BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 61, No. 1, 2013
DOI: 10.2478/bpasts-2013-0019

CIVIL ENGINEERING

Sensitivity analysis of sandwich beams and plates accounting for variable support conditions

R. STUDZIŃSKI*, Z. POZORSKI, and A. GARSTECKI

Poznan University of Technology, Institute of Structural Engineering, 5 Piotrowo St., 60-965 Poznań, Poland

Abstract. The paper addresses the problems of the sensitivity analysis and optimal design of multi-span sandwich panels with a soft core and flat thin steel facings. The response functional is formulated in a general form allowing wide practical applications. Sensitivity gradients of this functional with respect to dimensional, material and support parameters are derived using adjoint variable method. These operators account for the jump of the slope of a Timoshenko beam or a Reissner plate at the position of concentrated active load or reaction, thus extending the sensitivity operators known in literature. The jump of slope is the effect of shear deformation of the core. Special attention is focussed on sensitivity and optimisation allowing for variable support position and stiffness, because local phenomena observed in supporting area of sandwich plates often initiate failure mechanisms. Introducing optimally located elastic supports allows to reduce the unfavourable influence of temperature on the state of stress. Several examples illustrate the application of derived sensitivity operators and demonstrate their exactness.

Key words: sandwich panels, sensitivity analysis, optimal design, support conditions.

1. Introduction

The common aim of designers and technologists is to find solutions which combine high quality, safety and low price. It can be achieved in sandwich structures because the properties of the component materials can be used reasonably. Therefore the optimal design of sandwich structures is a topic frequently taken up in literature [1–14].

In the construction industry three-layered panels (sandwich panels) with a soft core and steel flat or profiled facings are most often used because they provide high load-bearing capacity coupled with small weight and good thermal insulation. In design of these panels the effects of shear flexibility of the core, wrinkling failure of the facings and essential influence of the temperature on the state of the stresses should be taken into account.

Dimensional parametric optimisation of three layered sandwich panels in the form of minimum weight design was discussed in [13], whereas in [11] the Pareto optimal solutions combining maximum range of applications with minimum cost were found for sandwich panels with soft core. In [7] the optimisation problems with stress concentrations at the interfaces were considered with the aim to maximize the bending moment and minimize the energy due to interlayer stresses. The proposed functionally graded sandwich panels allows energy to transfer from bending to shear and vice versa. In the papers [2, 3] core junctions were proposed to improve shear panel capacity. Simultaneously the geometric shape of the boundary of the adjacent core materials has been improved to significantly diminish local stress concentrations at the core. Some recent works have tackled the problem of different failure modes and their influence on the state of stresses. The comparison of the behaviour of sandwich panels with various combinations of materials for achieving minimum mass has been presented in [10]. In the study [12] failure maps have been created which illustrate the dependence of failure mechanisms on the structural parameters as also load and support conditions.

A sensitivity analysis has been considered by many authors. In the early paper of Courant [4] the basic variational formulation of sensitivity problems and possible applications were discussed. Later in [8] the adjoint variable method was developed. The influence of the initial distortions on the optimal design of support conditions in beams and frames was studied in [5]. In [6, 9] the sensitivity analysis in the case of dynamically loaded structures allowing for variable joint parameters has been presented.

The paper develops the theory of optimal design of sandwich panels by consideration of variable support conditions. In the derivation of the sensitivity operators the shear deformation of the core has been taken into account, thus extending the operators known in the literature. The response functional is defined in general form allowing optimisation of stress, strains or displacements. The sensitivity operators are expressed explicitly using continuous formulation and the adjoint variable method. The sensitivity gradients are often employed in hybrid optimisation algorithms, where genetic or particle swarm methods are combined with a gradient method. Lately, the importance of the robust design accounting for manufacturing and service tolerances has been stated in the literature [1] and here the sensitivity information can also be useful.

*e-mail: robert.studzinski@put.poznan.pl

2. Sandwich panel theory: assumptions and simplifications

There are several methods of the analysis of sandwich structures. In macroscopic scale sandwich structures can be examined using Equivalent Single Layer (ESL) theories (1D or 2D) which assume that mechanical properties of the surrogate layer are the resultant of the parameters of the particular layers. Many examples in literature prove, in spite of assumed simplifications, that ESL models are very useful in determining the mechanical response of sandwich structures in macro scale [14]. Moreover, because the number of degrees of freedom is independent from the number of layers the time of the numerical calculations is rather small. Among the ESL theories are: the Classical Laminated Plate Theory (CLPT) based on the Kirchoff-Love plate theory, the First Order Shear Deformation Theory (FSDT) and the High Order Shear Deformation Theory (HSDT). Three-dimensional elasticity theories can provide more precise results especially in the analysis of local effects. They are more sophisticated, engage large number of degrees of freedom and require refined definition of boundary conditions. Therefore, in the engineering practice these theories are not often used.

We analyse sandwich panels using the Timoshenko beam theory generalized to sandwich sections (FSDT). This theory assumed that the materials are isotropic, homogenous and linear. Because the Young modulus of the core is about 70 thousand less than Young modulus of the facings, the normal stresses in the core are negligible, hence shear stresses in the core are constant. It should be underscored that the response of sandwich panels with shear deformable core is quantitatively and qualitatively different from the response of a typical panel. It is illustrated by the example of the beam loaded by a concentrated force at the end of the cantilever, Fig. 1.

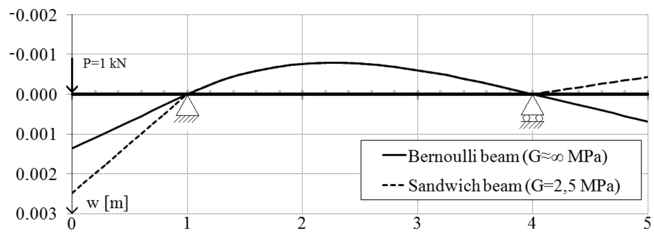

Fig. 1. Displacement lines for a sandwich and Bernoulli beam

The continuous line represents displacements of the Bernoulli beam where the shear rigidity is infinitely great. The dashed line represents displacements of the sandwich beam for which the shear rigidity is assumed to be finite. The bending and shear effects influence the total displacements of sandwich beam.

3. Problem formulation

Rational optimisation of structural elements made of drastically different materials (i.e. steel facings and polyurethane soft core) is rather difficult because the response can be counter-intuitive as shown in Fig. 1. Therefore, mathematical theory of optimisation can be particularly useful for practical engineers. Sensitivity analysis also provides useful information for this optimal design. In the present paper the problems of sensitivity analysis are formulated in a general form to make possible wide application of derived sensitivity operators. The analysed structures are loaded mechanically and/or thermally. The latter one induces initial distortions. In the paper a general form of initial distortions is allowed for.

