

Fixed final time and free final state optimal control problem for fractional dynamic systems – linear quadratic discrete-time case

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Abstract. The optimization problem for fractional discrete-time systems with a quadratic performance index has been formulated and solved. The case of fixed final time and a free final state has been considered. A method for numerical computation of optimization problems has been presented. The presented method is a generalization of the well-known method for discrete-time systems of integer order. The efficiency of the method has been demonstrated on numerical examples and illustrated by graphs. Graphs also show the differences between the fractional and classical (standard) systems theory. Results for other cases of the fractional system order (coefficient α) and not illustrated with numerical examples have been obtained through a computer algorithm written for this purpose.

Key words: fractional order systems, discrete-time systems, optimal control, linear quadratic performance index.

1. Introduction

Fractional calculus is an extension of the traditional calculus of the integer order, where the definition of derivatives and integrals are defined for a non-integer (real or complex) order. Using fractional calculus we can get a more detailed mathematical model of physical processes or experiments. A very good example of the use of the fractional calculus is modeling of an ultracapacitor [1] or the heating process [2]. Also, other areas of science and technology have started to pay more attention to these concepts and it may be noted that the fractional calculus is being adopted in the fields of signal processing, system modeling and identification, and control [3–6].

Dynamic optimization problems for integer (not fractional) order systems have been widely considered in literature (see e.g. [7–10]). Mathematical fundamentals of the fractional calculus, such as basic definitions of derivatives and integrals and their relationship, are given in the monographs [11–13] and the fractional differential equations and their applications have been addressed in [14–15]. A numerical simulation of the fractional order control systems has been investigated in [16]. One of the fractional discretization method has been presented in [17]. Some optimal control problems with fixed final time and a final state for continuous-time systems of the fractional order have been investigated in [18–26]. The fractional Kalman filter and its application have been addressed in [27–28]. Some recent interesting results in fractional systems theory and its applications to standard and positive systems can be found in [29–32].

In this paper the optimization problem with fixed final time for fractional discrete-time systems with quadratic performance index are formulated and solved. The case of a free final state with fixed final time is considered. The case of a fixed final state with fixed final time has been investigated in [33]. A method for a numerical computation of the solution of such an optimal control problem is presented. The presented method is a generalization of the well-known method for discrete-time systems of an integer order. The efficiency of the method is demonstrated on a numerical example and illustrated by graphs. Graphs also show the differences between the fractional and classical (standard) systems theory. It is shown that in a case when alpha is an integer number the presented method is equivalent to well known results for discrete-time systems of an integer order. Results for other cases of the fractional system order (coefficient α) and not illustrated with numerical examples are obtained through a computer algorithm written for this purpose [34].

The paper is organized as follows. In Sec. 2 some preliminaries are recalled and the problem is formulated. Also a general solution and a link to the classical theory is demonstrated in Sec. 2. The solution of the problem in case of a free final state is presented in Sec. 3. In Sec. 4 a procedure for computation of the solution is proposed and illustrated by a numerical example. Conclusions of the paper are given in Sec. 5.

The following notation is used: \mathcal{R} – the set of real numbers, $\mathcal{R}^{n \times n}$ – the set of $n \times n$ real matrices, Z_+ – the set of positive integers, $S \geq 0$ is a matrix with non-negative elements, $R > 0$ is a matrix with positive elements.

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2. Problem formulation and general solution

The fractional continuous-time system is described by the equations [31]

$${}^{RL}D_t^\alpha x(t) = Ax(t) + Bu(t),$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are respectively the state and control vectors, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and ${}^{RL}D_t^\alpha x(t)$ is Riemann-Liouville fractional derivative.

The Grunwald-Letnikov (shifted) approximation of fractional order derivative [17] is given as

$${}^{RL}D_t^\alpha x(t) \approx \frac{1}{h^\alpha} \sum_{\Sigma}^n j = 0 (-1)^\alpha \binom{\alpha}{j} x_{k-j+1}.$$

Using the above two relations we can obtain a fractional discrete-time system, described by the equations

$$x_{k+1} = \sum_{j=0}^k d_j x_{k-j} + Bu_k, \quad k \in \mathbb{Z}_+, \quad (1a)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are respectively the state and control vectors, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and

$$d_0 = A_\alpha = A + \alpha I_n, \quad 0 < \alpha < 1, \\ d_j = (-1)^j \binom{\alpha}{j+1} I_n, \quad j = 1, \dots, k, \quad (1b)$$

where I_N – the $n \times n$ identity matrix. We assume that the initial value x_0 of the state vector in discrete time $k = 0$ (initial conditions) is given and $h = 1$. The number of discrete time points is $N \in \mathbb{Z}_+$, at which the state vector has to be estimated, while the final value of the state vector at discrete time $k = N$, i.e. $x(k = N) = x_N$ (final conditions) is also pre-determined.

We consider a performance index of the form

$$J_k = S(x_N, N) + \sum_{k=0}^{N-1} F_k(x_k, u_k) = x_N^T S x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k), \quad (2)$$

where

$$R \in \mathbb{R}^{m \times m}, \quad Q \in \mathbb{R}^{n \times n}, \quad S \in \mathbb{R}^{n \times n}$$

and $S \geq 0$, $Q \geq 0$ and $R > 0$, $k = 0, \dots, N - 1$.

