BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 61, No. 1, 2013
DOI: 10.2478/bpasts-2013-0002

CIVIL ENGINEERING

Topology optimization in structural mechanics

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Abstract. Optimization of structural topology, called briefly: topology optimization, is a relatively new branch of structural optimization. Its aim is to create optimal structures, instead of correcting the dimensions or changing the shapes of initial designs. For being able to create the structure, one should have a possibility to handle the members of zero stiffness or admit the material of singular constitutive properties, i.e. void. In the present paper, four fundamental problems of topology optimization are discussed: Michell's structures, two-material layout problem in light of the relaxation by homogenization theory, optimal shape design and the free material design. Their features are disclosed by presenting results for selected problems concerning the same feasible domain, boundary conditions and applied loading. This discussion provides a short introduction into current topics of topology optimization.

Key words: structural optimization, topology optimization, free material design, anisotropic elasticity, compliance minimization, minimum weight design, funicular structures, optimal design of frames.

1. Introduction

Consider the following four problems of optimal design within linear theories of bar structures, plates, shells and solids:

a) find the lightest framework within a given feasible domain transmitting a given loading (possibly a set of concentrated forces) to a prescribed segment of the boundary such that the stresses arising are bounded by $-\sigma_C$ (the limit compression stress) and σ_T (the limit tension stress). This problem is referred to as (P_a);
b) assume that a body is to be constructed within a given domain Ω with prescribed loading applied at a given segment of the boundary and possibly fixed within a different boundary segment. Given two isotropic materials of Kelvin moduli k_α and Kirchhoff moduli μ_α (here $\alpha = 1, 2$) such that $k_2 > k_1$ and $\mu_2 > \mu_1$, place them into a given domain Ω with given amounts (i.e. volumes) V_1, V_2 such that $V_1 + V_2 = |\Omega|$ and that thus constructed non-homogeneous body is capable of transmitting the given loading to the prescribed segment of the boundary and it is less compliant than any other body composed of these two materials, subjected to the same conditions. Thus the layout of two materials, or the surface (contour) being the interface between the materials is to be found. The compliance is understood as the work done by the loading on the displacements of the loaded part of the boundary. This problem is referred to as (P_b);
c) the problem is formulated as (P_b), but the weaker material degenerates now to a void, viz. $k_1 = 0$, $\mu_1 = 0$. Thus the layout problem concerns distribution of one material within the domain Ω, whose volume is bigger than the volume V_2 of the material to be placed. The openings cannot appear on the loaded boundary, to keep the loading unchanged. This problem is referred to as (P_c);
d) the problem is similar to (P_b) but the feasible domain should be filled up with one elastic material of non-homogeneously distributed elastic moduli $C_{ijkl}(x)$ representing arbitrary anisotropy, forming the tensor field $\mathbf{C}(x)$, x being a point of the domain Ω. Tensor $\mathbf{C}$ is subject to the conditions of positive definiteness and usual symmetry of Hooke's tensor. One should impose a resource constraint on the distribution of $\mathbf{C}$ within Ω. It is assumed that the integral of $\Phi(\mathbf{C})$ over Ω is bounded, where $\Phi(\mathbf{C})$ is an isotropic function. Among such anisotropic and inhomogeneous bodies one seeks the least compliant body, satisfying all the required conditions. This problem is referred to as (P_d).

Problems (P_a)-(P_d) may be modified in the following directions:

m1) by changing the merit function; e.g. in (P_a) one can require that the compliance is minimized, while in (P_d) one can minimize a norm of the displacement field;
m2) by taking new bounds on the design into account; it is typical to impose the point-wise conditions of an effective stress (e.g. Huber's effective stress or the norm of the stress deviator);
m3) by admitting more than a single load; in the simplest extensions the merit function, here the compliance, is replaced by the linear combination of compliances corresponding to considered loads, under the condition of their independent application;
m4) by imposing constraints concerning partitioning (degree of multi-connectedness) of the layout; e.g. in (P_c) one

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can fix the maximal number of openings in the domain. In problem (P_b) we may additionally assume that the microstructures which emerge have a given degree of complexity or even a given shape, to make the composite manufacturable;

m5) by admitting transmissible loads or the loads sensitive to the boundary deformation.

The history of development of the methods constructed to attack the problems (P_a)-(P_d) is very interesting, as it accompanies the progress of the following subjects:

– structural mechanics,
– mechanics of deformable bodies,
– mechanics of composites and micromechanics,
– boundary value problems,
– control theory,
– linear algebra and tensor analysis,
– theory of homogenization for elliptic problems,
– theory of duality and theory of saddle-point problems.

Solutions to the problems (P_a)-(P_d) are applicable in various branches of industry, e.g. in automobile or aircraft production, or in civil engineering. This subject has been discussed in a great number of papers presented at biennial World Congresses of Structural and Multidisciplinary Optimization (WCSMO) which took place in Goslar (1995), Zakopane (1997), Buffalo (1999), Dalian (2001), Lido di Jesolo (2003), Rio de Janeiro (2005), Seoul (2007), Lisbon (2009) and Shizuoka (2011), the next to be held in Orlando (2013). The most important papers on this topic are regularly published in the Structural and Multidisciplinary Optimization journal.

Topology optimization has emerged in the paper by James Clerk Maxwell [1], where several solutions to problem (P_a) have been constructed. In these solutions all members are subject to stresses of uniform sign. This paper was an inspiration for Anthony George Maldon Michell (1870–1959) who discovered a broader class of solutions to problem (P_a). Optimal structures, called now Michell trusses, consist of bars, some of which can be in tension and some in compression, see [2]. This remarkable paper was ahead of its time.

The first correct formulations of problems (P_b) and (P_c) have been found in 1970's along with the progress in control theory and homogenization theory, cf. historical remarks by Luc Tartar [3] on how to approach (P_b) and (P_c) correctly. The most important results within the elasticity theory are now available. They are published in the monographs by Cherkaev [4] and Allaire [5], see also Lipton [6].

Historical remarks on the development of shape design – problem (P_c) – can be found in Sokołowski and Zolesio [7]. An important result is due to Wasiutyński [8]. An essential method to detect, where a new hole should be created to improve the design has been developed by Sokołowski and Żochowski [9], using the concept of the *bubble method* of Eschenauer et al. [10].

The problem (P_d) has been formulated and partly solved in Bendsøe et al. [11]. Despite the recent progress done in Haslinger et al. [12], Kočvara et al. [13], this field of topology optimization is still in the stage of infancy.

A common feature of problems (P_a)-(P_d) is a necessity for the a priori reformulation by extending the design spaces. In this way, problems become well-posed. The above-mentioned extension means that:

– in problem (P_a), one should admit pin-jointed frameworks of infinite number of bars, because the sequence of volumes of sub-optimal solutions is decreasing as the number of truss members increases,
– in problems (P_b) and (P_c), the mixture of two materials is admitted at the microstructural level, which leads to the composite-like solutions with anisotropic properties,
– in problem (P_d), the discrete-continuous solutions should be admitted, because this problem turns out to be governed by a minimization problem with the functional of the integrand of linear growth, as noted just recently in [14, 15].

The necessity of extension of the design space is not surprising, see comments in [3–5]. New solutions corresponding to extended design spaces are referred to as relaxed solutions and the problems themselves become *relaxed problems*. They involve notions beyond the scope of the initial formulations. Thus the first step in the process of solving any particular problem falling into any class mentioned above is its reformulation to the relaxed form. This should precede any numerical calculations. The relaxation stage cannot be omitted which was not known until the middle of the 1980s. Due to this fact, many incorrect pseudo-solutions had been published thus making the history of structural optimization extremely quixotic.

The specific feature of topics discussed here is that classical solutions in Michell's sense had inspired the work of architects and civil engineers: Wacław Zalewski, Wojciech Zabłocki, Stanisław Kuś – to mention the experts from Poland only – see the book [16] and the papers [17–20]. Although used in practice, the Michell-like solutions cannot be explained by elementary methods; to understand them correctly one should resort to the papers written by contemporary mathematicians.

Some industrial applications of the extended solutions to (P_d)-like problems have only recently been proposed, since only recently some techniques of controlling the microproperties of composites have been developed. One can say that the content of *topology optimization* lies in the triangle whose vertices are given by contemporary calculus of variations, materials technology and the canons of modern architecture.