Let us introduce the structural response functional in the form (1), where F is an arbitrary Gateaux differentiable function of displacements w, stresses $\mathbf{Q}$ and strains $\mathbf{q}$.

$$G(\xi) = \int_0^L F(w,\ \mathbf{Q},\ \mathbf{q}) \cdot dx. \tag{1}$$

The functional $G(\xi)$ can play the role of the objective function or a constraint. Using Dirac function $\delta(x-x_0)$ in F, the global functional can represent a point-wise response at $x = x_0$. The design vector ξ (2) consists of the position of the support x_s, stiffness of the support k_s, thickness of the upper t_{F1} and the lower t_{F2} facing, total thickness of the panel D, shear modulus of the core G_C and the Young modulus of the upper E_{F1} and the lower E_{F2} facing

$$\xi = [x_s,\ k_s,\ t_{F1},\ t_{F2},\ D,\ G_C,\ E_{F1},\ E_{F2}]. \tag{2}$$

The variation of the functional $G(\xi)$ takes the form

$$\delta G(\xi) = \int_0^L \left(\frac{\partial F}{\partial w}\delta w + \frac{\partial F}{\partial \mathbf{Q}}\delta \mathbf{Q} + \frac{\partial F}{\partial \mathbf{q}}\delta \mathbf{q} \right) dx, \tag{3}$$

where the variations δw, $\delta\mathbf{Q}$ and $\delta\mathbf{q}$ are implicit functions of the design vector ξ.

It is worth adding that in general, according to [5], we assume both the initial strain field $\mathbf{q}^i$ which is kinematically inadmissible and initial stress field $\mathbf{Q}^i$ which is statically inadmissible. In effect elastic strain $\mathbf{q}^e$ and stress $\mathbf{Q}^e$ (4) are induced (Fig. 2). They are interrelated by Hooks law (5):

$$\mathbf{q} = \mathbf{q}^i + \mathbf{q}^e, \qquad \mathbf{Q} = \mathbf{Q}^i + \mathbf{Q}^e, \tag{4}$$

$$\mathbf{Q}^e = \mathbf{K}\,\mathbf{q}^e, \qquad \mathbf{q}^e = \mathbf{K}^{-1}\,\mathbf{Q}^e, \tag{5}$$

where $\mathbf{K}$ is the stiffness matrix. The total strain $\mathbf{q}$ and stress $\mathbf{Q}$ are kinematically and statically admissible.

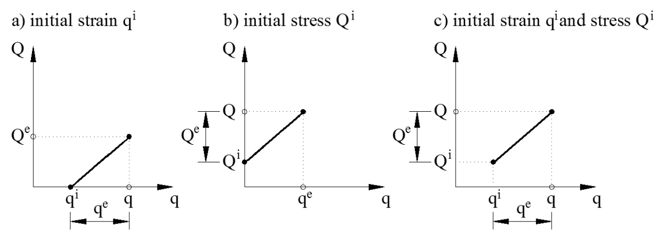

Fig. 2. Conception of the initial stress and strain, see (after Ref. 5)

Relations (4), (5) apply to general class of linear elastic structures. In the case of Bernoulli beam we have scalar valued quantities

$$Q = M_y, \qquad q = -w'', \qquad K = EI_y. \tag{6}$$

In the case of Timoshenko beam theory there is

$$\mathbf{Q}=\begin{bmatrix} M_y \\ V_z \end{bmatrix}, \qquad \mathbf{q}=\begin{bmatrix} \phi' \\ \phi+w' \end{bmatrix}, \qquad \mathbf{K}=\begin{bmatrix} B_s & 0 \\ 0 & S \end{bmatrix}, \tag{7}$$

where M_y and V_z denote bending moment and shear force, w is the vertical displacement, ϕ is rotation angle of the cross section and B_s, S represent the bending and shear rigidity of the cross section, respectively. Primes denote differentiation with respect to x. Note that setting $\phi = -w'$ in (7) we obtain (6), where influence of V_z is neglected. In the case of a three layered sandwich beam the bending and shear stiffness coefficients B_s and S are expressed as follows:

$$B_s=\frac{E_{F1}t_{F1}E_{F2}t_{F2}B}{E_{F1}t_{F1}+E_{F2}t_{F2}}e^2, \qquad e=D-\frac{t_{F1}+t_{F2}}{2}, \tag{8}$$

$$S=G_C A_C, \qquad A_C=B\left(D-t_{F1}-t_{F2}\right) \tag{9}$$

where B is the width of the beam.

In the case of Kirchhoff plate we have

$$\mathbf{Q}=\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix}, \qquad \mathbf{q}=\begin{bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -\dfrac{\partial^2 w}{\partial x \partial y} \end{bmatrix}, \qquad \mathbf{K}=\begin{bmatrix} D & D\nu & 0 \\ D\nu & D & 0 \\ 0 & 0 & (1-\nu)D \end{bmatrix}, \tag{10}$$

where D is the plate bending stiffness. In the case of a homogenous plate $D = Eh^3/12(1-\nu^2)$.

In the case of Reissner plate the relations (10) take the form

$$\mathbf{Q}=\left[M_{xx}M_{yy}M_{xy}V_xV_y\right]^T, \tag{11^1}$$

$$\mathbf{q}=\left[\frac{\partial \phi_x}{\partial x}\,\frac{\partial \phi_y}{\partial y}\,\frac{1}{2}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\phi_x+\frac{\partial w}{\partial x}\phi_y+\frac{\partial w}{\partial y}\right]^T. \tag{11^2}$$

Again, we note that by setting $\phi_x = -\partial w/\partial x$, $\phi_y = -\partial w/\partial y$ in (11) we arrive at (10) with shear effects neglected.

There is a large class of engineering applications e.g. in civil engineering, where the slabs are loaded uniformly, have the ratio length of span to width $L/B > 2$ and have line supports in the direction B. In this case Timoshenko beam theory can be successfully used for plates. For the brevity of the presentation we focus attention on this class of problems. However, the sensitivity operators derived in Sec. 4 based on Timoshenko theory can be easily generalized for Reissner plate theory, which is discussed in Sec. 5 and illustrated in the example 5.

4. Sensitivity operators

Consider a sandwich beam illustrated in Fig. 3, where three states are depicted: the actual structure (primary), the actual structure after variation of control parameters ξ (primary with superscript *) and the adjoint structure (denoted by superscript ^{a}). The adjoint structure is introduced in order to transform (3) to the explicit form.

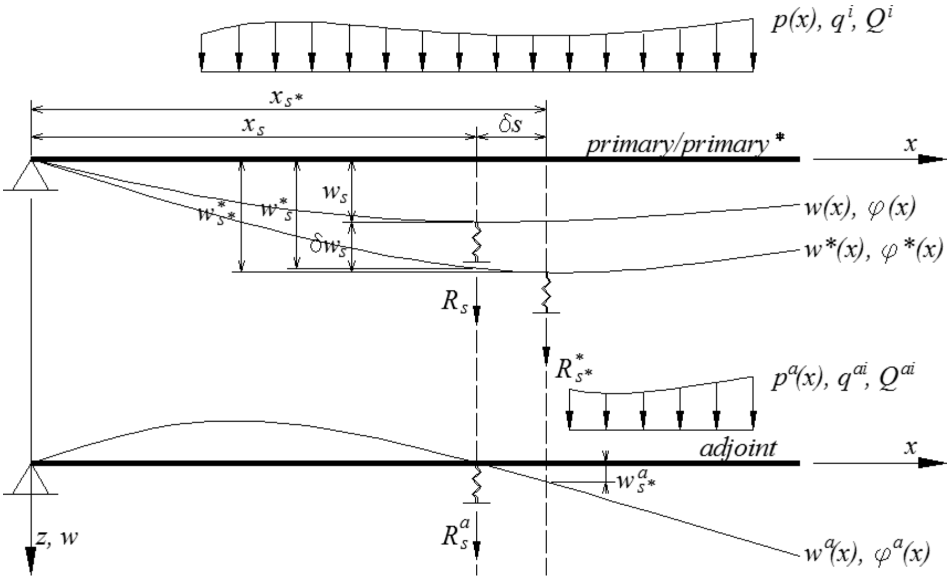

Fig. 3. Considered structures: actual (primary), after variation of control parameters (primary *) and adjoint

