} = \mathbf{C}^{opt} = \mathbf{C}(\boldsymbol{\pi})$ given by (63). The problem (91), involving the optimal non-homogeneous and (in our case - optimal) anisotropic elasticity tensor $\mathbf{C} = \mathbf{C}^{opt} = \mathbf{C}(\boldsymbol{\pi})$, has been solved by the classical FEM. Upon finding the numerical solution $\mathbf{u}$, one can compute the components of the stress tensor field $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u})$ and then calculate the optimal compliance from the formulae (32), (33) for $p = 1$ and (9), i.e. simply as

$$\int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{u})\, dx. \tag{92}$$

The three results: (89), (90) and (92) should be exactly the same. In our case we have got the following values (90) and (92):

$$\int_{\Gamma_1} \mathbf{T} \cdot \mathbf{u} ds = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{u})\, dx \approx \frac{3.9568}{E_0}, \tag{93}$$

which confirms the correctness of the numerical solution calculated from the analytical formula (89). The small differences between (89) and (93) can be e.g. explained as the result of the fact that FEM interpolation of the displacement fields $\mathbf{u}$ and $\mathbf{v}$ in (91) is defined by the same shape functions (75) used in the interpolation of the stress field $\boldsymbol{\tau}$ in (74) and (78). The partial derivatives of the shape functions (75) in (91) defining the interpolation of the strain and next of the stress field are evidently not the same as the shape functions (75) defining the interpolation of the stress field (74) directly.

As the last step in the post-processing analysis we make a comparison of the compliance of the optimal, non-homogeneous and anisotropic body Ω with the compliance of the homogeneous and isotropic body Ω. It can be easily shown that for an isotropic body the Kelvin moduli λ_1, λ_2, λ_3 are equal to $2k$, 2μ, 2μ respectively, where

$$k = \frac{E}{2(1-\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}, \tag{94}$$

here E and ν denote Young modulus and Poisson ratio, respectively. Let E_0 in (8) be given. Assume that the condition

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{ const } = E_0, \quad \text{i.e. } \frac{E(3-\nu)}{(1-\nu)(1+\nu)} = E_0 \tag{95}$$

holds pointwise in the design domain, thus satisfying the isoperimetric condition (34). From (95) we can calculate the Young modulus and next, for the given Poisson ratio ν, we can calculate the constants k and μ by (94). The constant components of the isotropic Hooke tensor $\mathbf{C}^{iso}$ are then expressed by

$$C^{iso}_{1111} = C^{iso}_{2222} = k + \mu, \qquad C^{iso}_{1122} = k - \mu, \qquad C^{iso}_{1212} = \mu, \qquad C^{iso}_{1112} = C^{iso}_{2212} = 0, \tag{96}$$

see e.g. [23]. Next we find the solution of the Eq. (91) for $\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{C}^{iso}(\boldsymbol{\pi})\,\boldsymbol{\varepsilon}(\mathbf{u})$ and calculate compliance from (90) or (92). In our case, for $\nu = 0.3$, the result is

$$\int_{\Gamma_1} \mathbf{T} \cdot \mathbf{u}\, ds = \int_{\Omega} \boldsymbol{\sigma} \cdot \left(\mathbf{C}^{iso}\right)^{-1}(\boldsymbol{\pi})\, \boldsymbol{\sigma}\, dx \cong \frac{28.5175}{E_0}. \tag{97}$$

Therefore, the compliance of the homogeneous, isotropic material with moduli given by (94), (96) is ~7.2 greater than the compliance of the optimal plate. In Fig. 4 below the components of the three stress fields are shown: minimizer $\boldsymbol{\pi}$ of the functional (57), solution $\boldsymbol{\sigma}$ of the equation (91) for optimal $\mathbf{C}^{opt} = \mathbf{C}^{opt}(\boldsymbol{\pi})$ and $\boldsymbol{\sigma}^{iso}$ for isotropic Hooke tensor $\mathbf{C}^{iso}$.