2. Problem (P_a) and related issues

2.1. On the solutions to Michell's problem. Majority of these solutions concern the plane case. Assume that a certain truss of n bars, capable of transmitting the given loading to given support, can be treated as a good candidate for the subsequent optimization step. The specific property of problem (P_a) is that there exists a truss of m bars, $m > n$, lighter than the previous truss of smaller number of bars. And this process never stops. This property is implicitly present in the original paper [2], but only the books by Cox [21], Hemp [22] and the reports and papers by A.S.L. Chan and H.S.Y. Chan cited therein established the underpinnings of the theory for the problem (P_a). Let us set the extended formulation of problem (P_a) in case of Ω being two-dimensional. Assume that the domain Ω is reinforced by a bar in tension along Γ_T while the reinforcing bar along Γ_C is compressed. The set $\Sigma(\Omega, \Gamma)$ consists of statically admissible pairs: $(\boldsymbol{\sigma}, F)$ where $\boldsymbol{\sigma}$ is understood in the homogenized (averaged) meaning with σ_{I}, σ_{II} denoting the principal values of $\boldsymbol{\sigma}$, and F stands for the axial force in reinforcing bars. Volume of fully stressed Michell structure is expressed as follows

$$I(\boldsymbol{\sigma}, F) = \frac{1}{\sigma_T}\left[\int_{\Omega}(|\sigma_{\mathrm{I}}| + \kappa|\sigma_{\mathrm{II}}|)\,dx + \int_{\Gamma_T}|F|\,ds + \kappa\int_{\Gamma_C}|F|\,ds\right] \tag{1}$$

and $\kappa = \sigma_T/\sigma_C$. The stress-based formulation of the extension of problem (P_a) reads

$$V = \min\left\{I(\boldsymbol{\sigma}, F)\big|(\boldsymbol{\sigma}, F) \in \Sigma(\Omega, \Gamma)\right\}. \qquad (P_M)$$

The information on the loading are concealed in the definition of $\Sigma(\Omega, \Gamma)$. The averaged field $\boldsymbol{\sigma}$ has nothing to do with the stress field inside the microstructure, where the stress bounds $-\sigma_C$, σ_T are attained. Let $\mathbf{u}$ stand for a kinematically admissible displacement field in Ω; $\boldsymbol{\varepsilon}(\mathbf{u})$ – a symmetric part of $\nabla\mathbf{u}$ and set ε_{I}, $\varepsilon_{\mathrm{II}}$ for the principal strains. Let $\varepsilon_\Gamma(\mathbf{u})$ denote the axial strain. Next, introduce the sets – the locking loci, as they are named in the theory of locking bodies, [23, 24], see also papers on rigid plastic bodies [25, 26]:

$$\begin{aligned} B_\kappa &= \left\{\boldsymbol{\varepsilon} \in E_s^2 \Big| |\varepsilon_{\mathrm{I}}| \le 1, |\varepsilon_{\mathrm{II}}| \le \kappa\right\}, \\ \beta_\kappa &= \{\varepsilon_\Gamma \in R\,| -\kappa \le \varepsilon_\Gamma \le 1\}. \end{aligned} \tag{2}$$

Write $f(\mathbf{u})$ for the work of the loading on the field $\mathbf{u}$. One can prove that the problem dual to (P_M) reads

$$\begin{aligned} V = \max\Big\{\frac{1}{\sigma_T} f(\mathbf{u})\Big|\,&\boldsymbol{\varepsilon}(\mathbf{u}(x)) \in B_\kappa, \\ &x \in \Omega, \varepsilon_\Gamma(\mathbf{u}(s)) \in \beta_\kappa, s \in \Gamma\Big\}. \end{aligned} \qquad (P_M^*)$$

The equivalence of (P_M) and (P_M^*), discussed in [27, 28], is referred to as *the Michell theorem*. The problem (P_M^*) has the structure of the statics problem of a body with locking. Since the integrands in (P_M) are of linear growth, the stress fields should be sought in the space of measures, see [23].

A net of parametric lines along which $\varepsilon_{\mathrm{I}} = 1$, $\varepsilon_{\mathrm{II}} = -\kappa$ is referred to as the Hencky net, since such nets are very well-known in the plasticity theory. The foundation of the theory of the Hencky nets has been created by Carathéodory and Schmidt [29]; just there the analytical construction based on Riemann's method has been given, upon deriving the governing equation of the net in the form of the hyperbolic equation: $\partial^2\phi(\alpha, \beta)/\partial\alpha\partial\beta = 0$.

Detailed analysis of solutions to problems (P_M) and (P_M^*) leads to the conclusion that the net of bars in optimal structures coincides with the Hencky net in these regions where the bars are mutually orthogonal. The optimal solution can include also the fan-like domains where the net is composed of bars in one direction, as well as domains which are completely empty. Each solution must be analyzed individually and corresponding construction needs an individual approach.

Given the set of bars one can tackle the problem of computing the stress components and the axial forces F in the reinforcing bars (they exist in problems with point loads, to prevent the stress field from being singular). These data suffice to compute the volume of the lightest structure. Alternatively, the same volume can be found by using the dual formula (P_M^*) or by computing the virtual work done on the adjoint displacement associated with the peculiar strain field, being the maximizer of this problem. Equivalence of two results found as described above is called *the condition of zero duality gap* between (P_M) and (P_M^*). This equivalence confirms correctness of the particular solution. Although it is recommended to have such a check it is not easy to find these verifications. The exceptions are: some most elementary problems solved in Hemp [22], the cantilever designed in the exterior of the circle [30] and the cantilevers designed in trapezoidal domains [31–36].

The majority of exact solutions to the Michell problem have been found by guessing the geometry of the nets of bars, this being the basis for checking statical admissibility with the given loading. Such a class of solutions – underpinned by the geometry of the Hencky nets – is referred to as *the G class*. However, other Michell structures exist, forming *the S class*, in which the geometry of the net is determined by the static equations, including the data on the loading. An important subclass of G are Michell cantilevers, transmitting a given point load to a given supporting line. Not every point of such line is then used as a supporting hinge – in this class of problems the position of supports is also unknown hence it has to be found via optimization! The cantilevers have a remarkable feature: the underlying net is stable with respect to some changes of the load direction and, to some extent – with respect to the position of the load. One can say that optimal cantilever is cut out of the given net. For other directions of loading, which are not admissible for this solution, the cantilever disappears and the topology of optimal solution jumps towards a completely different layout, sometimes reduced to one bar, see comments in [28, 33–41].

The S class is more complex and it is still not examined in full detail. Consider a particular problem of transmitting two forces to two fixed hinges. Two cases of symmetric position of two point loads, for the case of $\sigma_T = \sigma_C = \sigma_p$, are dealt with in Fig. 1. Depending on the position of the forces P the Hencky net changes its geometry. The problem cannot be separated into the geometric problem of the net construction and the static problem of finding the stresses. Analytical solutions in Figs. 1b) and 1c) have been found by Sokół and Lewiński in [40]. These solutions are characterized by different topologies with a lower horizontal bar appearing in the latter layout to make the structure equilibrated. Both structures are statically determinate, despite the existence of infinite number of bars in fan-like domains. The structure in Fig. 1b) is geometrically unstable but the possible zero-energy deformation (in which bars are not subject to elongations) is such that the points where loads are applied move horizontally. This in turn assures vanishing of the virtual work, the condition of correctness of the equilibrium problem.

One can say that both solutions are taken from the half-plane in which virtual vector field $\mathbf{u}$ is constructed as such that the corresponding principal strains are equal to $\pm 1/\sigma_p$ in bi-directional fan-like domains and the strains are smaller in other directions. An optimal structure is imbedded into such a field, the idea of imbedding was originated by Maxwell [1].

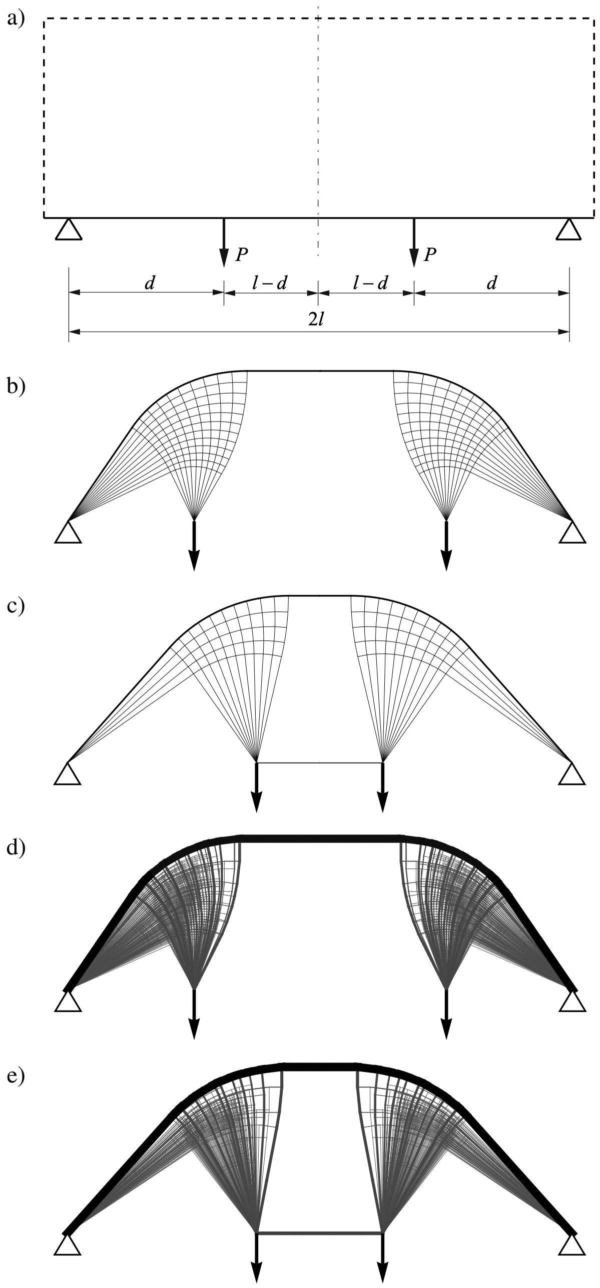

Fig. 1. The Michell problem: transmit two given forces to two fixed hinges; feasible domain is the upper half of a plane (dashed line): a) problem setting; b) exact solution for the case of $d = 0.5\ l$, after [40]; c) exact solution for $d = 0.75\ l$, after [40]; d) and e) numerical solutions by T. Sokół, for the ground structure composed of approximately 125 million potential members