Using the Lagrange multiplier theory we write (2) in the extended form as

$$J(u) = x_N^T S x_N + \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k + \left[\sum_{j=0}^k d_j x_{k-j} + Bu_k - x_{k+1} \right]^T \lambda_{k+1} \right). \quad (3)$$

We define the scalar functions H_k , called the Hamiltonians, which are defined as follows:

$$H_k = x_k^T Q x_k + u_k^T R u_k + \left[\sum_{j=0}^k d_j x_{k-j} + Bu_k \right]^T \lambda_{k+1}. \quad (4)$$

Using (4) and (2), we define a new performance index expressed by Hamiltonians, of the form

$$J(u) = x_N^T S x_N + \sum_{k=i}^{N-1} (H_k - x_{k+1}^T \lambda_{k+1}). \quad (5)$$

To the right-hand side of the above equation we add and subtract the term $x_i^T \lambda_i$. By making changes to indices in the second part of the sum we get a performance index of the form

$$J(u) = x_N^T S x_N + x_i^T \lambda_i - x_N^T \lambda_N + \sum_{k=i}^{N-1} (H_k - x_k^T \lambda_k). \quad (6)$$

We shall now examine the increment of the performance index J due to the increments in all the variables x_k , u_k and λ_k . The increment of the performance index we write as follows

$$dJ(u) = [(S + S^T)x_N - \lambda_N] dx_N^T + \lambda_i dx_i^T + \sum_{k=i}^{N-1} [(H_k - \lambda_k) dx_k^T + H_k du_k^T + (H_{k-1} - x_k) d\lambda_k^T]. \quad (7)$$

According to the Lagrange multiplier theory, at a constrained minimum this increment should be zero. Necessary conditions for a constrained minimum are given by

$$0 = \frac{\partial H_k}{\partial u_k^T} \quad \text{for } k = i, \dots, N - 1, \quad (8a)$$

$$\lambda_k = \sum_{k=i}^{N-1} \frac{\partial H_k}{\partial x_k^T} \quad \text{for } k = i, \dots, N - 1, \quad (8b)$$

$$x_{k+1} = \frac{\partial H_k}{\partial \lambda_{k+1}^T} \quad \text{for } k = i, \dots, N - 1 \quad (8c)$$

and

$$\lambda_N = \frac{\partial S(x_N, N)}{\partial x_N^T}, \quad \lambda_i \in \mathbb{R}. \quad (8d)$$

The conditions (8) for the considered performance index (6) and discrete-time fractional system (1) take the form

$$u_k = -[R + R^T]^{-1} B^T \lambda_{k+1}, \quad (9a)$$

$$\lambda_k = [Q + Q^T] x_k + \sum_{j=0}^{N-k-1} d_j^T \lambda_{k+j+1}, \quad (9b)$$

$$x_{k+1} = \sum_{j=0}^k d_j x_{k-j} + Bu_k \quad (9c)$$

and

$$\lambda_N = (S + S^T)x_N, \quad \lambda_0 \in \mathbb{R}. \quad (9d)$$

The conditions (9) for $\alpha = 1$ are equivalent to the conditions for continuous-time systems of integral order (not fractional) after the discretization. Substituting $\alpha = 1$ to (9) yields

$$u_k = -[R + R^T]^{-1}B^T \lambda_{k+1}, \tag{10a}$$

$$\lambda_k = [Q + Q^T]x_k + A_d^T \lambda_{k+1}, \tag{10b}$$

$$x_{k+1} = A_d x_k + B u_k, \tag{10c}$$

where $A_d = A + I_n$.

3. Problem solution in case of free final state and fixed final time

In the case of free final state x_N the variation dx_N is not equal to zero. Therefore, we consider Eqs. (9d) and (9a)–(9c). We assume that the number of discrete time points N is given. The initial conditions and final time are given as

$$x(k = 0) = x_0, \quad N \in \mathbb{Z}_+. \tag{11}$$

Taking into account initial conditions (11) and applying the z-transform to Eqs. (9a) and (9c) we obtain the equations in z domain. Applying then the inverse z-transform we obtain the solution as

$$x_k = \Psi_k x_0 - \sum_{i=0}^{k-1} \Psi_{k-i-1} B [R + R^T]^{-1} B^T \lambda_{i+1}, \tag{12}$$

where

$$\Psi_0 = I_n, \quad \Psi_k = \sum_{j=0}^{k-1} d_j \Psi_{k-j-1}. \tag{13}$$

Equation (12) in matrix form gives

$$\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_N \end{bmatrix} x_0 - \begin{bmatrix} \Psi_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \Psi_{N-1} & \cdots & \Psi_0 \end{bmatrix} B [R + R^T]^{-1} B^T \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix}. \tag{14}$$

Using the well-known matrix operations it can be easily shown that the solution of (9b) has the form

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} \Psi_{N-1}^T \\ \vdots \\ \Psi_0^T \end{bmatrix} \lambda_N + \begin{bmatrix} 0 & \Psi_0^T & \cdots & \Psi_{N-2}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Psi_0^T \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdot [Q + Q^T] \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}. \tag{15}$$

Substituting (9d) to (15) we obtain

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} \Psi_{N-1}^T \\ \vdots \\ \Psi_0^T \end{bmatrix} [S + S^T] x_N + \begin{bmatrix} 0 & \Psi_0^T & \cdots & \Psi_{N-2}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Psi_0^T \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdot [Q + Q^T] \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}. \tag{16}$$

From the above equation it follows that this time range of the vector λ_k does not depend from x_0 . Taking as $T_1 = [R + R^T]$, $T_2 = [Q + Q^T]$ and $T_3 = [S + S^T]$ the above relationship is written in the form

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{N-1} \\ \lambda_N \end{bmatrix} = \begin{bmatrix} \Psi_0^T T_2 & \cdots & \Psi_{N-3}^T T_2 & \Psi_{N-2}^T T_2 & \Psi_{N-1}^T T_3 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \Psi_0^T T_2 & \Psi_1^T T_2 & \Psi_{N-2}^T T_3 \\ 0 & \cdots & 0 & \Psi_0^T T_2 & \Psi_1^T T_3 \\ 0 & \cdots & 0 & 0 & \Psi_0^T T_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}. \tag{17}$$