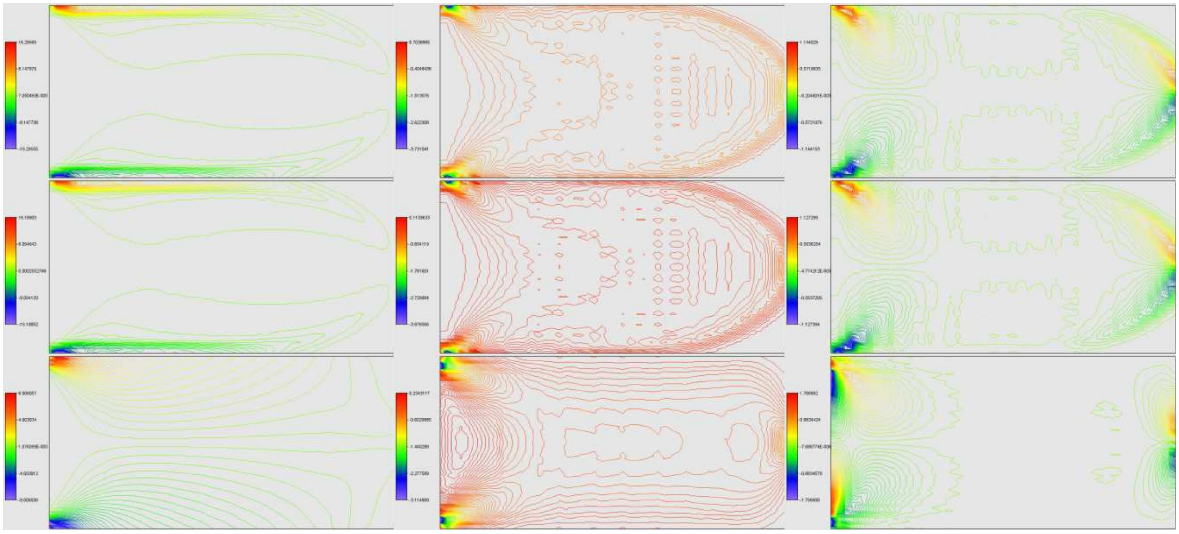

Fig. 4. The first row shows the contour plots of the components $\pi_{11}, \pi_{12}, \pi_{22}$ of the statically feasible stress field π being the minimizer in problem (57). The second row shows the components $\sigma_{11}, \sigma_{12}, \sigma_{22}$ of the stress field $\boldsymbol{\sigma}$ corresponding to the optimal, non-homogeneous, anisotropic Hooke tensor $\mathbf{C}^{opt}$ given by (63). The last row presents the components $\sigma^{iso}_{11}, \sigma^{iso}_{12}, \sigma^{iso}_{22}$ of the stress field $\boldsymbol{\sigma}^{iso}$ corresponding to the homogeneous, isotropic Hooke tensor $\mathbf{C}^{iso}$, cf. (96)

It is numerically confirmed that the fields $\boldsymbol{\pi}$, $\boldsymbol{\sigma}$ are similar: they have common trajectories but differ in the magnitudes of their components, see (65) and a remark below this formula. The field $\boldsymbol{\sigma}^{iso}$ is clearly different.

Example 2. The second example concerns a half-annular plate being an image of a rectangle of dimensions $L_x \approx 6.0\,[m]$ and $L_y = 2.0\,[m]$. The finite element mesh is defined by $n_x \times n_y = 60 \times 20 = 1200$ quadrilateral elements (see Fig. 5). The total number of nodes equals $N = (n_x + 1)(n_y + 1) = 1281$; the total number of the columns and rows in matrix $\mathbf{B}$ equal to $n = 3N = 3843$ and $M = 2N = 2562$, respectively. The half-annular plate is clamped along the lower, vertical edge and subject to a vertical, constant unit load $T_y = -1.0\,[N/m^2]$ at the top edge (tangent to the vertical top edge) in the first case and to a horizontal, constant pressure load $T_x = 1.0\,[N/m^2]$ perpendicular to the same top, vertical edge, see Fig. 5.

The optimal distributions of the modulus $\lambda_1 = E(\boldsymbol{\pi})$ are shown for two load cases considered, see Fig. 6. The initial and optimal values of the objective function (for all initial values $\alpha_i = 0$) for the first case of loading are 28.5 [Nm] and 20.8 [Nm], respectively, after 64 iterations. In the second case of loading these values are 43.6 [Nm] and 35.3 [Nm], respectively, after 102 iterations.