We note that the process of optimization of trusses extends the family of trusses to a broader one consisting of certain 2D continuum bodies reinforced by boundary ribs. This extension links the truss theory with the theory of plane stress. The result also suggests that one should account for the theory of discrete-continuous structures from the very beginning of the optimization process. The specific feature of problem (P_a) is that material constants are absent in the formulation. Consequently, optimal structures become statically determinate, because only then the stresses can be found directly from equilibrium conditions.

Analytical solutions shown in Fig. 1b) and Fig. 1c) can be confirmed by using the ground structure method, originally proposed in Dorn et al. [42]. Problem (P_a) can be reduced to the following remarkable form, very attractive from the numerical viewpoint, see Hemp [22], Achtziger [43], Pritchard et al. [44], Gilbert and Tyas [45], Tyas et al. [46], Sokół [47]

$$\begin{aligned} \min \{ \ & \mathbf{L}^T\mathbf{T} + \kappa \mathbf{L}^T\mathbf{C} \mid \mathbf{B}^T(\mathbf{T}-\mathbf{C}) = \mathbf{P}, \\ & \mathbf{T} \geq \mathbf{0}, \mathbf{C} \geq \mathbf{0},\ \mathbf{T}, \mathbf{C} \in R^m \}, \end{aligned} \qquad (N_a)$$

where $\mathbf{L}$ is the column of bar lengths, $\mathbf{B}$ is the geometric matrix and $\mathbf{P}$ is the column of loads. Sokół [47] has developed a computer program (in Mathematica) based on the formulation above. An access to the program is open, see [48]. This program makes it possible to approximate the analytical solution of Michell problem with accuracy of a fraction of 0.1%. For instance, the numerical solution of Figs. 1d, e, used over 125 million of possible bars and 20 thousand of nodes; the volume of the optimal truss shown and the exact volume equal $V = 3.77092\ Pl/\sigma_p$ and $V = 4.64170!\ Pl/\sigma_p$, respectively. The truss constructed numerically is structurally unstable, but such solutions are admissible within the formulation (N_a) in which no matrix is inverted, the equilibrium equations playing the role of the subsidiary conditions.

Since Michell's solutions cannot be constructed algorithmically, numerical predictions like that shown in Fig. 1d, are of essential importance: they inspire us to predict exact solutions. A check of correctness is obviously easier than the creation of an exact layout. Let us now turn to a similar problem to that of Fig. 1, i.e. let us consider a bigger number of equal vertical forces, still applied along the line linking unmovable supporting hinges. The feasible domain is still the top half-plane. Numerical solutions in Fig. 2b), 2c) and 2d), obtained by T. Sokół, were never published before (the available numerical solutions in McConnel [49] are much less accurate). They disclose the sequence of layouts tending to a very strange and complicated layout for the case of the vertical load uniformly distributed between the supports. This numerical layout delivers a strong suggestion of how the exact layout looks like, yet the latter is still unknown.

Only limited number of exact analytical solutions of Michell class are available. The simplest problems have been published by Cox [21] and Hemp [22], where important results found by A.S.L. Chan and H.S.Y. Chan in the 1960s are also reported, including the cantilever, transmitting a point load to a straight segment of the line support. The cantilever within the exterior and interior of a circle have been found in

[30]. Cantilevers within trapezoidal domains have been discussed in [28, 31–36]. The cantilevers in the exterior of polygons were found in [37, 39], while the structures within L-shape domains were dealt with in [38] and [50]. The problem of transmitting a uniform load to the successive supports located at equal distances has been solved in [51]. The problem in Fig. 2d) for a uniform load has been partly solved in [46, 52, 53]. The three forces problem has been solved partly in [40]; where also the related problem illustrated in Fig. 1 is solved. The problem similar to that of Fig. 1 with the feasible domain being the full plane has been recently solved in Sokół and Rozvany [54]. The problem related to Fig. 1 but with one roller has been solved in Sokół and Lewiński [55, 56] and Sokół and Rozvany [54]. New Michell's structures for different permissible stresses in tension and compression with taking into account the cost of supports have been put forward recently by Rozvany and Sokół [57].

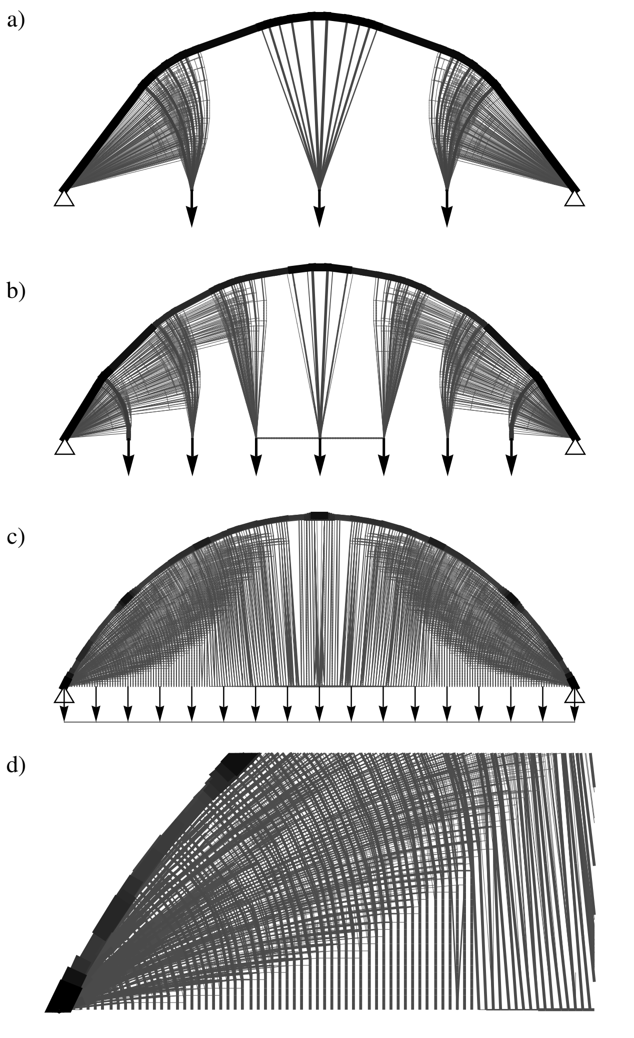

Fig. 2. The Michell problem: transmit given vertical forces to two fixed hinges; feasible domain is the upper half of a plane: a) numerical solution for three forces of magnitude P; b) numerical solution for seven forces of magnitude P; c) numerical solution for the case of 199 forces of equal magnitude; d) magnified part of a structure near the left support. Distances between forces are kept equal in each example. All solutions found by T. Sokół

The only spatial Michell type solution concerns the axisymmetric torsion problem: find the lightest fully stressed structure between two co-axial rings subject to a torsional loading. The proof that Michell's solution is a grid-work in the shape of a shell of revolution is unknown; we know only the proof that the spherical shell is the lightest among shells of revolution in torsion, see [58].

Minimization in (P_a) leads to the lightest truss of minimal compliance $f(\mathbf{u})$ where $\mathbf{u}$ is the elastic solution, see Theorem 2.7 in Achtziger [43]. Therefore, designing for least weight is in a certain sense equivalent to designing for maximal stiffness.

The idea of imbedding a structure into a displacement field, lying behind the (P_M^*) formulation, finds its application in topology optimization of trusses of finite number of bars. In the thesis by Bojczuk [59] a kinematic criterion is used in the updating process.

Exact solutions of problem (P_a) help us in finding sub-optimal trusses satisfying the desired conditions with appropriate accuracy. Already William Prager noted some remarkable discrete analogies between geometric properties of Hencky nets and geometric properties of trusses of finite number of members which mimick exact layouts, see [60, 61]. Recent work by Mazurek et al. [62] follows this direction.

Other method is applied in Rychter and Musiuk [63], where the truss layouts are a priori assumed to be composed of rectangular cells.