Substituting (17) to (14) we obtain

$$\begin{bmatrix} W_{11}^0 & W_{12}^0 & \cdots & W_{1,N-1}^0 & W_{1,N}^0 \\ W_{21}^0 & W_{22}^0 & \cdots & W_{2,N-1}^0 & W_{2,N}^0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ W_{N-1,1}^0 & W_{N-1,2}^0 & \cdots & W_{N-1,N-1}^0 & W_{N-1,N}^0 \\ W_{N,1}^0 & W_{N,2}^0 & \cdots & W_{N,N-1}^0 & W_{N,N}^0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_{N-1} \\ \Psi_N \end{bmatrix} x_0, \tag{18}$$

where

$$\begin{bmatrix} W_{11}^0 & W_{12}^0 & \cdots & W_{1,N-1}^0 & W_{1,N}^0 \\ W_{21}^0 & W_{22}^0 & \cdots & W_{2,N-1}^0 & W_{2,N}^0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ W_{N-1,1}^0 & W_{N-1,2}^0 & \cdots & W_{N-1,N-1}^0 & W_{N-1,N}^0 \\ W_{N,1}^0 & W_{N,2}^0 & \cdots & W_{N,N-1}^0 & W_{N,N}^0 \end{bmatrix} = \begin{bmatrix} I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \\ 0 & 0 & \cdots & 0 & I_n \end{bmatrix} \cdot \begin{bmatrix} \Psi_0 & 0 & \cdots & 0 & 0 \\ \Psi_1 & \Psi_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Psi_{N-2} & \Psi_{N-3} & \cdots & \Psi_0 & 0 \\ \Psi_{N-1} & \Psi_{N-2} & \cdots & \Psi_1 & \Psi_0 \end{bmatrix} T_1$$

$$\cdot \begin{bmatrix} \Psi_0^T T_2 & \cdots & \Psi_{N-3}^T T_2 & \Psi_{N-2}^T T_2 & \Psi_{N-1}^T T_3 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \Psi_0^T T_2 & \Psi_1^T T_2 & \Psi_{N-2}^T T_3 \\ 0 & \cdots & 0 & \Psi_0^T T_2 & \Psi_1^T T_3 \\ 0 & \cdots & 0 & 0 & \Psi_0^T T_3 \end{bmatrix} \cdot$$

Using the Gauss-Jordan elimination method we can determine the inverse matrix and write (18) in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} M_{11}^N & M_{12}^N & \cdots & M_{1,N-1}^N & M_{1,N}^N \\ M_{21}^N & M_{22}^N & \cdots & M_{2,N-1}^N & M_{2,N}^N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-1,1}^N & M_{N-1,2}^N & \cdots & M_{N-1,N-1}^N & M_{N-1,N}^N \\ M_{N,1}^N & M_{N,2}^N & \cdots & M_{N,N-1}^N & M_{N,N}^N \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_{N-1} \\ \Psi_N \end{bmatrix} x_0 = \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \vdots \\ \Theta_{N-1} \\ \Theta_N \end{bmatrix} x_0, \tag{20}$$

where M_{ij}^N for $i, j = 1, \dots, N$ are given as

$$W_{pr}^k = \begin{cases} W_{pr}^{k-1} - W_{pk}^{k-1}(W_{kk}^{k-1})^{-1}W_{kr}^{k-1} & \text{for } p \neq k \\ (W_{kk}^{k-1})^{-1}W_{kr}^{k-1} & \text{for } p = k \end{cases}$$

$$M_{ij}^k = \begin{cases} M_{ij}^{k-1} - W_{ik}^{k-1}(W_{kk}^{k-1})^{-1}M_{kj}^{k-1} & \text{for } i \neq k \\ (W_{kk}^{k-1})^{-1}M_{kj}^{k-1} & \text{for } i = k \\ M_{ij}^0 = I_N & \text{for } i = j \\ M_{ij}^0 = 0 & \text{for } i \neq j \end{cases} \tag{21}$$

for $k = 1, 2, \dots, N$; $i, j, p = 1, 2, \dots, N$ and $r = k + 1, \dots, N$.

From (20) we get the optimal value of the vector x_k . Substituting (20) to (17) we can determine the values of the vector λ_k . Then, substituting vector λ_k to (9a) we can determine the value of the optimal control vector u_k .

4. The procedure and example for fixed final state case

From the above considerations, the following procedure for solving the dynamic optimization problem follows:

Procedure 1.

Step 1. For given discrete-time fractional system (1), performance index (2) and initial conditions x_0 write the performance index in the extended form (5) expressed by the Hamiltonian.

Step 2. Determine the necessary conditions (8), which in the quadratic case of performance index are given in the form (9).

Step 3. Using known methods for solving systems of equations, determine the vector x_k relative to the initial conditions x_0 and matrices Ψ_k . Also determine the vector λ_k relative to the x_k and matrices Φ_k^T .

Step 4. Determine the matrices Θ_k from (20). Knowing the initial conditions x_0 and matrices Θ_k determine the values of the vector x_k and λ_k . Knowing the value of the vector λ_k determine the optimal control vector u_k satisfying (1).

Example 1. Consider a discrete-time fractional system (1) with matrices

$$A = \begin{bmatrix} 0.1 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{22}$$

$n = 2, \quad m = 1$

and performance index (4) with matrices

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}, \quad R = [1] \tag{23}$$

with initial conditions given as

$$x_0 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}. \tag{24}$$

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We assume $N = 5$ and $\alpha = 0.7$. Using the foregoing considerations, we obtain

$$\begin{aligned} x_0 &= \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, & x_1 &= \begin{bmatrix} -0.4377 \\ 0.5011 \end{bmatrix}, \\ x_2 &= \begin{bmatrix} -0.2301 \\ 0.2257 \end{bmatrix}, & x_3 &= \begin{bmatrix} -0.1526 \\ 0.1454 \end{bmatrix}, \\ x_4 &= \begin{bmatrix} -0.1075 \\ 0.1064 \end{bmatrix}, & x_5 &= \begin{bmatrix} -0.069 \\ 0.09 \end{bmatrix}, \\ \lambda_0 &= \begin{bmatrix} 7.7728 \\ 9.4877 \end{bmatrix}, & \lambda_1 &= \begin{bmatrix} -0.3464 \\ 2.1705 \end{bmatrix}, \\ \lambda_2 &= \begin{bmatrix} -0.3372 \\ 0.9682 \end{bmatrix}, & \lambda_3 &= \begin{bmatrix} -0.2434 \\ 0.5947 \end{bmatrix}, \\ \lambda_4 &= \begin{bmatrix} -0.1632 \\ 0.3852 \end{bmatrix}, & \lambda_5 &= \begin{bmatrix} -0.0961 \\ 0.2221 \end{bmatrix}, \\ u_0 &= \begin{bmatrix} -0.7389 \\ -0.0539 \end{bmatrix}, & u_1 &= \begin{bmatrix} -0.1469 \\ -0.0294 \end{bmatrix}, \\ u_2 &= \begin{bmatrix} -0.0539 \\ -0.0149 \end{bmatrix}, & u_3 &= \begin{bmatrix} -0.0294 \\ 0 \end{bmatrix}, \\ u_4 &= \begin{bmatrix} -0.0149 \\ 0 \end{bmatrix}, & u_5 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

The minimum values of performance index J_{\min} are given as follows

$$\begin{aligned} J_0 &= [6.1269], & J_1 &= [0.6609], \\ J_2 &= [0.1886], & J_3 &= [0.0818], \\ J_4 &= [0.0364], & J_5 &= [0.0133]. \end{aligned}$$

Figures 1–3 show the above considerations for the system (1) with matrices (22) and the performance index (2) with matrices (23) for four different values of $\alpha = 0.5, 0.7, 0.9, 1.0$ and $N = 5$. Individual results were obtained with the help of a dedicated computer program implementing the above issues.