We write the virtual work equation using the statically admissible stress fields from the adjoint structure and kinematically admissible variations of kinematic fields in the primary structure

$$\int_0^L p^a\left(w^*-w\right)dx+R_s^a\left(w_s^*-w_s\right) - \int_0^L \mathbf{Q}^a\left(\mathbf{q}^*-\mathbf{q}\right)dx=0. \tag{12}$$

Conversely, using adjoint kinematic fields and variations of primary stress fields we obtain

$$\int_0^L \left(p^*-p\right)w^a dx+R_{s*}^* w_{s*}^a - R_s w_s^a - \int_0^L \left(\mathbf{Q}^*-\mathbf{Q}\right)\mathbf{q}^a dx=0. \tag{13}$$

The quantities denoted by star refer to the structure after variation of the design parameters. Particularly, the subscript s^* denotes that a quantity is measured after the variation of the support position $x_s^*=x_s+\delta x_s$. By developing the variations of displacements in Taylor series and retaining only linear terms we obtain the following relations

$$\begin{aligned} w_{s*}^*-w_s&=\delta w_s\left(def.\right),\\ w_s^*-w_s&=\delta w_s-w_{s,x}^-\delta x_s,\\ w_{s*}^a-w_s^a&=w_{s,x}^{a+}\delta x_s,\\ R_{s*}^*-R_s&=\delta R_s(def.). \end{aligned} \tag{14}$$

Because of the slope discontinuity, the left and right derivatives of the displacement are not equal at the point of the concentrated load. This makes necessary to use left- $(\cdot)^-$ and right-handed $(\cdot)^+$ derivatives.

Subtracting (12) from (13) and introducing (4), (5), (14), we finally obtain the following relation

$$\begin{aligned} 0=&\int_0^L \{\left(p^*-p\right)w^a-p^a\left(w^*-w\right)\\ &-\delta\mathbf{Q}^i\mathbf{q}^{ai}-\delta\mathbf{Q}^i\mathbf{q}^{ae}-\mathbf{q}^{ai}\left(\delta\mathbf{K}\mathbf{q}^e+\mathbf{K}\delta\mathbf{q}^e\right)\\ &-\mathbf{q}^{ae}\left(\delta\mathbf{K}\mathbf{q}^e+\mathbf{K}\delta\mathbf{q}^e\right)+\mathbf{Q}^{ai}\delta\mathbf{q}^i\\ &+\mathbf{Q}^{ai}\delta\mathbf{q}^e+\mathbf{K}\mathbf{q}^{ae}\delta\mathbf{q}^i+\mathbf{K}\mathbf{q}^{ae}\delta\mathbf{q}^e\}\,dx+R_s^a w_{s,x}^-\delta x_s\\ &+R_s w_{s,x}^{a+}\delta x_s-R_s^a w_{s*}^*+R_s^a w_s+\delta R_s w_s^a, \end{aligned} \tag{15}$$

where the functions $\mathbf{Q}$, $\mathbf{q}$, $\mathbf{Q}^{ae}$ and $\mathbf{q}^{ae}$ can be discontinuous in the point s (i.e. point of applied concentrated load) as well as function $\mathbf{Q}^*$ and $\mathbf{q}^*$ in the point s^*; the functions w, w^* and w^a are continuous of class C^0, whereas p, p^* and p^a could have distribution of the Heaviside or Dirac type. Additionally, we assume that $p=p^*$ and $\mathbf{K}^a=\mathbf{K}$. In our problem we consider also the elastic supports, which can be introduced in the following forms:

$$\begin{aligned} R_s&=-k_s w_s,\\ \delta R_s&=-\delta k_s w_s-k_s\delta w_s,\\ R_s^a&=-k_s^a w_s^a,\\ k_s^a=k_s,&\qquad k_s=f_s^{-1}, \end{aligned} \tag{16}$$

where k_s and f_s are respectively stiffness and flexibility of the support. Now, we can rewrite the (15) by introducing the (16) in the following way

$$\begin{aligned} 0=&\int_0^L\{-p^a\delta w-\mathbf{q}^{ae}\delta\mathbf{Q}^i-\mathbf{q}^{ae}\delta\mathbf{K}\mathbf{q}^e+\mathbf{K}\mathbf{q}^{ae}\delta\mathbf{q}^i\}\,dx\\ &+\int_0^L\Bigg\{-\underbrace{\mathbf{q}^{ai}\delta\mathbf{Q}^i-\mathbf{q}^{ai}\delta\mathbf{Q}^e}_{\mathbf{q}^{ai}\delta\mathbf{Q}}+\underbrace{\mathbf{Q}^{ai}\delta\mathbf{q}^i+\mathbf{Q}^{ai}\delta\mathbf{q}^e}_{\mathbf{Q}^{ai}\delta\mathbf{q}}\Bigg\}dx\\ &+\left(R_s^a w_{s,x}^-+R_s w_{s,x}^{a+}\right)\delta x_s-w_s w_s^a\delta k_s. \end{aligned} \tag{17}$$

Because operations on the k_s and δk_s are numerically inefficient, the use of flexibility of the support f_s is more convenient. Hence, the later component of the (17) can be written as

$$-w_s w_s^a\delta k_s=R_s R_s^a\delta f_s. \tag{18}$$

In our considerations we also assume that the load and initial strains and stresses do not change with the variation of the control parameters. Hence $p=p*$, $\mathbf{q}^i=\mathbf{q}*^i$ and $\mathbf{Q}^i=\mathbf{Q}*^i$. It results in $\delta\mathbf{q}^i=0$, $\delta\mathbf{Q}^i=0$. Summing up, we obtain fundamental Eq. (19), which expresses the Eq. (3) explicitly towards to δx_s, δf_s and $\delta\mathbf{K}$:

$$\begin{aligned} \delta G\left(\xi\right)=&\int_0^L\left(p^a\delta w+\mathbf{q}^{ai}\delta\mathbf{Q}-\mathbf{Q}^{ai}\delta\mathbf{q}\right)dx\\ =&\left(R_s^a w_{s,x}^-+R_s w_{s,x}^{a+}\right)\delta x_s\\ &+R_s R_s^a\delta f_s-\int_0^L(\mathbf{q}^{ae}\mathbf{q}^e\delta\mathbf{K})dx. \end{aligned} \tag{19}$$

In the derivation of (19) the step right of the support position was assumed, i.e. $\delta x_s^+=x_{s*}-x_s>0$. In the case of step left $\delta x_s^-=x_{s*}-x_s<0$ the terms with the superscripts $(\cdot)^+$ and $(\cdot)^-$ change the sign i.e. $w_{s,x}^-$ changes into $w_{s,x}^+$ and $w_{s,x}^{a+}$ changes into $w_{s,x}^{a-}$. In a special case when only the position of the support is subject to variation δx_s^+ or δx_s^- we have

$$\delta G\left(x_s\right)=\left(R_s^a w_{s,x}^+ + R_s w_{s,x}^{a-}\right)\delta x_s^-, \tag{20^1}$$

$$\delta G\left(x_s\right)=\left(R_s^a w_{s,x}^- + R_s w_{s,x}^{a+}\right)\delta x_s^+. \tag{20^2}$$