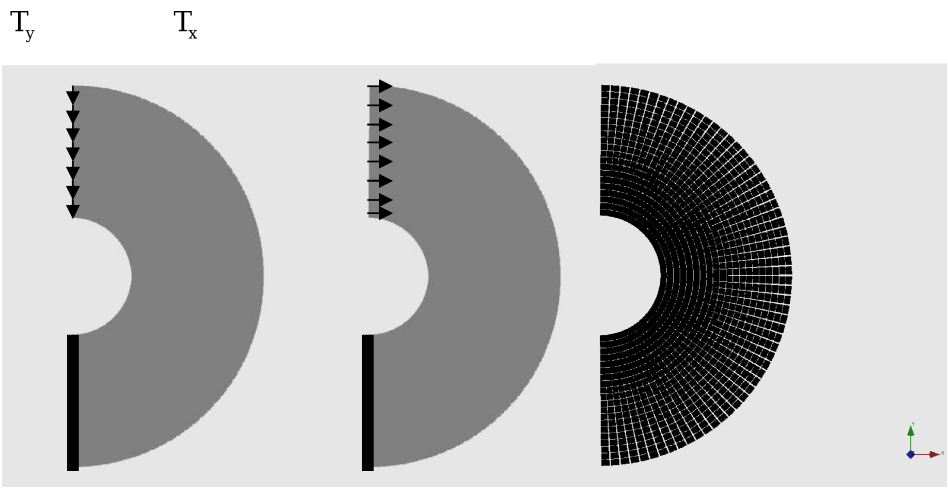

Fig. 5. The half-annular plate with lower edge clamped. Two cases of loads (T_y, T_x) is discussed. The FEM mesh $n_x \times n_y$ is shown at the right-hand side

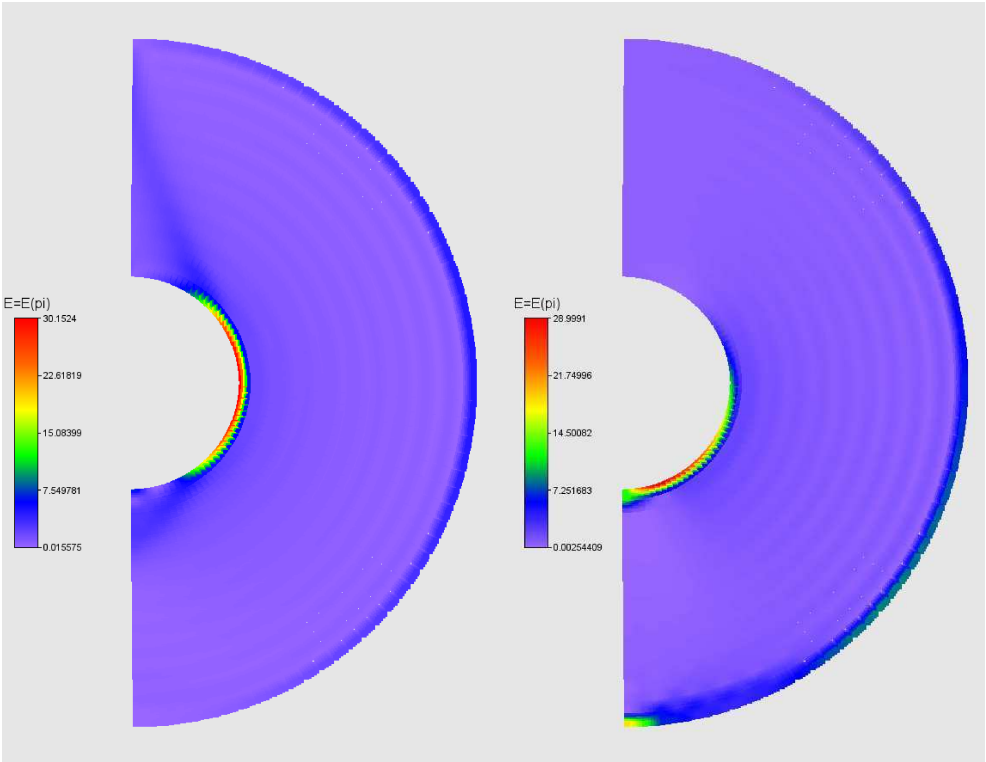

Fig. 6. Solution of example 2. Distribution of the optimal modulus $\lambda_1(\boldsymbol{\pi})/E_0$ (denoted as E(pi)) for the first case of loading (left figure) and second case of loading (right figure)

To make the result of the left layout of Fig. 6 (repeated at left of Fig. 7) easily manufacturable, this layout is filtered in black and white to obtain the right-hand layout in Fig. 7. This sharpened layout can be interpreted as the internal stiffening of the half-annular plate, which is designed to resist the vertical force, resembling a bow prestressed by a rope. Vertical displacements are then minimized. Similar filter sharpening can be performed for the second case of load, in which the sandwich-like structure emerges to minimize the horizontal displacements.