From Fig. 3, we conclude that the performance index takes smaller values for fractional discrete-time systems for $\alpha = 0.5, 0.7, 0.9$ than for discrete-time system of integer order $\alpha = 1.0$. In terms of physical and technical properties performance indexes present such values as energy consumption, fuel consumption, cost of production, profit, time, accuracy, etc. Consideration of the optimization problems in a case of fractional systems can bring greater benefits than for the systems of integer order.

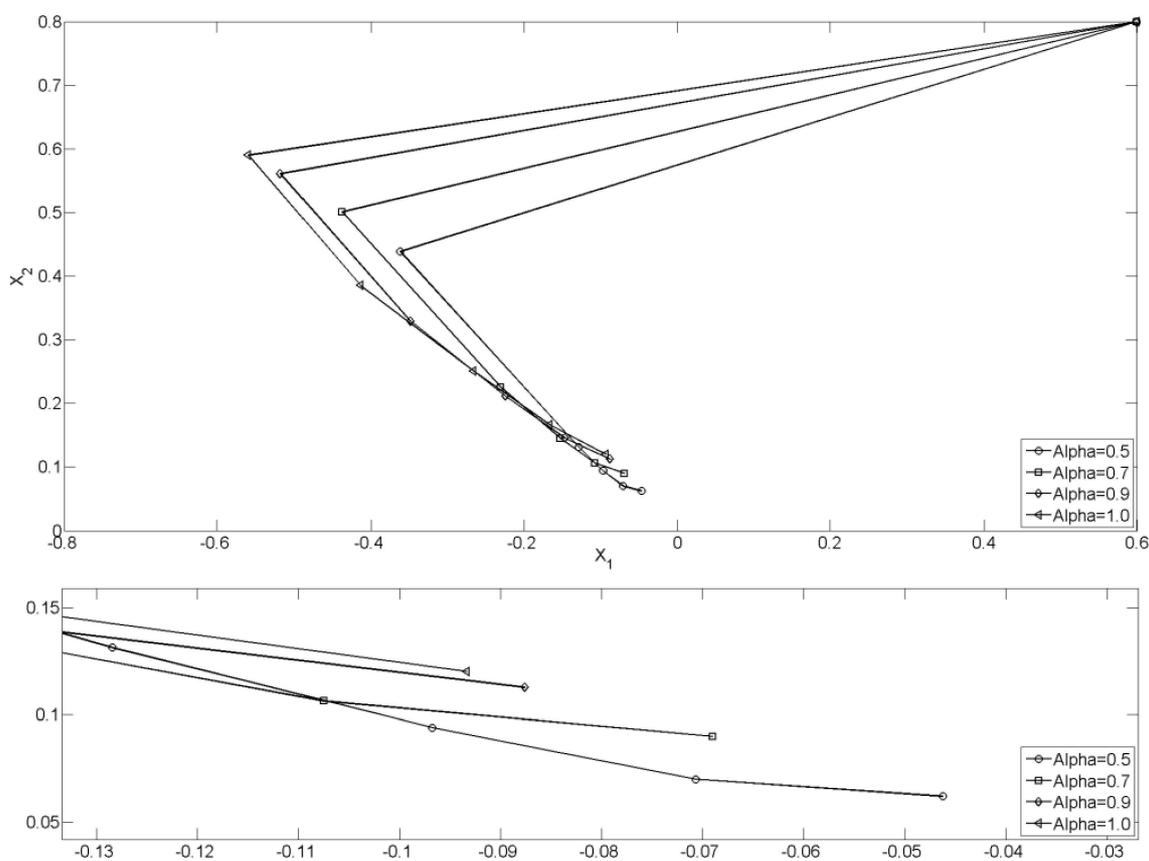


Fig. 1. Optimal trajectory and its zoom of end points for $\alpha = 0.5, 0.7, 0.9, 1.0$ and $N = 5$