Lemma

$$R_s^a w_{s,x}^+ + R_s w_{s,x}^{a-}=R_s^a w_{s,x}^- + R_s w_{s,x}^{a+}. \tag{21}$$

Proof. Equality (20) can be written as

$$R_s^a\left(w_{s,x}^{+}-w_{s,x}^{-}\right)-R_s\left(w_{s,x}^{a+}-w_{s,x}^{a-}\right)=0$$

or briefly by introduction of the notation $[\dots]$ for the jump of a function

$$R_s^a\left[w_{s,x}\right]-R_s\left[w_{s,x}^a\right]=0. \tag{22}$$

From (7) it follows that

$$M_y=B_S\phi' \qquad \text{and} \qquad V_Z=S\left(\phi+w'\right).$$

At a hinge support $x = x_s$ the bending moment M_y is C^0 continuous, hence ϕ is C^1. Therefore, the jumps of functions can be written as

$$\left[V_z\right]=\left[S\left(\phi+w'\right)\right]=S\left[w'\right]=S\left[w_{s,x}\right].$$

But $[V_z] = R_s$ hence,

$$\left[w_{s,x}\right]=\frac{1}{S}R_s \qquad \text{and} \qquad \left[w_{s,x}^a\right]=\frac{1}{S}R_s^a. \tag{23}$$

Introducing (23) into (22) one obtains $0 = 0$. **End of the proof**.

Using the lemma, the sensitivity operators (20^1) and (20^2) simplify to

$$\begin{gathered}\delta G\left(x_S\right)=\left(R_s^a w_{s,x}^{+}+R_s w_{s,x}^{a-}\right)\delta x_s\\ \text{or}\quad \delta G\left(x_S\right)=\left(R_s^a w_{s,x}^{-}+R_s w_{s,x}^{a+}\right)\delta x_s.\end{gathered} \tag{24}$$

Necessary condition $\delta G(x_s) = 0$ for optimal support position in *Timoshenko beam* takes now the form

$$\begin{gathered}R_s^a w_{s,x}^{+}+R_s w_{s,x}^{a-}=0\\ \text{or} R_s^a w_{s,x}^{-}+R_s w_{s,x}^{a+}=0.\end{gathered} \tag{25}$$

The sensitivity gradient and optimality condition derived in [5] *for Bernoulli beam* were:

$$\begin{gathered}\delta G=\left(R_s^a w_{s,x}+R_s w_{s,x}^a\right)dx_S\\ \text{and}\qquad R_s^a w_{s,x}+R_s w_{s,x}^a=0.\end{gathered} \tag{26}$$

In case when the functional G_1 represents the global stiffness of the structure described by its total potential energy, its variation and optimality condition (26) reduce to

$$\begin{gathered}\delta G_1=\left(R_s w_{s,x}\right)dx_S\\ \text{and}\qquad R_s w_{s,x}=0.\end{gathered} \tag{27}$$

In the Example 4 we demonstrate that for Timoshenko beam the formulae (27) change to (42)

$$\begin{gathered}\delta G_1=R_s\cdot 0.5\left(w_{s,x}^{+}+w_{s,x}^{-}\right)\delta x_s\\ \text{and}\qquad R_s\cdot 0.5\left(w_{s,x}^{+}+w_{s,x}^{-}\right)=0.\end{gathered}$$

The derivative $\cdot w_{s,x}$ is replaced by the arithmetic mean of left- and right-hand derivatives.

Now let us focus attention on the last term in (19). According to (7) and (8) we can write

$$\int_0^L\left(\mathbf{q}^{ae}\mathbf{q}^{e}\delta\mathbf{K}\right)dx=\int_0^L\left(w_M''^{ae}w_M''^{e}\delta B_s+w_V'^{ae}w_V'^{e}\delta S\right)dx. \tag{28}$$

In case of support translation, discontinuities of kinematic and static fields at $x = x_S$ are observed. Therefore in special cases the integrals in x domain must be divided into three integration ranges $\int_0^L\cdots dx=\int_0^{x_s^-}\cdots dx+\int_{x_s^-}^{x_s^+}\cdots dx+\int_{x_s^+}^{L}\cdots dx$, where the integral $\int_{x_s^-}^{x_s^+}\cdots dx$ can be evaluated using 1^{st} mean value theorem for functions. This is discussed in [5] and [9] for Bernoulli beam.

The variations of the δB_s and δS in (28) in case of sandwich beams take the form:

$$\begin{gathered}\delta B_s=B_1\delta E_{F1}+B_2\delta E_{F2}+B_3\delta t_{F1}+B_4\delta t_{F2}+B_5\delta D,\\
B_1=\frac{B}{4}\frac{t_{F1}E_{F2}^2t_{F2}^2\left(2D-t_{F1}-t_{F2}\right)^2}{\left(E_{F1}t_{F1}+E_{F2}t_{F2}\right)^2},\\
B_2=\frac{B}{4}\frac{E_{F1}^2t_{F1}^2t_{F2}\left(2D-t_{F1}-t_{F2}\right)^2}{\left(E_{F1}t_{F1}+E_{F2}t_{F2}\right)^2},\\
B_3=\frac{B}{4}\frac{E_{F1}E_{F2}t_{F2}\left(2D-t_{F1}-t_{F2}\right)\left(2DE_{F2}t_{F2}-2E_{F1}t_{F1}^2-3t_{F1}E_{F2}t_{F2}-E_{F2}t_{F2}^2\right)}{\left(E_{F1}t_{F1}+E_{F2}t_{F2}\right)^2},\\
B_4=\frac{B}{4}\frac{E_{F1}t_{F1}E_{F2}\left(2D-t_{F1}-t_{F2}\right)\left(2DE_{F1}t_{F1}-2E_{F2}t_{F2}^2-3E_{F1}t_{F1}t_{F2}-E_{F1}t_{F1}^2\right)}{\left(E_{F1}t_{F1}+E_{F2}t_{F2}\right)^2},\\
B_5=\frac{BE_{F1}t_{F1}E_{F2}t_{F2}\left(2D-t_{F1}-t_{F2}\right)}{E_{F1}t_{F1}+E_{F2}t_{F2}},\end{gathered} \tag{29}$$

$$\delta S=B\left(D-t_{F1}-t_{F2}\right)\delta G_C-G_CB\delta t_{F1}-G_CB\delta t_{F2}+G_CB\delta D. \tag{30}$$

According to (3) and (19), the adjoint actions are defined as follows:

$$p^a = \frac{\partial F}{\partial w}, \qquad \mathbf{q}^{ai} = \frac{\partial F}{\partial \mathbf{Q}}, \qquad \mathbf{Q}^{ai} = -\frac{\partial F}{\partial \mathbf{q}}, \tag{31}$$

where p^a is the adjoint loading, $\mathbf{q}^{ai}$ and $\mathbf{Q}^{ai}$ are initial strain and stress in the adjoint structure.

In optimal design and sensitivity analysis there can be different classes of problems depending on the form of the response functional G. They will be illustrated by the way of examples presented in Chapter 6.