2.2. Topology and geometry optimization of spatial trusses and frames of finite number of bars. Let us consider a truss composed of m members and N nodes, subjected to the conservative nodal forces $\mathbf{Q} \in R^D$. The problem of finding node coordinates $\mathbf{X} \in R^D$ (geometry optimization) and cross sections $\mathbf{A} \in R^m$ of members (topology optimization) for which nodal displacement vector $\mathbf{q} \in R^D$ minimizes the truss compliance $\mathbf{Q} \cdot \mathbf{q}(\mathbf{A}, \mathbf{X})$ under the condition of the truss volume being smaller than a fixed volume V is formulated as follows

$$\min_{(\mathbf{A},\mathbf{X}) \in R^m \times R^N} \left\{ \mathbf{Q} \cdot \mathbf{q}(\mathbf{A},\mathbf{X}) \middle| \mathbf{K}(\mathbf{A},\mathbf{X})\mathbf{q}(\mathbf{A},\mathbf{X}) = \mathbf{Q}, \right. \qquad (P_T)$$
$$\left. \mathbf{A} \geq \mathbf{A}_{\min}, \mathbf{X}_{\min} \leq \mathbf{X} \leq \mathbf{X}_{\max}, \mathbf{A} \cdot \mathbf{L}(\mathbf{X}) \leq V \right\}.$$

In a spatial truss, $D = 3 \times N$ represents a total number of degrees of freedom in a truss, **K**(**A**,**X**) represents its stiffness matrix, **L**(**X**) represents the vector of all bar lengths, $\mathbf{A}_{\min} \in R^m$, $\mathbf{X}_{\min}$, $\mathbf{X}_{\max} \in R^D$ respectively represent vectors of feasible minimal cross sections, minimal "left" and maximal "right" coordinates (box limitations). Elastic equilibrium equation for the geometrically non-linear case in the total Lagrange formulation has the form **K[A,X,q(A,X)]q(A,X)=Q**. Dependence on the boundary conditions should be properly taken into account in (P_T). The problem of designing the stiffest space frame structure, constructed by straight bars with a given and fixed number of joints and elements connections can be formulated similarly as in (P_T). Due to the dyadic form of a stiffness matrix of a truss, formulae for the sensitivity analysis can be easily derived by using well-known theorems of the advanced calculus. The derivation of analytical formulae for all partial derivatives $\partial(\ldots)/\partial\mathbf{A}$ and $\partial(\ldots)/\partial\mathbf{X}$ for the spatial frame structure is more complicated, but still possible because very efficient symbolic computation systems such as e.g. Maple Computer Algebra System, are available. Analytical and symbolic formulae found with the help of Maple can be next automatically converted into the code of the functions in C or in Fortran language. Sensitivity formulae enable to implement any gradient oriented optimization algorithm, e.g. the Method of Moving Asymptotes (MMA). Paper [64] discusses in detail many issues of the formulated above problem (P_T) for geometrically linear and geometrically non-linear spatial trusses. Papers [65, 66] present the problem of designing the stiffest, geometrically linear spatial frame structure, together with the complete and analytical formulae of the sensitive analysis. Examples below show optimal layouts of few trusses and frames.

In the first example, optimal topology and geometry of a ground structure having a form of a cantilever truss is found by MMA, see Fig. 3. The initial cross-sections are taken as equal to $1.963{\cdot}10^{-5}$ [m^2]. Young modulus equals $200.0{\cdot}10^9$ [Pa]. A single vertical force = $1.0{\cdot}10^4$ [N] is applied at the middle of the right side of a truss. The initial dimension of a "repetitive" quadratic cell is 1.0 [m]. Initial and optimal compliances in geometrically linear and non-linear cases are equal to 92.7, 14.4 and 14.5 [Nm], respectively.

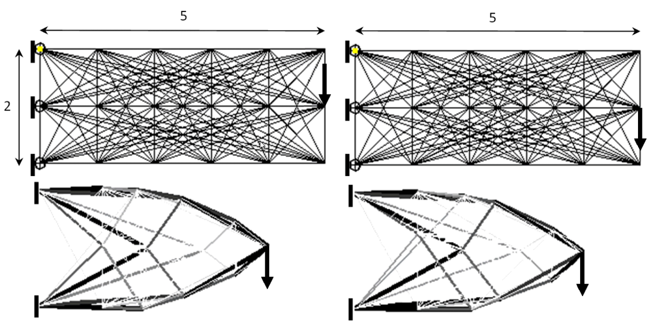

Fig. 3. Geometry and topology optimization. Initial and optimal layouts are shown in upper and lower figures respectively. Solutions to the geometrically linear and non-linear cases are illustrated in left and right figures respectively. The results were obtained by S. Czarnecki with help of the MMA method

In the second example, optimal topology and geometry of the "cylindrical" latticed cantilever shell ("radius" of the cylinder = 1.0 [m]) is found, see Fig. 4. Nodes at the bottom side of the truss are fixed. The horizontal unit forces = 1.0 [kN] (along the horizontal axis at top view) are assumed to be placed at all nodes (sort of a simplified wind load model). The initial compliance is equal to 8.3 [kNm] and the best compliance found by MMA is equal to 2.1 [kNm]. An optimal layout reminds, to some extent, the tulip-like shape of the high-rise, multi-storeyed building, taller than 200 m, proposed by W. Zalewski and W. Zabłocki [18–20] as a counterpart of the Michell-Hemp cantilever in three dimensions, (c.f. [67]).

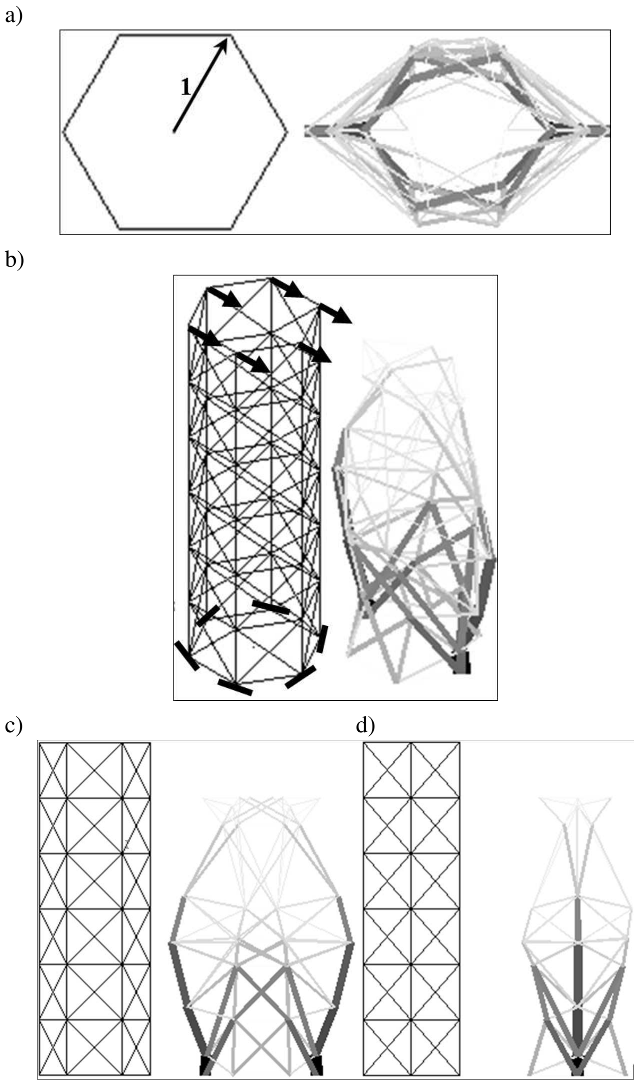

Fig. 4. Geometry and topology optimization of the "cylindrical" latticed cantilever shell. Initial and optimal layout: a) top view, b) right isometric view, c) front view, d) right view. The results were obtained by S. Czarnecki with help of the MMA method

The optimal shape much more clearly motivates the tulip-like shape of the high-rise building proposed by W. Zalewski and W. Zabłocki if the number of design parameters is significantly reduced to a small number of variables defining the horizontal positions of nodes represented by a vector $\mathbf{X}$ (vertical positions of nodes are assumed to be constant), e.g. to unknown Bernstein polynomials coefficients defining the meridional shape of the axial symmetrical cylindrical latticed shell, see [68]. In the third example, see Fig. 5, a very simplified truss model of a skyscraper is defined in such a way that the appropriate Cartesian pattern (similar as in the previous example, but much denser), is projected onto a cylindrical layout of the latticed shell (envelope structure). Horizontal forces are assumed to be placed at all top nodes while all bottom nodes are supported. Initial data: height = 185.0 [m] and "radius" = 30 [m]. Area of each cross section and Young modulus of bars are equal to 0.1 [m^2] and $1.0{\cdot}10^9$ [N/m^2], respectively. Value of each horizontal force placed at top node of the cylinder equals $3.0 \cdot 10^3$ [N]. Value of the radius of axial symmetrical structure can change from 20 [m] to 50 [m]. Initial compliance equals 240772 [Nm]. Optimal compliance equals 235888 [Nm]. The initial and optimal shape of the latticed shell is shown in Fig. 5 (c.f. [68]). In this case, the optimal shape can clearly motivate the tulip-like shape of the high-rise building. The reduction of compliance (increase of stiffness) is not as significant as in case of classical plane Michell cantilever (contrary to the previous example, only geometrical design was analysed and topological changes were not allowed) but the shape of a great deal of trees and many other plants nearing cylindrical one suggests that its modifications and improvement has significantly limited range.