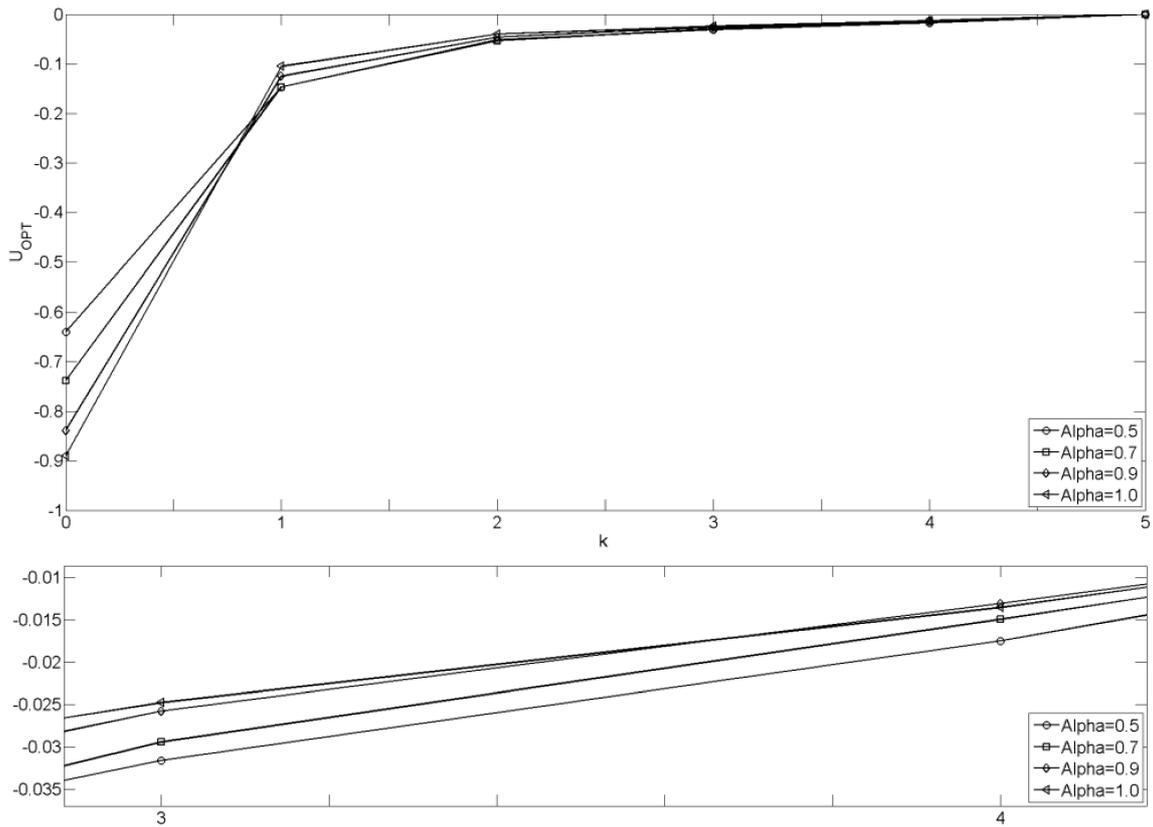


Fig. 2. Optimal control and its zoom of end points for $\alpha = 0.5, 0.7, 0.9, 1.0$ and $N = 5$

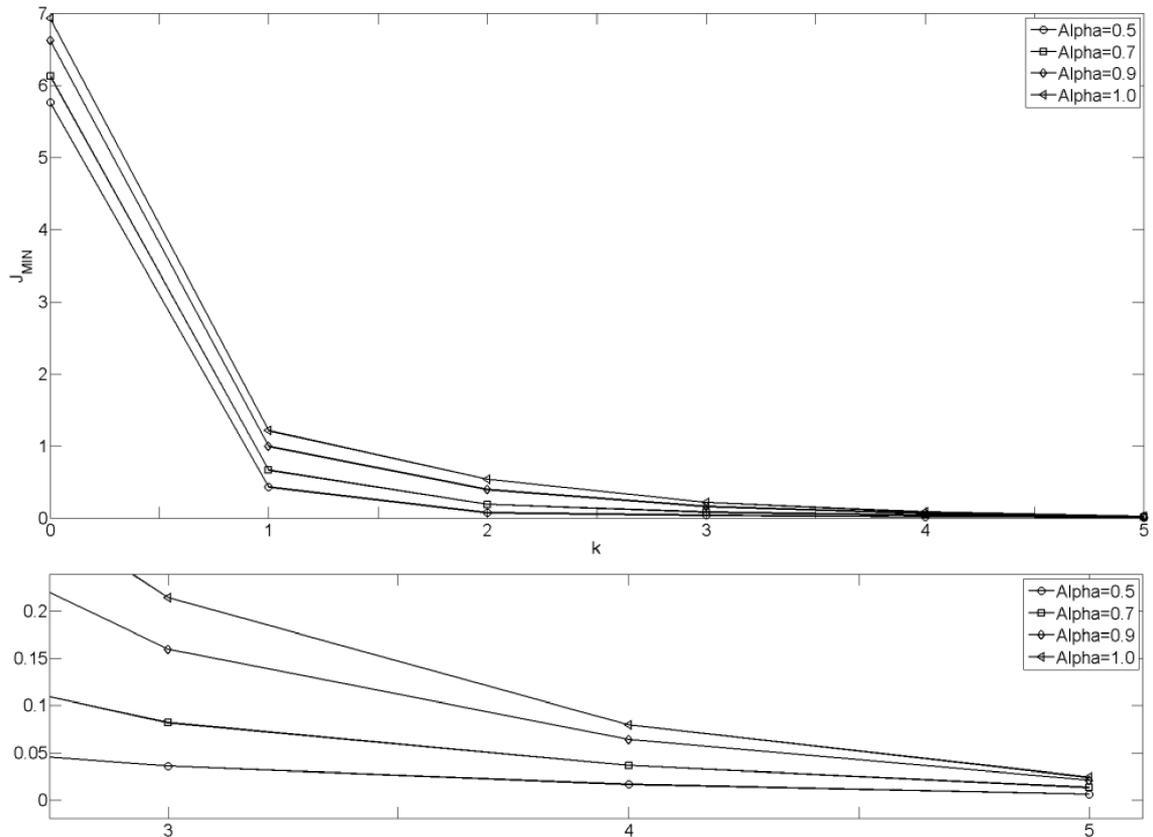


Fig. 3. The minimum values of the performance index and its zoom of end points for $\alpha = 0.5, 0.7, 0.9, 1.0$ and $N = 5$

Example 2. Consider a discrete-time fractional system (1) with matrices (22) and performance index (4) with matrices (23). The initial conditions are given as (24). We assume $N = 10$ and $\alpha = 0.5$. Using the foregoing considerations, we obtain optimal control in form

$$\begin{aligned}
 u_0 &= [-0.6417], & u_1 &= [-0.1469], \\
 u_2 &= [-0.0518], & u_3 &= [-0.0315], \\
 u_4 &= [-0.0216], & u_5 &= [-0.0161], \\
 u_6 &= [-0.0127], & u_7 &= [-0.0103], \\
 u_8 &= [-0.0086], & u_9 &= [-0.054], & u_{10} &= [0].
 \end{aligned}
 \tag{25}$$