All formulae derived hitherto can be directly used to rectangular slabs provided that the length-to-width ratio, loading and support conditions make possible to apply the Timoshenko layered beam theory, as described in Chapter 3. This limitation follows from the assumption that kinematic and static fields are constant in the width direction B of the slab. If these conditions are not satisfied and bending and torsional moments in width direction exist, then 2-D theory must be employed. This issue is discussed in Chapter 5 and in the example 5.

5. Generalization

The derived sensitivity operators can be generalized for a broader class of slabs, where the loading and hence the stress and strain fields are not constant in the width direction. The 2-D Reissner plate model must be used, where constitutive Eqs. (11) are introduced instead of (7). Distributed load p is defined per unit surface area (kN/m^2) and reaction force is expressed in kN/m. Let the support with a position x_s, which is subject to variation δx_s, be a line support perpendicular to L (Fig. 4). The sensitivity operators derived in Chapter 4 remain valid provided that the integrals in the length domain $\int_0^L dx$ are substituted by integrals over the middle surface of the slab $\int_0^L \int_{-B/2}^{B/2} dydx$ and the sensitivity operators with respect to variation of the support position $[..]\delta x_s$ are substituted by the same term integrated in the range $B/2 \leq y \leq B/2$, namely $\left(\int_{-B/2}^{B/2} [..]\, dy\right) \delta x_S$. The Eq. (19) takes the form

$$\delta G(\xi) = \int_0^L \int_{-B/2}^{B/2} \left(p^a \delta w + \mathbf{q}^{ai} \delta \mathbf{Q} - \mathbf{Q}^{ai} \delta \mathbf{q}\right) dydx =$$
$$= \left(\int_{-B/2}^{B/2} \left(R_s^a w_{s,x}^- + R_s w_{s,x}^{a+}\right) dy \right) \delta x_s \tag{32}$$
$$+ \left(\int_{-B/2}^{B/2} R_s R_s^a dy \right) \delta f_s - \int_0^L \int_{-B/2}^{B/2} (\mathbf{q}^{ae} \mathbf{q}^e \delta \mathbf{K}) dydx.$$

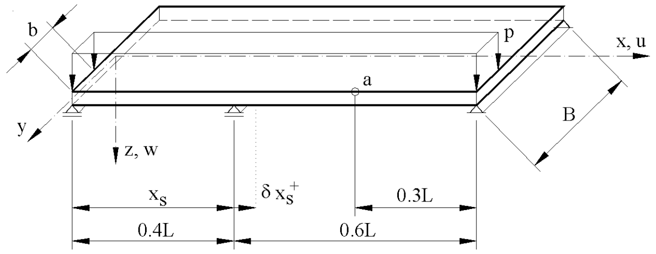

Fig. 4. Rectangular sandwich panel with variable support position x_s

Here the variations δx_s and δf_s are scalar valued. The optimality condition (24) can be written as

$$\int_{-B/2}^{B/2} \left(R_s^a w_{s,x}^+ + R_s w_{s,x}^{a-}\right) dy = \int_{-B/2}^{B/2} \left(R_s^a w_{s,x}^- + R_s w_{s,x}^{a+}\right) dy = 0. \tag{33}$$

6. Examples

Practical application and the correctness of the sensitivity operators derived above will be demonstrated by the following examples. Therefore several special cases are considered and the sensitivity operators evaluated by means of the presented theory are compared with simple finite difference calculation (central or forward). We focus attention on effects typical for sandwich panels and important in structural design, namely deflection of the panel, value of reaction force and bending moment at the middle support. The latter two quantities play important role in design, since they can result in local failure mechanisms at the support. Following the main idea of the paper, the attention will be focussed on variation of parameters specifying the support, namely its position and stiffness, although the derived sensitivity gradients account for variation of all design variables defined in (2). Example 4 demonstrates usefulness of sensitivity operators when in optimisation algorithm the genetic and gradient methods are combined. The analyses in examples 1 to 4 are carried out using the authors' program FEM a fine mesh which ensured exactness within the limits of Timoshenko beam theory. Example 5 demonstrates the correctness of the sensitivity operator when applied to Reissner plate. Here the Abaqus system with the FEM mesh 5×5 cm was used.

Example 1. Consider a sandwich beam shown in Fig. 5. It is subjected to uniformly distributed load $p(x) = 1.0$ kN/m. Total length of the beam is $L = 2.0$ m. We will study the sensitivity of displacement at the point a is $x_a = 1.0$ m with respect to variation of the support position. We will study the sensitivity of the deflection in the whole range of support positions x_s. The thickness of the facings is $t_{F1} = t_{F2} = 0.00045$ m, the thickness of the core is $d = 0.12$ m, and the shear modulus is $G_C = 2500$ kPa. The adjoint action, according to Eq. (31), is $P^a = 1$. The design vector is limited to x_S. We are checking both forms of the operator (24), namely

$$\delta G(x_s) = \left(R_s^a w_{s,x}^+ + R_s w_{s,x}^{a-}\right) \delta x_s$$
$$\text{and} \qquad \delta G(x_s) = \left(R_s^a w_{s,x}^- + R_s w_{s,x}^{a+}\right) \delta x_s. \tag{34}$$

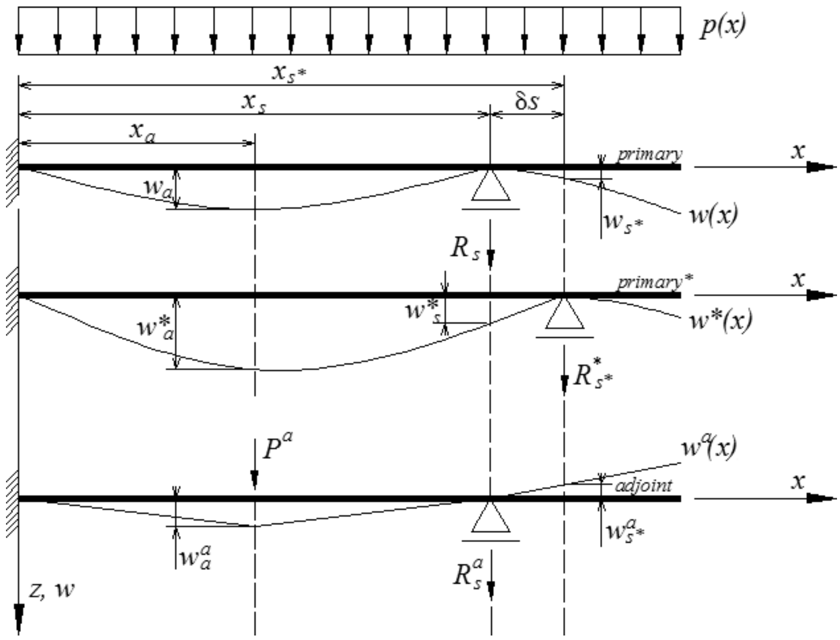

Fig. 5. Variation of the displacement of sandwich beam with respect to the variation of support position: primary structure, structure after variation of the support position (primary *) and adjoint structure