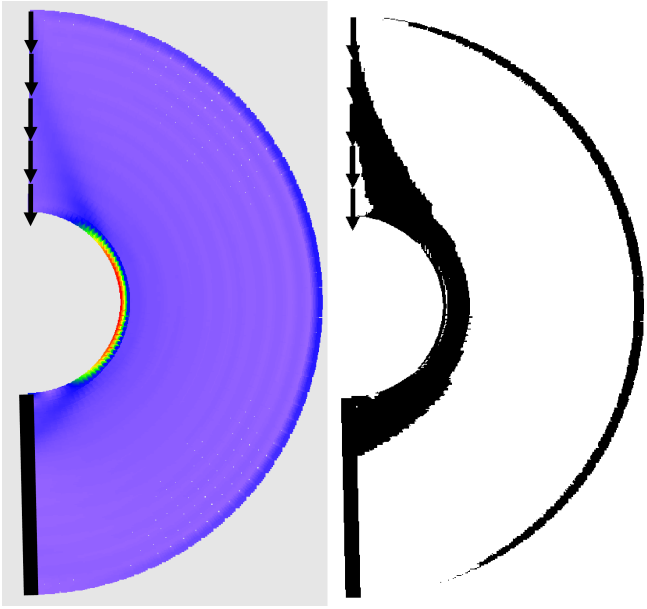

Fig. 7. The resulting distribution of $\lambda_1(\boldsymbol{\pi})/E_0$ (at left) and its black-white filter image

7. Final remarks

The stress-based approach to FMD problem with the trace constraint has made it possible to reduce the optimization problem over the Hooke's tensor at each point to the minimization problem (P) or (57), its formulation being situated between equilibrium problem of the body (then the norm of stress should be squared) and the Michell problem (P_M) in which the integrand is expressed in terms of principal stresses and is of linear growth as in (P). The stress-based approach used is advantageous over the displacement – based approach, since the whole optimization has been performed analytically, the remaining problem (P) being reminiscent of the equilibrium problem which always should be satisfied. The problem (P) to which the FMD problem has been reduced is numerically solved by virtue of the SVD decomposition method and gradient oriented non-constrained optimization solvers implementing e.g. Fletcher-Reeves, Polak-Ribiere or Broyden-Fletcher-Goldfarb-Shanno algorithms. The numerical results obtained compare favourably with the known FMD results thus proving their correctness.

The paper has dealt with a single load case, for which the dimension of the problem (2D or 3D) was not an essential factor. The dimension of the problem becomes an essential parameter if more than one loading case is discussed. The aim of the current research and of the forthcoming papers is to extend the results of [16, 23], where two load cases have been considered, to the 3D setting. The results concerning the 2D setting were obtained by virtue of the geometric visualization of plane stress tensors as vectors in space. To obtain the desired generalization to 3D one should treat stress tensors as vectors in R^6, including their rules of rotations.

Appendix

The details of the numerical the numerical treatment of the semi-equilibrium problem (P) will be presented. The numerical approach is based upon the following theorem of linear algebra: any $m \times n$ matrix $\mathbf{A} = (A_{ij}) \in M_{m\times n}$ can be written as the product of an $m \times n$ column-orthogonal matrix $\mathbf{U} = (U_{ij}) \in M_{m\times n}$, an $n \times n$ diagonal matrix $\mathbf{W} = (W_{ij}) \in M_{n\times n}$ with positive or zero elements and the transpose of an $n \times n$ orthogonal matrix $\mathbf{V} = (V_{ij}) \in M_{n\times n}$. Thus $\mathbf{A} = \mathbf{U}\ \mathbf{W}\ \mathbf{V}^T$. If $m < n$, the values W_{jj} for $j = m+1, \ldots, n$ are all zero and the corresponding columns of $\mathbf{U}$ are also zero. The matrices $\mathbf{U}$ and $\mathbf{V}$ satisfy the following orthonormality conditions:

$$\begin{aligned} &\forall 1 \le k \le m, \qquad \forall 1 \le l \le m, \\ &\sum_{i=1}^{m} U_{ik}U_{il} = \delta_{kl}, \qquad \sum_{j=1}^{n} V_{jk}V_{jl} = \delta_{kl}. \end{aligned} \tag{A1}$$

The decomposition $\mathbf{A} = \mathbf{U}\ \mathbf{W}\ \mathbf{V}^T$ can always be done, no matter how singular the matrix $\mathbf{A}$ is. The numerically stable routine svdcmp(...) that performs SVD on an arbitrary matrix $\mathbf{A}$ (see [15], p. 67) was implemented in our program.