The state vector is given as follows

$$\begin{aligned}
 x_0 &= \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, & x_1 &= \begin{bmatrix} -0.3633 \\ 0.4384 \end{bmatrix}, & x_2 &= \begin{bmatrix} -0.1298 \\ 0.1297 \end{bmatrix}, \\
 x_3 &= \begin{bmatrix} -0.0987 \\ 0.0917 \end{bmatrix}, & x_4 &= \begin{bmatrix} -0.0736 \\ 0.0667 \end{bmatrix}, \\
 x_5 &= \begin{bmatrix} -0.0589 \\ 0.0528 \end{bmatrix}, & x_6 &= \begin{bmatrix} -0.0487 \\ 0.0436 \end{bmatrix}, \\
 x_7 &= \begin{bmatrix} -0.0412 \\ 0.0372 \end{bmatrix}, & x_8 &= \begin{bmatrix} -0.0353 \\ 0.0325 \end{bmatrix}, \\
 x_9 &= \begin{bmatrix} -0.0303 \\ 0.0293 \end{bmatrix}, & x_{10} &= \begin{bmatrix} -0.0222 \\ 0.029 \end{bmatrix}.
 \end{aligned}$$

The minimum values of performance index J_{\min} are given as follows

$$\begin{aligned}
 J_0 &= [5.7746], & J_1 &= [0.4429], & J_2 &= [0.0859], \\
 J_3 &= [0.0495], & J_4 &= [0.0303], & J_5 &= [0.0198], \\
 J_6 &= [0.0133], & J_7 &= [0.0088], & J_8 &= [0.0056], \\
 J_9 &= [0.0032], & J_{10} &= [0.0014].
 \end{aligned}$$

Now we assume that elements of matrices (22) of fractional discrete-time system (1) have been changed. We consider two cases, i.e.

$$\begin{aligned}
 \text{a) } A_1 &= \begin{bmatrix} 0.15 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}, \\
 \text{b) } B_1 &= \begin{bmatrix} 2.05 \\ 1 \end{bmatrix}.
 \end{aligned}
 \tag{26}$$

We apply optimal control (25) to fractional discrete-time system with matrices (26) and we compute new state vector and minimal values of performance index.

The Figs. 4–8 show the above considerations for the system (1) with matrices (22) and (26) and the performance index (4) with matrices (23) for four different values of $\alpha = 0.5, 1.0$ and $N = 10$. Individual results were obtained with the help of a dedicated computer program implementing the above issues.

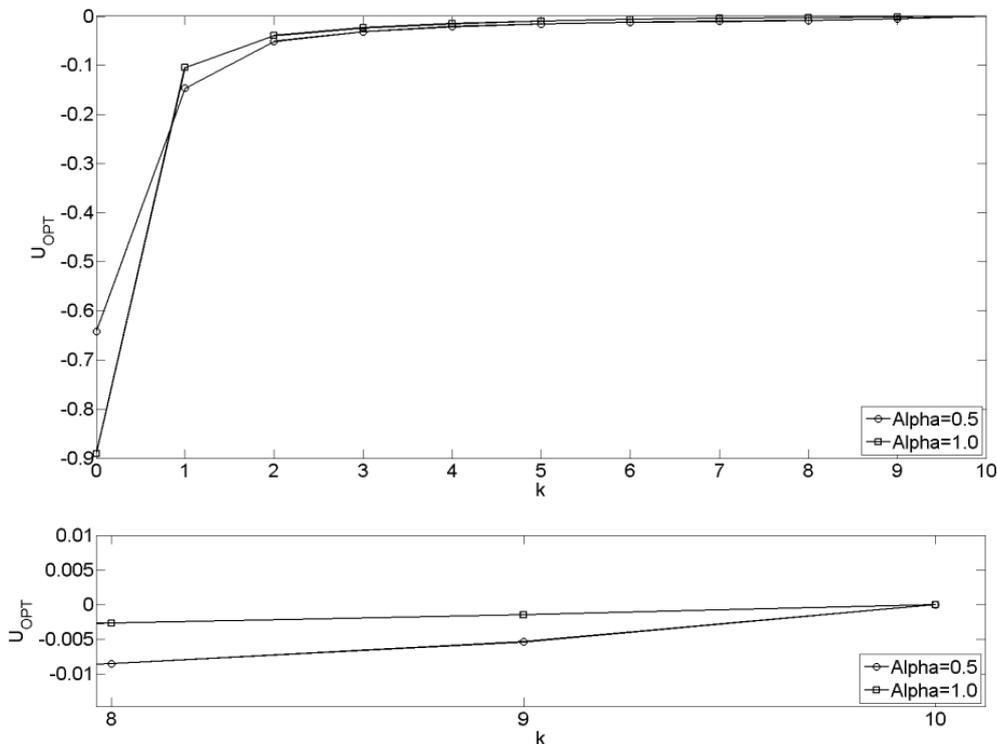


Fig. 4. Optimal control for $\alpha = 0.5, 1.0$ and $N = 10$

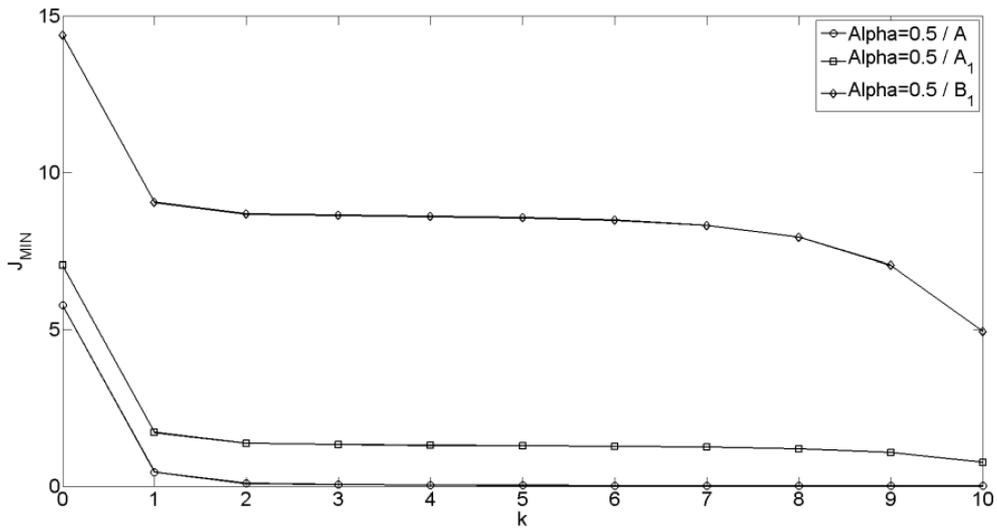


Fig. 5. The minimum values of the performance index for $\alpha = 0.5$ and $N = 10$