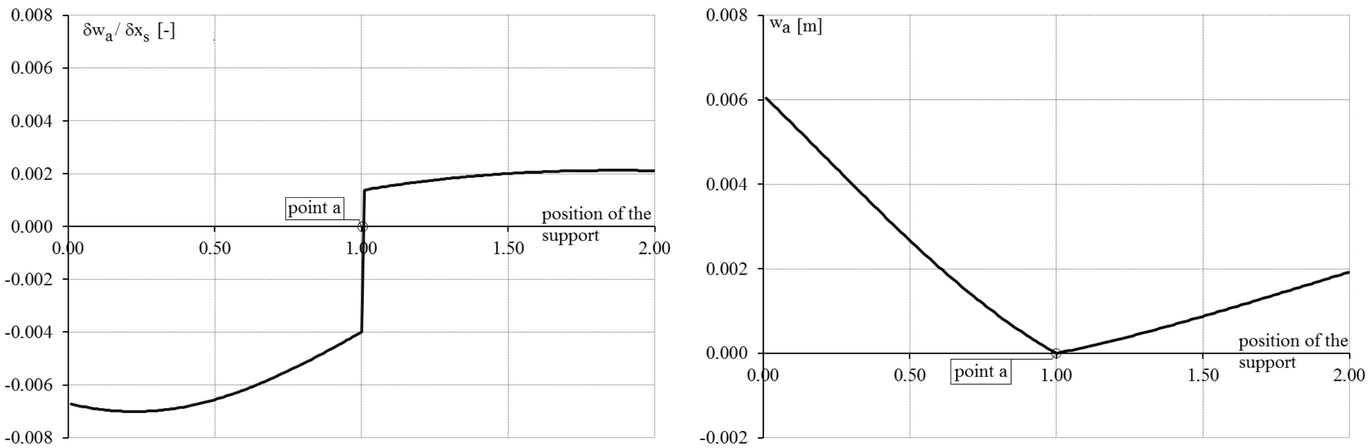

Fig. 6. Variable support position x_s: a) sensitivity gradient $\delta w_a/\delta x_s$, b) displacement w_a

This example proved that in the whole range of support positions x_s both forms of (24) provided the same values of the sensitivity operator and these values were compared with simple finite difference computation. The differences were less than 3%. The displacement w_a and the sensitivity gradient $\delta w_a/\delta x_s$ are shown in Fig. 6. The singular point at $x_s = 1.0$ m represents a trivial solution when the deflection is measured at the support.

Example 2. Consider again the sandwich beam analysed in the example 1, with the same dimensions and loading. Now we will study the sensitivity of the reaction force with respect to variation of support position i.e. $(\delta R_s/\delta x_s)$. Determination of δR_s requires introducing the settlement of the support Δ_s^a in the adjoint structure. Hence, the primal structure is in the state of stress and strain induced by the load p, whereas stresses and strains in the adjoint structure are induced by kinematic distortion in the form of displacement Δ_s^a of the rigid support located at $x = x_s$. Variation of the functional G is expressed by two equivalent formulae

$$\delta G(x_s) = -\delta R_s \Delta_s^a = \left(R_s^a w_{s,x}^{+} + R_s w_{s,x}^{a-}\right)\delta x_s, \qquad (35^1)$$

$$\delta G(x_s) = -\delta R_s \Delta_s^a = \left(R_s^a w_{s,x}^{-} + R_s w_{s,x}^{a+}\right)\delta x_s. \qquad (35^2)$$

In Fig. 7 the value of the support reaction R_s and the sensitivity gradient $(\delta R_s/\delta x_s)$ are depicted for the whole range of support positions $0 \le x_s \le L$. Again, both formulae (35^1) and (35^2) gave the same results. These results were compared with simple finite difference computation. The differences were less than 2%. The values of R_s in Fig. 7b are negative, because we assumed positive direction of R_s downwards in agreement with displacement w.

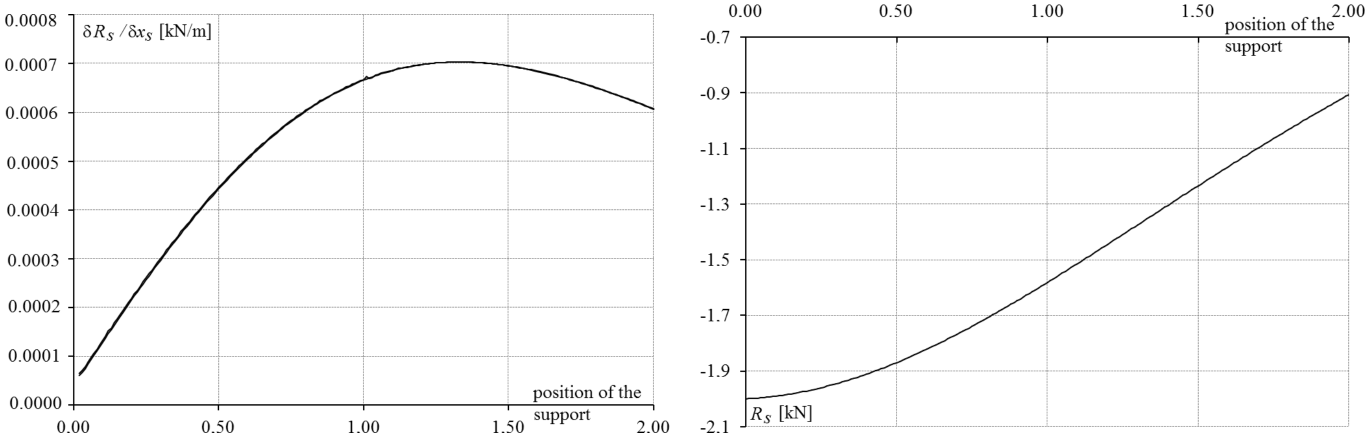

Fig. 7. Variable support position x_s: a) sensitivity gradient $\delta R_s/\delta x_s$, b) value of reaction R_s at $x = x_s$

Example 3. Consider a symmetrical two-span sandwich beam with the length of spans L shown in Fig. 8. Let us assume that the middle support is perfectly rigid, i.e. its flexibility $f = 0$. The steel facings are $t_{F1} = t_{F2} = 0.00045$ m and the thickness of the core $d = 0.10$ m. Hence the bending stiffness of the beam is $B_S = 531.3$ kNm^2. The shear modulus is $G_C = 3000$ kPa. We will analyse three load cases: (a) mechanical load, (b) thermal load and (c) interaction of mechanical and thermal load. The analyses will be carried out for length of span L equal to 3 m, 4 m and 5 m.

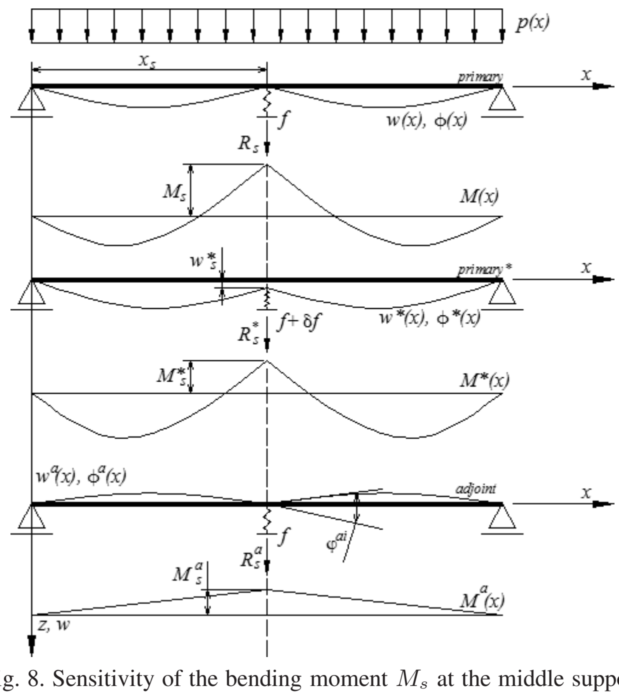

Fig. 8. Sensitivity of the bending moment M_s at the middle support with respect to variation of support flexibility f_s and bending stiffness B_S : primary structure, structure after variation of the design parameters (primary *) and adjoint structure