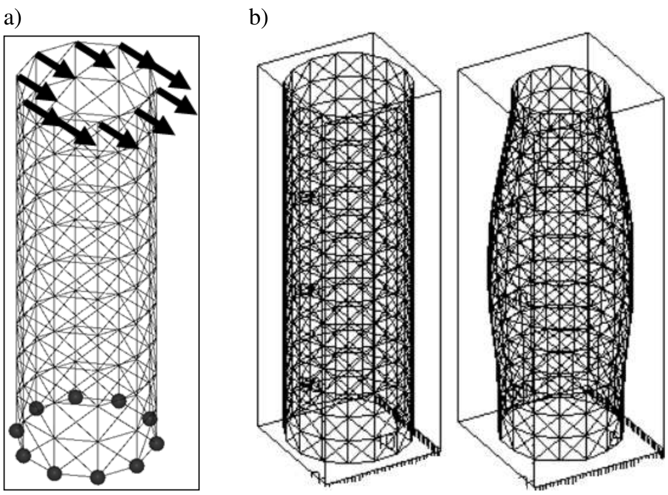

Fig. 5. Geometry optimization. Simplified model of the initial pattern of the cylindrical latticed shell: a) loading and supports, b) initial and optimal shape. The results were obtained by S. Czarnecki in Ref. 67

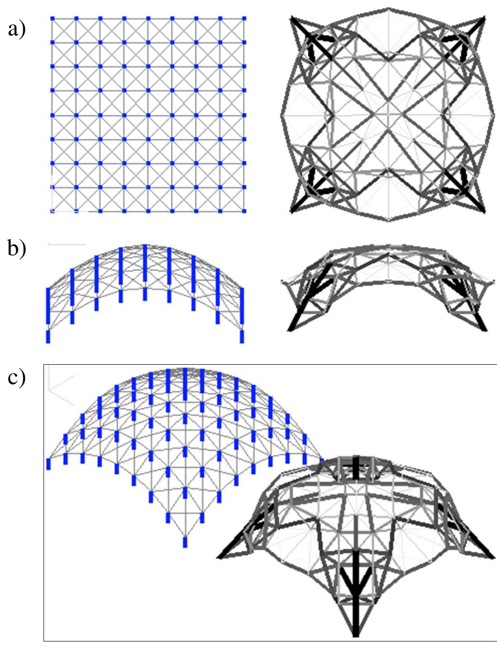

Fig. 6. Geometry and topology optimization of a spatial frame structure modelling a shell-like ceiling. Initial and optimal layout: a) top view, b) front view, c) right isometric view. The results were obtained by S. Czarnecki in Ref. 65

In the last example, fully optimal solution (topology and shape) was found for the geometrically linear, spatial frame structure modelling the shell-like ceiling, Fig. 6. The initial structure has the shape of the flat quadrilateral pattern composed of cylindrical bars shown in Fig. 6. Equal vertical forces are placed at all nodes (simplified gravitational loading).

Let us note, that other interesting numerical layouts of optimal spatial frameworks have been reported in Pritchard et al. [44].

2.3. Prager and Michell structures for transmissible loads. Assume that the loading in a plane problem (P_a) is one-directional and can change its application points along this direction. Usually this direction is vertical, due to gravity. Assume additionally that stresses in the structure are of uniform sign. Then the problem (P_a) is referred to as *the William Prager problem*. Old results teach us that the solutions are funiculars, or the arches (with hinges) of shapes $y = f(x)$ proportional to the shape of the diagram of the bending moment in the simply supported beam subjected to the same load acting directly on the beam. Funicular structures are not bent and are not sheared across their sections; the axial force alone is capable of assuring the equilibrium of any segment of the funicular. Among all funiculars determined by a given loading, the lightest one is unique. It is called the Prager structure. Its rise is given by the formula

$$\frac{1}{L}\int_0^L \left(\frac{df}{dx}\right)^2 dx = 1 + 2\left(\frac{h}{L}\right)^2, \tag{3}$$

where h stands for the vertical distance between the levels of the supports and L denotes the span of the funicular. The derivation of (3) from the formulation similar to (P_M^*) has been given by Rozvany and Wang [69, 70]. The locking locus is here an infinite domain due to the condition of the stress being of uniform sign. The elementary derivation of (3) can be found in [71]. In case of vertical, uniform and transmissible load, the Prager structure is the parabolic arch inscribed into the equilateral triangle. Thus the Prager structures are funiculars of appropriate rise. A slight change of the formulation – omitting the condition of all the bars being in compression (or tension) – changes the Prager class of solutions to Michell class (in which both tension and compression are allowed) but for the transmissible loads. This change in formulation for the uniform transverse load problem changes the solution: from the parabolic arch to a highly complicated structure in which the arch is reinforced by a net of bars, some of them being hangers for the vertical load; only the middle part of the structure is represented by an arch, this new spectacular solution has been recently discovered by Tyas et al. [46]. The solution is remarkable, since adding the bars in tension to the funicular in compression brings about a decrease in weight.

2.4. Prager-Rozvany grillages. The problem (P_a) can be reformulated to the out-of-plane case if the loading is assumed perpendicular to the plane design domain, see the books by George Rozvany [72, 73], the literature discussed there, especially the papers by Lowe and Melchers cited therein. The principal stresses are then replaced by principal moments and the axial force is replaced by the normal moment bending the edges along the ribs. A similar yet not identical formulation of the grillage problem can be inferred from the theory of relaxation by homogenization of the minimal compliance problem of thin perforated plates with small volume, see [24].

2.5. Applications in civil engineering. The problem (P_a) is a typical problem for structural engineer: transmit a given loading to given support in an optimal manner. A naive approach to this problem is to assume that the design domain is filled up with an isotropic and homogeneous material, then to solve the elasticity problem and finally to find the principal stress trajectories. In case of steel structures – to design a system of bars which roughly follow the trajectories of stress. In case of reinforced concrete – to design a truss and then a system of reinforced bars along the lines related to tension and neglect the bars in compression by using the ability of concrete to resist it. The method described above is incorrect, since the trajectories of stress should refer to the optimized structure. An essential difference between stress plots corresponding to bodies before and after optimization of material properties is shown in Sec. 4. Contemporary progress in topology optimization makes it possible to re-formulate the process of design of the reinforced concrete structures, taking into account the non-linear behaviour of concrete and tension stiffening effects, see Amir and Sigmund [74].

The Prager solutions (Sec. 2.3) are of fundamental importance for designing roofs and girders to cover large spans, since such structures should be simultaneously stiff and light. One can say that the whole book by Zalewski and Allen [16] is devoted to the problem of how to cover the large span and this problem is mainly solved with using funiculars, if self-weight prevails.

Designing of high-rise buildings needs other optimization tools and other concepts. The area of the support is relatively small, its high prize suggests tulip-like shapes, as those proposed by Zalewski and Zabłocki [18–20]. The wind loading becomes an important factor, it determines the shape in space. Since Michell's cantilevers fixed on small support areas are tulip-like to resist the lateral loading, just these shapes appear in Refs [18–20].

Exact solutions to problems (P_a) had inspired engineers in the past and will inspire them in the future to find a compromise between weight and stiffness of a structure. Numerous applications are given in Kuś and Zalewski [17]. Fantastic buildings have been recently designed by the architects and engineers from SOM LLP. They are inspired by topology optimization solutions and by Michell's trusses in particular, see [75, 76].