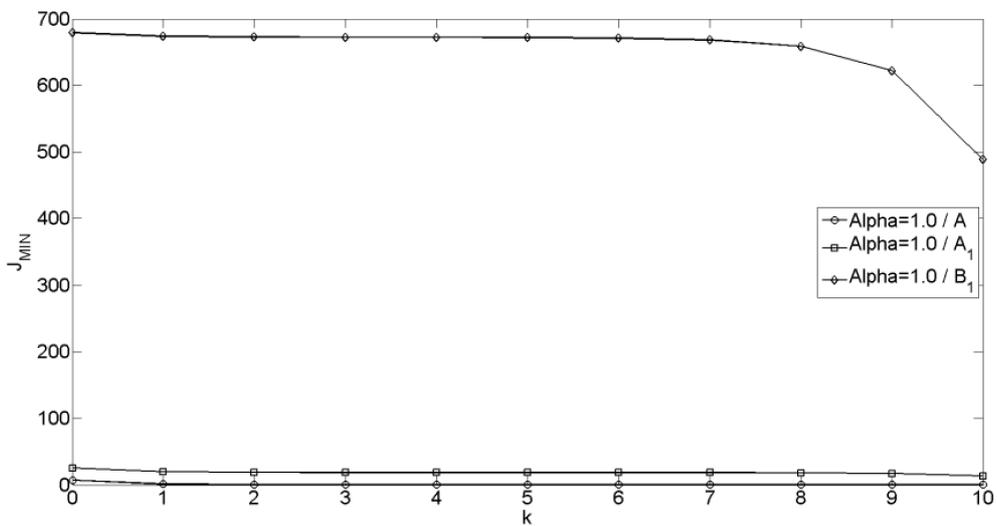


Fig. 6. The minimum values of the performance index for $\alpha = 1.0$ and $N = 10$

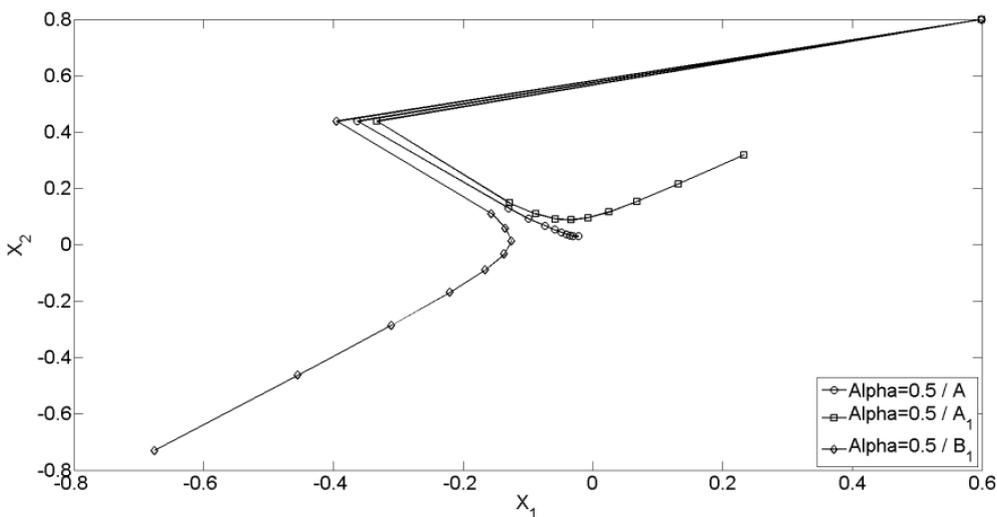


Fig. 7. Optimal trajectory for $\alpha = 0.5$ and $N = 10$

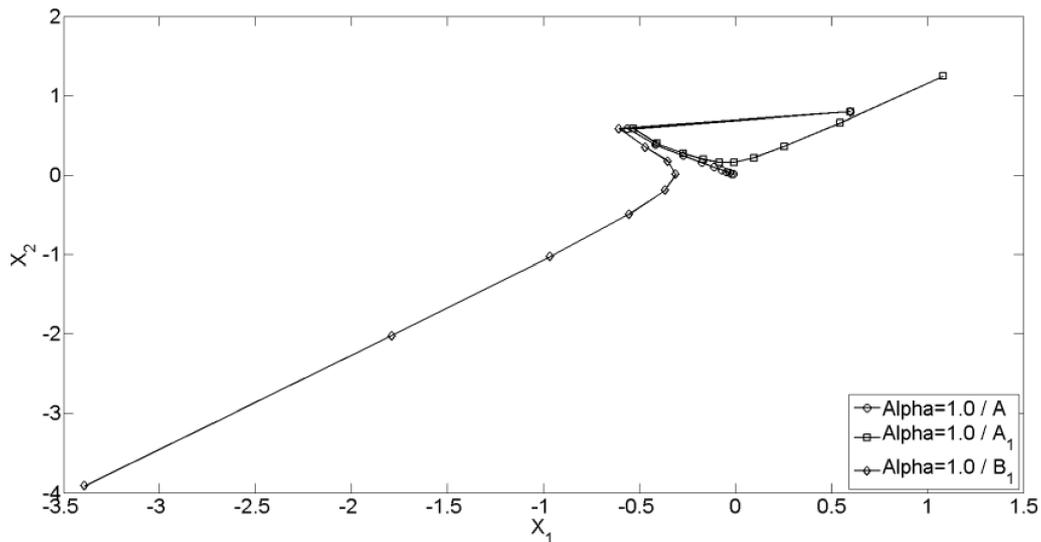


Fig. 8. Optimal trajectory for $\alpha = 1.0$ and $N = 10$

5. Conclusions

An optimal control problem for fractional discrete-time systems with quadratic performance index has been formulated and solved. A method for numerical computation of optimal control problems in the case of the free final state and fixed final time for discrete-time fractional systems has been presented. The presented method is a generalization of the well known method for discrete-time systems of integer order. The efficiency of this method has been demonstrated by a numerical examples and computer simulations. It has been shown that in a case when alpha is an integer number the presented method is equivalent to well known results for discrete-time systems of an integer order. The differences between results of the fractional and integer order systems theory have been shown. A computer algorithm for solving this issue with a quadratic performance index for fractional discrete-time systems has been tested for different values of the coefficient alpha.