We are concerned in the value of the bending under the middle support. This is an important issue since this moment produces concentration compressive stress in the upper layer which can result in wrinkling and failure. We will study the variation of moment M_s with respect to variation of support flexibility δf_s and to variation of the bending stiffness δB_s of the beam. The latter one can result from variations of the thickness of cover plates δt_{F1} and δt_{F2}. The design vector is $\xi = [f_S, B_S]$ and (19) takes the form

$$\delta G(\xi) = \int_0^L \left(\mathbf{q}^{ai}\delta\mathbf{Q}\right) dx = +R_s R_s^a \delta f_s - \int_0^L (\mathbf{q}^{ae}\mathbf{q}^e\delta\mathbf{K})dx, \tag{36}$$

or setting $\boldsymbol{Q}(x) = M_S\delta(x - x_S)$ and $\boldsymbol{q} = w''$ we obtain

$$\delta G(\xi) = \varphi^a \delta M_s = R_s^a R_s \delta f_s - \int_0^L \left(w_M''^{ae} w_M''^{e} \delta B_s\right) dx. \tag{37}$$

Figure 8 shows the primary structure with rigid support, and the structure after variation of design parameters δf and δB_S, where static and kinematic fields are denoted by star. Both structures are subjected to mechanical load p and/or temperature load. Next we assume that the adjoint structure is subjected only to initial kinematic distortion in the form of an angle of rotation $\varphi^{ai} = 1$ of cross section over the middle support (Fig. 8).

We solve using FEM the primary and adjoint problems. Next the reaction forces and kinematic fields are introduced into (37) to provide the sensitivity gradients The results for $L = 3.0$ m and rigid support $f = 0$ are:

– *case* (a), when $p(x) = 6.0$ kN/m, the sensitivity gradients are

$\partial M_S/\partial f = 2128.7$ kN^2 (note that kNm/(m/kN)=kN^2) and

$\partial M_S/\partial B_S = 0.003173$ m^{-1} (since kNm/(kNm^2) = m^{-1}),

– *case* (b), when $\Delta T = -30°$C, there is $\partial M_S/\partial f = -112.6$ kN^2,

– *case* (c), when $p(x) = 6.0$ kN/m and $\Delta T = -30°$C, it is $\partial M_S/\partial f = 420.9$ kN^2.

In cases (b) and (c) the gradients $\partial M_S/\partial B_S$ were not computed. Optimisation of sandwich panels with respect to B_S was discussed by the authors in [11].

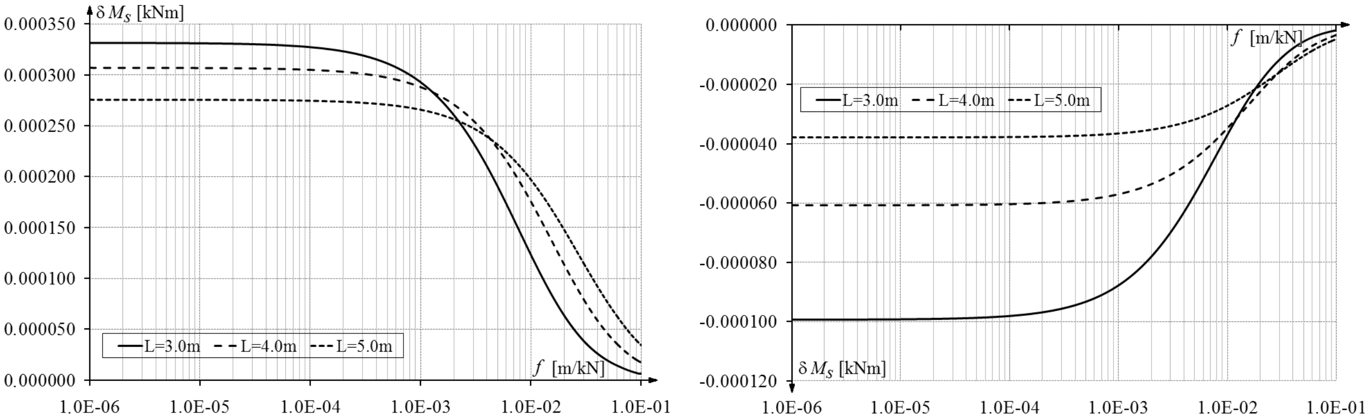

Fig. 9. Variation of the moment at the support δM_s for various flexibility f of support, for three span lengths: a) mechanical load $p(x) = 1.0$ kN/m b) thermal excitation $\Delta T = -30.0°$C

All presented above gradients were compared with the ones obtained from finite difference method. The differences were less than 1.5%.

Figure 9 presents the increments of the moment at the support (δM_s) for a specified small variation of the support flexibility for three lengths of span and for various flexibility coefficients f. It is interesting that all functions are nearly constant in a large range of small flexibility f. For great flexibility all curves tend to zero. Crossing of functions for different L is observed. Note, that in both load cases $p(x) = 1.0$ kN/m (Fig. 9a) and $\Delta T = -30°$C (Fig. 9b) introduction of flexible support is advantageous. In the first load case p, negative moment at the support is induced therefore positive increment $\delta M_S > 0$ improves the response. In the latter load case ΔT there is positive moment M_S, hence negative increment $\delta M_S < 0$ is advantageous, too.

Example 4. Let us check the usefulness of the derived sensitivity operators by the way of the example of support position optimisation. Consider the sandwich beam shown in Fig. 5 with all parameters and loading p described in the Example 1. Our task is to find the optimal position x_s of the support which provides maximum stiffness measured by the total potential energy Π. For a linear elastic material the equivalent formulations are

$$\max \Pi = \max \left(\int_L \left(\frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q} - pw \right) dx \right) \quad \text{or} \quad \min \left(\frac{1}{2} \int_L pw dx \right). \tag{38}$$

We formulate the following problem: find the optimal support position x_s which provides minimum of G

$$G = \int_L F(w) dx = \int_L pw dx. \tag{39}$$

Comparing (39) with (1) and in view of (3), (30), the adjoint problem is identical as the primal one. Hence $R_s^a = R_s$, $w_{s,x}^{a+} = w_{s,x}^{+}$ and $w_{s,x}^{a-} = w_{s,x}^{-}$. The sensitivity operators (24) simplify to one formula

$$\delta G(x_S) = R_s \left(w_{s,x}^{+} + w_{s,x}^{-} \right) \delta x_s. \tag{40}$$

Note that in case of minimization of internal energy

$$G_1 = U = 0.5 \int_L \mathbf{q}^T \mathbf{K} \mathbf{q} dx = 0.5 \int_L pw dx \tag{41}$$

the sensitivity gradient (40) and optimality condition take the forms

$$\delta G_1(x_S) = \delta U = R_s \cdot 0.5 \left(w_{s,x}^{+} + w_{s,x}^{-} \right) \delta x_s \quad \text{and} \quad R_s \cdot 0.5 \left(w_{s,x}^{+} + w_{s,x}^{-} \right) = 0. \tag{42}$$

The total number of the fitness function calls n is limited to 80. For this range of n the best fitness function value equals 1.233884 kNm and occurs for $n = 71$ (point 1 in Fig. 10). It corresponds to the support position $x_s = 1.366$ m. This will be the starting point for next using the operator (40) with the step $\delta x_s = 0.001$ m. Table 1 presents the computational steps required to obtain the optimal global solution with the precision 0.001 m. This precision is assumed to demonstrate the correctness and effectiveness of the algorithm, though in engineering practice so high precision is not needed.