3. Problems (P_b) and (P_c) and related issues

3.1. Two-material problem and generalized shape design. Consider a sequence of partitions of given domain Ω into two subdomains Ω_1, Ω_2, occupied by materials "1" and "2", respectively. Assume that the compliance corresponding to each partition in a sequence is smaller than the one related to the previous division of Ω. Numerical tests show that subsequent partitioning leads to the decrease of compliance. In other words, the process of consecutive partitioning, driven by the need to decrease the compliance, never stops. This experiment suggests that an inhomogeneous body composed of two materials is not the most compliant one but this role is played by a composite of certain microstructure. At each point x of Ω one can reveal a representative volume element (RVE) with a clear-cut partition into both constituents. Therefore, the problem of compliance minimization must be a priori reformulated to its relaxed setting, in this way admitting two-phase composites as possible solutions from the very beginning of an optimization process. Explicit formulation of this relaxed problem in 2D in the context of linear elasticity has been found in the 1980s. Detailed lecture on this topic is provided in the books by Cherkaev [4] and Allaire [5]. All details concerning the thin plate optimization can be found in Lewiński and Telega [77], Lewiński [78] and Dzierżanowski [79, 80]. The relaxed 2D problem of minimal compliance for the linear elasticity case reduces to

$$\min\left\{ J(m_2) \middle| m_2 \in L^\infty\left(\Omega,[0,1]\right), \int_\Omega m_2\left(x\right)dx = V_2 \right\}, \tag{$P_{1,2}$}$$

$$J(m_2) = \min\left\{ \int_\Omega 2W^*\left(\boldsymbol{\sigma}, m_2\right)dx \middle| \boldsymbol{\sigma} \in \Sigma(\Omega)\right\}, \tag{4}$$

where $\Sigma(\Omega)$ represents the set of statically admissible tensor fields, while the potential $W^*\left(\boldsymbol{\sigma}, m_2\right)$ is a given, explicitly defined isotropic function of the first argument and a rational function of the second argument. Its definition can be found in [77–79]. Scalar m_2 represents density of material "2".

Let us stress here the following: reformulation of (P_b) into the form $(P_{1,2})$ is a great success of many experts engaged in the homogenization theory, optimum design and the control theory. Potential $W^*\left(\boldsymbol{\sigma}, m_2\right)$ conveys the information on the underlying microstructure. There are many microstructures leading to this potential, the simplest among them are in-plane laminates of 1st and 2nd rank, see Cherkaev [4]. Loosely speaking, $W^*\left(\boldsymbol{\sigma}, m_2\right)$ is defined by three functions, depending on the value of stress invariant $\varsigma_\sigma = \left|\left(\sigma_{\rm I} - \sigma_{\rm II}\right) / \left(\sigma_{\rm I} + \sigma_{\rm II}\right)\right|$ in three domains $[0, \varsigma_2]$, $[\varsigma_2, \varsigma_1]$, (ς_1, ∞). Here ς_2, ς_1 denote the material parameters depending on m_2, k_2, k_1, μ_2, μ_1, see [4]. The process of optimization leads to the division of Ω into three subdomains in which ς_σ belongs to $[0, \varsigma_2]$, or $[\varsigma_2, \varsigma_1]$, or (ς_1, ∞). Interfaces between subdomains correspond to $\varsigma_\sigma = \varsigma_1$ or $\varsigma_\sigma = \varsigma_2$. These above-mentioned ranges for ς_σ are referred to as *regimes*, as proposed in [4]. In each regime, the underlying optimal microstructure is different, but

– the passage between regimes is smooth,
– trajectories of stress and strain coincide everywhere in Ω,
– optimal microstructures are orthotropic – as are the Hencky nets in problem (P_a),
– microstructural orthotropy does not contradict isotropy of the potential $W^*(\boldsymbol{\sigma}, m_2)$ with respect to the first argument.

Remark 3.1. The problem $(P_{1,2})$ allows for the passage to the limit: $k_1 \to 0$, $\mu_1 \to 0$, which means that resulting composite is a porous medium whose solid fraction is characterized by moduli k_2, μ_2. The majority of papers on topology optimization refers to the problem of optimal distribution of one material within the given domain to achieve minimal compliance of a structure. This problem is denoted $(P_{0,2})$ in the sequel and it is referred to as *generalized shape design*. Instead of m_2 we write $\rho = m_2$. Potential $W^*(\boldsymbol{\sigma}, m_2)$ simplifies to the form

$$2W^*(\boldsymbol{\sigma}, \rho) = 2W_0^*(\boldsymbol{\sigma}) - G(\boldsymbol{\sigma}) + \underline{\frac{1}{\rho}G(\boldsymbol{\sigma})}, \tag{5}$$

where

$$\begin{aligned} 2W_0^*(\sigma) &= \frac{1}{E_2}\left(\sigma_{\mathrm{I}}^2 + \sigma_{\mathrm{II}}^2 - 2\nu_2\sigma_{\mathrm{I}}\sigma_{\mathrm{II}}\right), \\ G(\sigma) &= \frac{1}{E_2}\left(|\sigma_{\mathrm{I}}| + |\sigma_{\mathrm{II}}|\right)^2 \end{aligned} \tag{6}$$

and E_2, ν_2 respectively denote the Young modulus and the Poisson ratio of material "2".

Remark 3.2. If the volume V_2 tends to zero, $(P_{0,2})$ reduces to the problem which is very similar to (P_M) since the underlined element in (5) is a dominating term, see Sec. 4.2.3 in Allaire [5]. Thus we expect that solutions to $(P_{0,2})$ with small value of $V_2/|\Omega|$ have a skeletal structure.

Solutions to problem $(P_{1,2})$ involve composite domains, hence are difficult to manufacture. Thus many authors have developed numerical schemes to penalize composite areas in a structure thus forcing the density ρ to be equal either to 0 or to 1; these techniques are discussed in [81–83] and the papers cited therein. The most popular technique to penalize the composite domains in problems $(P_{0,2})$ is called SIMP (Solid Isotropic Material with Penalization), cf. Bendsøe [84] and Zhou and Rozvany [85]. Phrase "Isotropic Material" is a part of an acronym hence the penalization factor p in $E(\rho) = \rho^p E_2$ needs to satisfy $p \geq \max(2/(1-\nu_2), 4/(1+\nu_2))$, if SIMP is to be used for mimicking the constitutive behaviours of a solid-void isotropic composite in two-dimensional elasticity framework, see [84]. Authors try to obtain an agreement between numerical results and skeletal-like solution to (P_a), which is justified due to Remarks 3.1 and 3.2. Thus the SIMP approximation of the potential $W^*(\boldsymbol{\sigma}, \rho)$ in problem $(P_{0,2})$ reads

$$W^*_{\mathrm{SIMP}}(\boldsymbol{\sigma}, \rho) = \frac{1}{\rho^p} W_0^*(\boldsymbol{\sigma}). \tag{7}$$

As an alternative to the heuristic scheme provided by SIMP, a different solid-void material model with penalization may be derived directly from (5) by smoothing the expression for the energy potential. Consequently, one obtains a two-parameter family of functions approximating $W^*(\boldsymbol{\sigma}, \rho)$. Details of this approach were proposed by Dzierżanowski, see [86]. The simplest, one-parameter formula in the above-mentioned family reads

$$W^*_{\mathrm{GRAMP}}(\boldsymbol{\sigma}, \rho) = W_0^*(\boldsymbol{\sigma}) + \frac{(1+q)(1-\rho)}{2E_2}\left(\sigma_{\mathrm{I}}^2 + \sigma_{\mathrm{II}}^2\right), \tag{8}$$

where GRAMP stands for "Generalized Rational Approximation of Material Properties" as it is shown in [86] that (8) justifies and generalizes the RAMP scheme of Stolpe and Svanberg, see [87]. In the GRAMP scheme it is assumed that $q \geq 3$ to assure isotropy of the material. Comparison of various solid-void interpolation models shows that GRAMP provides better accuracy of approximation of the exact solution to $(P_{0,2})$, especially for auxetic materials, see the discussion in [86].

Let us come back to the problem illustrated in Fig. 1a. We admit loading, supports and feasible domain as in Sec. 2, but the problem is formulated now as $(P_{0,2})$ or by using either of the approximations: SIMP, see (7), or GRAMP, see (8). Position of forces are assumed as in Fig. 1b, viz. $d/l = 0.5$ or as in Fig. 1c, i.e. $d/l = 0.75$. Computations were performed for the isoperimetric condition in the form: $V_2/|\Omega| = 0.1$. Plots of the density ρ, predicted by $(P_{0,2})$, SIMP or GRAMP are shown in Fig. 7.

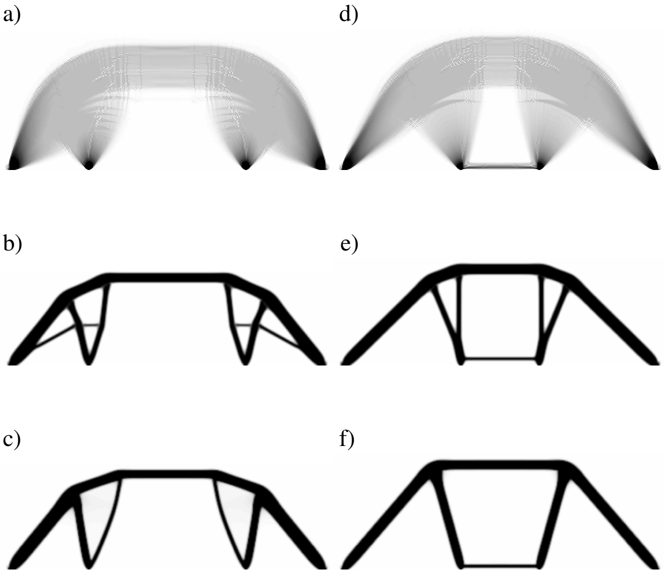

Fig. 7. The problem illustrated in Fig. 1a in the $(P_{0,2})$ setting. Plots of the material density based on exact solution to $(P_{0,2})$ and approximate "0-1" topologies provided by SIMP and GRAMP models: a) $d = 0.5l$ – exact; b) $d = 0.5l$ – SIMP; c) $d = 0.5l$ – GRAMP; d) $d = 0.75l$ – exact; e) $d = 0.75l$ – SIMP, f) $d = 0.75l$ – GRAMP. All results obtained by G. Dzierżanowski

Plane design area Ω was discretized by the Q1 elements, i.e. 4-node square finite elements with bi-linear shape functions. The number of elements was equal to 68000.