Consider the set of equations $\mathbf{A}\ \mathbf{x} = \mathbf{b}$, where $\mathbf{A} \in M_{m\times n}$, $\mathbf{x} \in R^n$, $\mathbf{b} \in R^m$ in the case when we have fewer linear equations m than unknowns n. Upon performing the SVD decomposition, besides $n - m$ zero diagonal elements $W_{jj} = 0$ (since $m < n$), there may exist additional zero or negligible W_{jj}. The SVD decomposition yields an equivalent set of equations $\mathbf{W}\boldsymbol{\xi} = \boldsymbol{\beta}$ where $\boldsymbol{\xi} = (\xi_j) = \mathbf{V}^T\mathbf{x} \in R^n$, $\boldsymbol{\beta} = (\beta_i) = \mathbf{U}^T\mathbf{b} \in R^n$. First, we calculate vector $\boldsymbol{\beta} = \mathbf{U}^T\mathbf{b}$ taking no account of columns $\mathbf{U}_j\,(j = 1, ..., n)$ of the matrix $\mathbf{U}$ with indices $j = j_k\ \ (k = 1, ..., s)$ defining $W_{j_kj_k} = 0$. Next we calculate vector $\boldsymbol{\xi}$, dividing all elements $\beta_j\ \ (j \ne j_k,\ k = 1, ..., s)$ by non-zero diagonals W_{jj}, i.e.

$$\begin{aligned} &\xi_j = \frac{\beta_j}{W_{jj}}\ \ (j \ne j_k,\ k = 1, ..., s) \\ &\cdot(\boldsymbol{\xi} = \mathbf{W}^{-1}\ \boldsymbol{\beta} \text{ only if all diagonals } W_{jj} \ne 0) \end{aligned} \tag{A2}$$

All remaining elements $\xi_j\ \ (j = j_k,\ k = 1, ..., s)$ will be equal zero. Then, the first (non-unique) basic solution $\mathbf{x}^* \in R^n$ of the non-homogeneous set of equations $\mathbf{A}\ \mathbf{x} = \mathbf{b}$ will be calculated as $\mathbf{x}^* = \mathbf{V}\boldsymbol{\xi}$. The simple numerical "back-substitution" routine svbksb(...) (see [15]) for obtaining a fundamental solution vector $\mathbf{x}^*$ was implemented in our program. The solutions $\widehat{\mathbf{x}} \in R^n$ (or rather a complete family of the solutions) of the non-homogeneous set of equations

$\mathbf{A}\,\mathbf{x}=\mathbf{b}$ can be written as $\widehat{\mathbf{x}}=\mathbf{x}^{*}+\mathbf{x}^{\mathbf{o}}$, where the linear combinations $\mathbf{x}^{\mathbf{o}}=\sum_{k=1}^{s}\alpha_k\,\mathbf{V}_{j_k}\in R^n$, $\alpha_k\in R$ of the columns $\mathbf{V}_j$ $(j=j_k, k=1,...,s)$ of the matrix $\mathbf{V}$ that correspond to $W_{j_kj_k}=0$, are the arbitrary solutions of the homogeneous set of equations $\mathbf{A}\,\mathbf{x}=\mathbf{0}$. Denoting the base of the kernel of $\mathbf{A}$ by $\{\mathbf{h}_k\}_{k=1,...,s}$, we define the set of all solutions of the equation $\mathbf{A}\,\mathbf{x}=\mathbf{b}$ as

$$\Im=\left\{\mathbf{x}=\mathbf{x}\,(\alpha_1,...,\alpha_s)\in R^n\,|\mathbf{x}=\mathbf{x}^{*}+\sum_{k=1}^{s}\alpha_k\,\mathbf{h}_k,\ \alpha_k\in R\right\}, \tag{A3}$$

where $\mathbf{h}_k=\mathbf{V}_{j_k}$, $k=1,...,s$. This construction makes it possible to represent the statically admissible stress fields in the numerical treatment of problem (P), which is a crucial tool in the present paper.

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