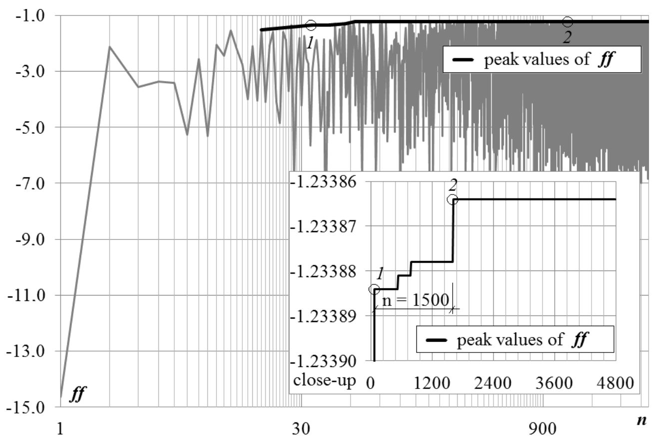

Fig. 10. Course of the fitness function (*ff*) vs. number of computation n by EA

Table 1
Computational steps in the gradient method

Step no.	x_s [m]	$\nabla G(x_s)$	Decision
1	start $x_s = 1.366$	$2.52693 \cdot 10^{-5} > 0$	go left
2	$x_s = 1.365$	$1.30598 \cdot 10^{-5} > 0$	go left
3	$\boldsymbol{x_s}$ **= 1.364**	$0.08448 \cdot 10^{-5} > 0$	go left
4	$x_s = 1.363$	$-1.13820 \cdot 10^{-5} < 0$	return to 3 and stop

For the obtained support position $x_s = 1.364$ m the fitness function equals 1.233864 kNm. Table 1 demonstrates that optimal global solution can be obtained faster using the derived sensitivity operators and combining them with evolutionary algorithm. The sensitivity operators in this hybrid method found the optimal solutions in 4 steps after switching from EA. For comparison solitary EA had to perform almost 1500 computations more, see point 2 in Fig. 10.

Example 5. Consider the variation of the sandwich plate displacement δw_a at the point a, induced by variation δx_s of the support position, see Fig. 4. The sandwich plate is subjected to uniformly distributed load $p(x, y) = 1.6$ kN/m^2 over the strip of a width $b = 0.20$ m at the boundary of the plate. The total length of the plate is $L = 3.0$ m, the width $B = 1.0$ m, the thickness of the facings is $t_{F1} = t_{F2} = 0.0005$ m, the thickness of the core is $d = 0.08$ m, and the shear modulus is $G_C = 2000$ kPa. The adjoint action, according to Eq. (21), is $P^a = 1$.

The structural analysis was performed by use of the Abaqus software. Shell composite elements with a fine mesh 5×5 cm were employed. The main goal of this example was verification of the derived operators (33) in the case of their application to Reissner plates. The sensitivity gradient obtained using the operators (33) equals $(\delta w_a / \delta x_s) = 0.001236$, whereas finite difference method with the step right $\Delta x_s = 0.05$m provided 0.001226. The difference did not exceed 2%.

7. Concluding remarks

In the paper sensitivity operators accounting for variation of geometrical and material parameters of the layers and variation of the support conditions of the thermally and mechanically loaded sandwich panel have been derived. The response functional is formulated in a general form allowing wide practical applications of derived sensitivity gradients. A special attention has been focussed on variation of support conditions since local phenomena at the supports often initiate the failure mechanisms in sandwich plates. The derived sensitivity operators are valid for Timoshenko beam and Reissner plate. They account for the discontinuity of the slope of the displacements at the support of shear deformable beam or plate. Therefore they further develop that is known in the literature, which are valid for the Bernoulli beam. By the way of examples the application of derived sensitivity operators has been illustrated. The sensitivity gradients were computed with two methods, namely using a derived formulae and a simple finite difference method. An excellent agreement has been obtained. The derived sensitivity formulae are more general and numerically more efficient. Computation of structural sensitivity with respect to many variables requires only two solutions (primary and adjoint), whereas in the finite difference approach each variable must be perturbed. The derived sensitivity operators well illustrate physical and engineering meaning of optimality conditions.

REFERENCES

[1] H. Beyer and B. Sendhoff, “Robust optimisation – a comprehensive survey”, *Comp. Meth. in Appl. Mechanics and Engineering* 196, 3190–3218 (2007).
[2] E. Bozhevolnaya and A. Lyckergaard, “Local effects at core junctions of sandwich structures under different types of loads”, *Composite Structures* 73, 24–32 (2006).
[3] E. Bozhevolnaya, A. Lyckegaard, and O.T. Thomsen, “Novel design of foam core junctions in sandwich panels”, *Composites: Part* B 39, 185–190 (2008).
[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience Publisher Inc., New York, 1953.
[5] A. Garstecki and Z. Mróz, “Optimal design of supports of elastic structures subjected to loads and initial distortions”, *Mech. Struct. & Mach.* 15, 47–68 (1987).
[6] A. Garstecki and K. Thermann, “Sensitivity of frames to variations of hinges in dynamic and stability problems”, *Structural Optimization* 4, 108–114 (1992).
[7] U. Icardi and L. Ferrero, “Optimisation of sandwich panels with functionally graded core and faces”, *Composite Science and Technology* 69, 575–585 (2009).
[8] Z. Mróz and A. Mironov, “Optimal design for global mechanical constraints”, *Arch. Mech.* 32 (4), 505–516 (1980).
[9] Z. Pozorski, “Sensitivity analysis of frames with unspecified dynamic load and joint parameters accounting for damping”, *Computer Assisted Mechanics and Eng. Sciences* 13, 641–653 (2006).
[10] C.A. Steeves and N.A. Fleck, “Material selection in sandwich beam construction”, *Scripta Materialia* 50, 1335–1339 (2004).
[11] R. Studziński, Z. Pozorski, and A. Garstecki, “Optimal design of sandwich panels with soft core”, *J. Theoretical and Applied Mechanics* 47 (3), 685–699 (2009).
[12] R. Studziński, Z. Pozorski, and A. Garstecki, “Failure maps of sandwich panels with soft core”, *Proc. 10th Int. Conf. Modern Building Materials, Structures and Techniques* 1, 1060–1065 (2010).
[13] L. Valdevit, J.W. Hutchinson, and A.G. Evans, “Structurally optimized sandwich panels with prismatic cores”, *Int. J. Solids and Structures* 41, 5105–5124 (2004).
[14] C.M. Wang, J.N. Reddy, and K.H. Lee, *Shear Deformable Beams and Plates. Relationships with Classical Solutions*, Elsevier Science, Oxford, 2001.