Homogenization-based solution of problem $(P_{0,2})$ was obtained with help of numerical integration scheme with 4 integration points in each element. These points were also used to calculate updated values of material density. In this way, numerical instability known as "the checkerboard effect", see Sigmund and Petersson [88] was significantly reduced. The SIMP- and GRAMP-based solutions were obtained on the same 68000 Q1 element mesh but the updated values of the material distribution field were calculated in the middle of each element. In order to avoid the checkerboard instability and to obtain the manufacturable, albeit sub-optimal, solution the density filter was applied, see Bourdin [89].

Alternative SIMP versions have been proposed by Kutyłowski [90].

Remark 3.3. The relaxation by homogenization method to make the problem $(P_{1,2})$ well-posed can also be applied for thin plate optimization [77–79, 91], sandwich plates optimization [92], see also Studziński et al. [93] and Díaz et al. [94] and shallow shells [77–79, 95], see also Krog and Olhoff [96]. Most of the homogenization formulae for plates and shells can be found in [77].

Remark 3.4. The most important theorems on problems (P_b), (P_c) in the 3D setting can be found in Cherkaev [4] and Allaire [5]. In particular, one can prove that optimal microstructures are orthotropic and take a form of a rank-3 laminate. Numerical methods have been developed in Allaire et al. [97], Borrvall and Petersson [98] and in [99].

3.2. On the shape design. The method of relaxation discussed in Sec. 3.1 predicts composite materials in optimal solutions. In case of one-material optimal distribution, the relaxed form a problem admits porous designs, and such designs indeed appear, as shown in Fig. 7. Yet the shape design problem can be formulated in a different manner. One can assume that the design domain preserves the index of connectedness. Consequently, no small voids appear in the solution and it is not necessary to look for the generalized shape design. Conventional methods of shape design can be found in Pironneau [100]. They are based on the sensitivity results, see Sokołowski and Zolesio [7]. In problem (P_c), where the shape is unknown, the necessary condition of optimality says that the density of energy is constant along the boundary. This condition can be linked with the name of Wasiutyński, see [8]. In 2D problems it means that the hoop stress is constant along the contour of optimal shape, and this feature is valid also along the contour of optimally shaped openings. Effective numerical schemes in 3D shape optimization with applications to biomechanics, developed by Nowak [101], was based on this observation.

Classical shape optimization methods and sensitivity theory were unable to improve the solution by changing topology, viz. by introducing an opening. This task is now possible to tackle by the topological derivative method, developed by Sokołowski and Żochowski [9]. The method is closely related to some techniques developed in mechanics of composites, as discussed in [102].

4. On free material design (FMD) – problem (P_d)

Let the design domain be plane. If $\mathbf{u}$ is a solution associated with given distribution of moduli represented by tensor field $\mathbf{C}(x)$, then

$$\wp(\mathbf{C}) = f(\mathbf{u}(\mathbf{C})), \tag{9}$$

denotes the compliance treated as a function of $\mathbf{C}$. Let $\langle f \rangle = |\Omega|^{-1} \int_\Omega f dx$. Let E_0 be a reference Young's modulus. We consider distributions of $\mathbf{C}$ within the domain Ω such that

$$\langle \Phi(\mathbf{C}) \rangle = E_0, \tag{10}$$

where $\Phi(\mathbf{C})$ is an isotropic function of nonnegative values. We thus assume $\Phi(\mathbf{C}) = \|\boldsymbol{\lambda}(\mathbf{C})\|_p$, $p \geq 1$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ stand for the eigenvalues of $\mathbf{C}$, called Kelvin moduli. Let us note that

a) $p = 1$;

$$\begin{gathered} \Phi(\mathbf{C}) = \operatorname{tr} \mathbf{C}, \\ \Phi(\mathbf{C}) = \lambda_1 + \lambda_2 + \lambda_3, \end{gathered} \tag{11}$$

b) $p = 2$;

$$\begin{gathered} \Phi(\mathbf{C}) = \left(C_{\alpha\beta\lambda\mu} C_{\alpha\beta\lambda\mu}\right)^{1/2}, \\ \Phi(\mathbf{C}) = \left((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2\right)^{1/2}. \end{gathered} \tag{12}$$

The version of FMD with $p = 1$ is referred to as *FMD with trace constraint.*

Let $H(\Omega)$ be the space of admissible Hooke tensors $\mathbf{C}$ defined in given domain. Consider the following problem:

$$Y_p = \min \left\{ \wp(\mathbf{C}) \,|\, \mathbf{C} \in H(\Omega), \quad \langle \Phi(\mathbf{C}) \rangle = E_0 \right\}. \tag{13}$$

One can prove that for $p = 1$, see [15, 103]

$$\begin{gathered} Y_1 = \frac{1}{|\Omega| \, E_0} (Z_1)^2, \\ Z_1 = \min \left\{ \int_\Omega \|\boldsymbol{\tau}\| \, dx \,\middle|\, \boldsymbol{\tau} \in \Sigma(\Omega) \right\}. \end{gathered} \tag{14}$$

Let $\boldsymbol{\tau} = \boldsymbol{\pi}$ stand for the minimizer of the latter problem. The optimal Hooke tensor $\mathbf{C}$ is expressed by this minimizer as follows

$$\begin{gathered} \mathbf{C} = E_0 \frac{\|\boldsymbol{\pi}\|}{\langle \|\boldsymbol{\pi}\| \rangle} \widehat{\boldsymbol{\pi}} \otimes \widehat{\boldsymbol{\pi}}, \\ \widehat{\boldsymbol{\pi}} = \frac{1}{\|\boldsymbol{\pi}\|} \boldsymbol{\pi}, \qquad \lambda_1 = E_0 \frac{\|\boldsymbol{\pi}\|}{\langle \|\boldsymbol{\pi}\| \rangle}. \end{gathered} \tag{15}$$

Thus two of the optimal Kelvin moduli vanish; the material is optimally adjusted to a given loading, and stiffnesses orthogonal to the loading disappear as they are redundant. The problem (14) has an important common feature with problem (P_M) – in both cases the integrand is of linear growth. Thus this problem can also be interpreted as the locking problem. The minimizer $\boldsymbol{\tau} = \boldsymbol{\pi}$ determines not only the components of $\mathbf{C}$ but also the trajectories of the stress state. Note that material moduli do not appear in (14); trajectories of $\boldsymbol{\tau} = \boldsymbol{\pi}$ are determined by the loading, the domain and the support.

The problem (13) for $p = 1, 2$ was introduced in Bendsøe et al. [11]; the theorem on correctness of this problem was given in Haslinger et al. [12]. Numerical methods based on the strain-based formulation (not shown here) were developed e.g. in Kočvara et al. [13]. The result (14) is new. It was reported in [15], and published in [103] in full detail. Generalization of the results to the case of two loads applied non-simultaneously can also be found in [15], while [104] and [105] provide a discussion of the problem in the context of thin plates subjected to simultaneous bending and in-plane loading.

Consider now the problem illustrated in Fig. 1 within the setting of FMD with trace constraint. A numerical solution to the problem (14) has been found by constructing the representation of the solution to the equilibrium problem $\boldsymbol{\tau} \in \Sigma(\Omega)$. The procedure starts from an underdetermined system of equilibrium equations, the solution of which is then represented by the SVD decomposition, see Press et al. [106]. Design variables which appear in this representation are not constrained. They are determined by the minimization procedure hence one can say that a certain version of the force method is applied.