REFERENCES

- [1] A. Dzieliński, D. Sierociuk, and G. Sarwas, "Some applications of fractional order calculus", *Bull. Pol. Ac.: Tech.* 58 (4), 583–592 (2010).
- [2] D. Sierociuk, A. Dzieliński, G. Sarwas, I. Petras, I. Podlubny, and T. Skovranek, "Modelling heat transfer in heterogeneous media using fractional calculus", *Phil. Trans. R. Soc. A* (371), 20130146 (2013).
- [3] D. Sierociuk and B.M. Vinagre, "Infinite horizon state-feedback LQR controller for fractional systems", *Decision and Control (CDC), 2010 49th IEEE Conf.* 1, 6674–6679 (2010).
- [4] C. Tricaud and Y.Q. Chen, "An approximate method for numerically solving fractional order optimal control problems of general form", *Comput. Math. Appl.* 59, 1644–1655 (2010).
- [5] C. Tricaud and Y.Q. Chen, "Time-optimal control of systems with fractional dynamics", *Int. J. Differ. Equ.* 1, 1–16 (2010).
- [6] M. Busłowicz, "Stability analysis of continuous-time linear systems consisting of n subsystems with different fractional orders", *Bull. Pol. Ac.: Tech.* 60 (2), 279–284 (2012).
- [7] R. Bellman, *Dynamic Programming*, University Press, Princeton, 1957.
- [8] T. Kaczorek, *Control Theory*, vol. II, PWN, Warsaw, 1981, (in Polish).
- [9] F.L. Lewis and V.L. Syrmos, *Optimal Control*, 2nd ed, Wiley-IEEE, London, 1995.
- [10] D.S. Naidu, *Optimal Control Systems*, Electrical Engineering, CRC Press, Inc., Boca Raton, 2002.
- [11] P. Ostalczyk, *Epitome of the Fractional calculus: Theory and Its Applications in Automatics*, Lodz University of Technology Publishing House, Łódź, 2008.
- [12] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*, Academic Press, San Diego, 1999.
- [13] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science, New York, 1993.
- [14] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier Science Inc., New York, 2006.
- [15] F. Liu, M.M. Meerschaert, S. Momani, N.N. Leonenko, C. Wen, and O.P. Agrawal, "Fractional differential equations", *Int. J. Differ. Equations* 464321, CD-ROM (2010).
- [16] X. Cai and F. Liu, "Numerical simulation of the fractional-order control system", *J. Appl. Math. Comput.* 23, 229–241 (2007).
- [17] M.M. Meerschaert and C. Tadjeran, "Finite difference approximations for fractional advection-dispersion flow equations", *J. Computational and Applied Mathematics* 172 (1), 65–77 (2004).
- [18] O.P. Agrawal, "Formulation of Euler – Lagrange equations for fractional variational problems", *J. Mathematical Analysis and Applications* 272 (1), 368–379 (2002).
- [19] O.P. Agrawal, "A general formulation and solution scheme for fractional optimal control problems", *Nonlinear Dynamics* 38, 323–337 (2004).
- [20] O.P. Agrawal, "Fractional variational calculus and the transversality conditions", *J. Physics Math. Theor.* 39, 10375–10384 (2006).

- [21] O.P. Agrawal, "Fractional variational calculus in terms of Riesz fractional derivatives", *J. Physics. Math. Theor.* 40 (24), 6287–6303 (2007).
- [22] O.P. Agrawal, "A general finite element formulation for fractional variational problems", *J. Mathemat. Analysis and Appl.* 337 (1), 1–12 (2008).
- [23] G.S.F. Frederico and D.F.M. Torres, "Fractional conservation laws in optimal control theory", *Nonlinear Dynamics* 53 (3), 215–222 (2008).
- [24] Z. D. Jelcic and N. Petrovacki, "Optimality conditions and a solution scheme for fractional optimal control problems", *J Struct Multidisciplinary Optimization* 38 (6), 571–581 (2008).
- [25] R.K. Biswas and S. Sen, "Fractional optimal control problems with specified final time", *ASME J. Comput. Nonlinear Dyn.* 6, 021009.1–021009.6 (2011).
- [26] R.K. Biswas and S. Sen, "Fractional optimal control problems: a pseudo-state-space approach", *J. Vib. Control* 17, 1034–1041 (2010).
- [27] D. Sierociuk and A. Dzieliński, "Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation", *Int. J. Appl. Math. Comput. Sci.* 16 (1), 129–140 (2006).
- [28] D. Sierociuk, I. Tejado, and B.M. Vinagre, "Improved fractional Kalman filter and its application to estimation over lossy networks", *Signal Processing* 91 (3), 542–552 (2011).
- [29] T. Kaczorek, "New stability tests of positive standard and fractional linear systems", *Circuits and Systems* 2, 261–268 (2011).
- [30] T. Kaczorek, "Positive linear systems consisting of n subsystems with different fractional orders", *IEEE Trans. on Circuits and Systems* 58, 1203–1210 (2011).
- [31] T. Kaczorek, *Selected Problems of Fractional Systems Theory, Lecture Notes in Control and Information Sciences* 411, Springer, Berlin, 2011.
- [32] T. Kaczorek, "Positive fractional continuous-time linear systems with singular pencils", *Bull. Pol. Ac.: Tech.* 60 (1), 9–12 (2012).
- [33] A. Dzieliński and P. M. Czyronis, "Fixed final time optimal control problem for fractional dynamic systems – linear quadratic discrete-time case", *Advances in Control Theory and Automation* 1, 71–80 (2012).
- [34] A. Dzieliński and P. M. Czyronis, "Computer algorithms for solving optimization problems for discrete-time fractional systems", *Eur. Control Conf. ECC* 1, CD-ROM (2013).