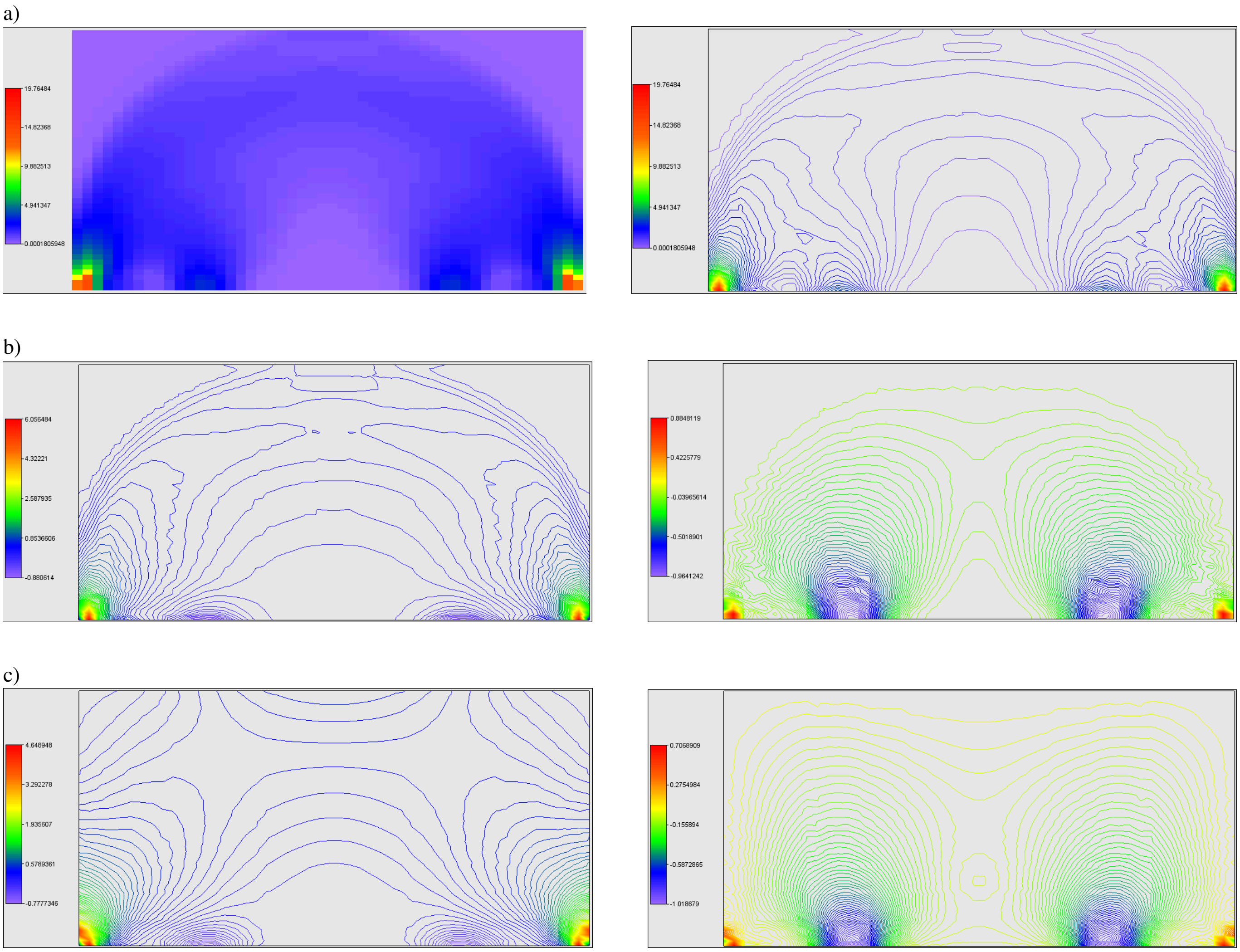

Fig. 8. Problem illustrated in Fig. 1a. Case of $d = 0.5l$ – solution of the FMD problem with trace constraint: a) scatter plots (left) and contours (right) of the Kelvin modulus λ_1; b) contour plots of the maximal (left) and minimal (right) principal stresses corresponding to the optimal, non-homogeneous, anisotropic distribution of material properties; c) contour plots of the maximal (left) and minimal (right) principal stresses corresponding to the non-optimal, homogeneous, isotropic body satisfying the same isoperimetric condition. All results obtained by S. Czarnecki

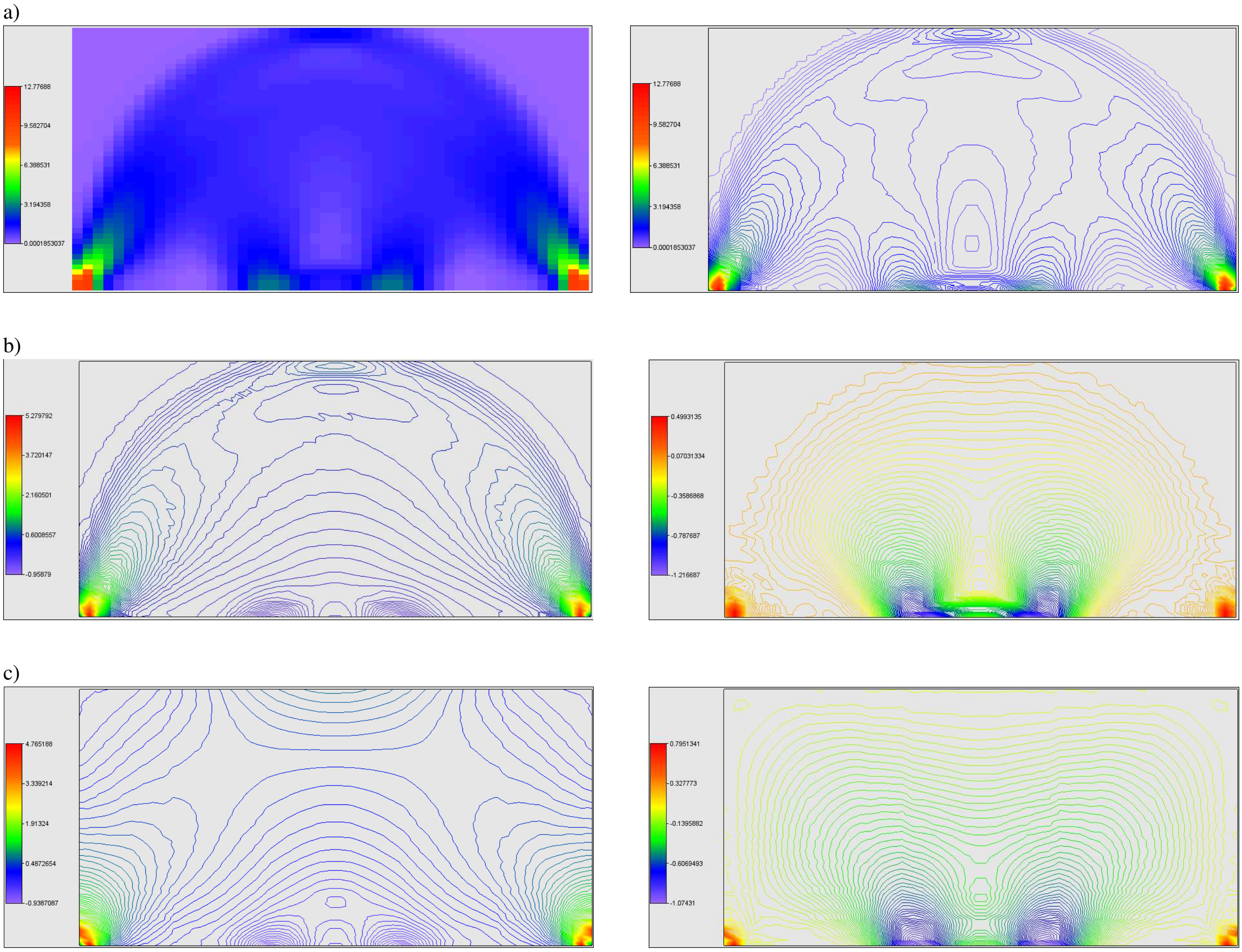

Fig. 9. Problem illustrated in Fig. 1a. Case of $d = 0.75l$ – solution of the FMD problem with trace constraint: a) scatter plots (left) and contours (right) of the Kelvin modulus λ_1; b) contour plots of the maximal (left) and minimal (right) principal stresses corresponding to the optimal, non-homogeneous, anisotropic distribution of material properties; c) contour plots of the maximal (left) and minimal (right) principal stresses corresponding to the non-optimal, homogeneous, isotropic body satisfying the same isoperimetric condition. All results obtained by S. Czarnecki

5. Final remarks

Topology optimization provides certain methods of controlling microstructures and structural shape to make the material and geometric characteristics ideally adjusted to given loading. Multiple load case can be also considered, the easiest way is to treat the loading cases separately and minimize a weighted sum of corresponding compliances. Thus obtained solutions are not as clear as those related to the single load case.

Solutions to topology optimization problems should be further improved to fulfil other design requirements, some of them refer to the stress level. In case of thin structures, local and global stability conditions should be satisfied by appropriate stiffening of the initial design.

The present review does not cover the evolutionary methods developed for the topology optimization problems. In this respect, the reader is referred to recent articles published in the Bulletin of the Polish Academy of Sciences, Technical Sciences, 60(2), 2012, and references therein, see e.g. papers by Długosz and Burczyński [107], Mrzygłód [108] or Szczepanik and Burczyński [109].

Acknowledgements. The paper was prepared within the Research Grant no N506 071338, financed by the Polish Ministry of Science and Higher Education, entitled: Topology Optimization of Engineering Structures. Simultaneous shaping and local material properties determination.

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