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Squaring down plant model and I/O grouping strategies for a dynamic decoupling of left-invertible MIMO plants

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Abstract. In the paper problems with dynamic decoupling of the left-invertible multi-input multi-output dynamic (MIMO) linear time invariant plants using a squaring down technique are considered. The procedure of squaring down the plant model and grouping of plant inputs and outputs are discussed. The final part of the paper includes a few examples of different strategies of synthesis of a decoupled system along with conclusions and final remarks.

Key words: multivariable systems, MIMO, dynamic decoupling, left-invertible systems, input-output pairing, polynomial approach.

1. Introduction

The main characteristic property of the multi-input multi-output (MIMO) dynamic plants is the cross coupling of theirs inputs and outputs, which can make the process of designing the control system seriously difficult. There are two ways to deal with such a problem. The first is to establish the level of coupling, match the appropriate inputs and outputs and treat the system as a set of single input single output systems (SISO). In that case one has to ignore the cross influence between such SISO systems. The second one is to decouple the system, i.e. to find a control system for which a specific group of inputs affect a specific group of outputs and no element of this input group have influence on any other output component of the system. After decoupling the transfer function matrix of the system becomes diagonal (or block diagonal), thus the system is divided into small subsystems, which can be analyzed irrespectively of each other.

Even if the plant is going to be decoupled it is essential to establish the expected level of coupling between the selected control loops, as it would be easier to operate the system from the practical point of view. In the course of years several interaction measuring methods have been proposed. The most popular is, presented in [1], Relative Gain Array (RGA) and its further modifications, Effective Relative Gain Array (ERGA) [2], Dynamic Relative Gain Array (DRGA) [3], Nonsquare Relative Gain Array (NSRGA) [4] and Nonlinear Relative Gain Array (NRGA) [5]. A detailed description of the above mentioned methods may be found in [6]. There are also used Gramian-based measures Hankel Interaction Index Array (HIIA) [7], Participation Matrix (PM) [8] and some others. In [9] the closed loop performance in terms of the output variance is computed for each control structure and the pairing corresponding to the lowest output variance is selected. However, the disadvantage of this method is that the number of possible pairings for a system with m inputs and m outputs is $m!$ and the computational burden grows accordingly. All RGA, the HIIA and PM methods do not suffer from this disadvantage.

The dynamic decoupling for the MIMO systems was intensively studied in the past [10–17]. However most of the proposed methods are often limited to the square or right invertible plants with minimum phase transmission zeros only. Moreover, most of them allow one to exist in the decoupled system some fixed poles, which can result in its instability. The problem of dynamic block decoupling of the left invertible plants was rarely studied in the past [16, 18] however in everyday practice there is often a need to maintain and control processes with more outputs than inputs. It may happen, e.g. during system failure when one loses some actuators and the precise control of all outputs is not possible. A common strategy in such situations may be squaring down the plant. That is, necessary number of outputs or inputs are deleted or added from the transfer function matrix to obtain a square plant model. However, adding or deleting unnecessary outputs or inputs means more costs, reduction of the degrees of freedom, less reliable measured information deteriorating control performance and so on.

Nonsquare plants with more outputs than inputs are functionally uncontrollable and their outputs cannot be perfectly derived towards the desired set points. This case may occur when one or more actuators are saturated or damaged. Then the squaring down procedure often contains NSRGA analysis and selection due to perfect control in the least square sense [4, 6]. What plays a larger role is the singularity of the nonsquare and squared down plant transfer matrices. Singularity of the nonsquare plant is rare due to that it requires a simultaneous vanishing of many minors depending on the dimensions of the matrix. Thus it is well known that transmission zeros for nonsquare plants are very uncommon. However, singularity of the squared down plant may dramatically change control system synthesis conditions. Especially when such zeros may decide on closed-loop system stability. It is well

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known in the decoupling theory that some poles of the decoupled (compensated) system, related to the so-called interconnection transmission zeros of the plant, are fixed. These can generate uncontrollable and/or unobservable parts of the feedback closed-loop system. Cancelations of such nonminimum phase zeros (unstable hidden modes) make the system unstable [14, 16, 19].

The paper deals with the problem of the interconnection transmission zeros, which may occur when the decoupler for the nonsquare plant is being calculated. It complements the results obtained in [18] and [20], which present an universal algorithm for dynamic decoupling for linear plants that can be unstable, non-minimum phase or both and problems with decoupler recalculation after plants actuator damage and loss of operability, respectively. Such recalculations of the control systems may be important from a practical point of view, e.g. in the fault tolerant control systems when the recalculations are done after actuator fault and loss of the plant operability [21]. As a plant actuator fault and/or switching in control systems may result in a loss of the system stability and/or controllability [22] the synthetized control system has to be carefully checked before its implementation. The off-line analysis presented in the paper may allow one to prepare alternative control structures which, after fault, may replace the standard one and make possible to e.g., switch off the system in an controlled way.

The paper is organized as follows. The problem statement and decoupling concept have been brought in Secs. 2. and 3. Section 4 presents conditions for decoupling of the left-invertible plants while in Sec. 5 the squaring down and I/O grouping procedure are given. An example which demonstrates the discussed problems is presented in Sec. 6. Finally the paper is concluded in Sec. 7.

2. Problem statement

We consider a controllable and observable LTI MIMO model of the plant defined by the state and output equations

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t), \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in R^n$, $\boldsymbol{u}(t) \in R^m$ and $\boldsymbol{y}(t) \in R^l (l > m)$ are the state, input and output vectors, respectively. In the polynomial matrix approach the transfer matrices (MFD) of all elements of the system are defined by pairs of polynomial matrices usually as relatively right prime (*r.r.p.*) for plants, and relatively left prime (*r.l.p.*) for other elements of the control system. Applying this approach, the plant model (1) can be transformed into the *r.r.p.* matrix fraction description in the frequency s-domain as follows

$$\boldsymbol{y} = \boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s)\boldsymbol{u}, \tag{2}$$

where

$$\boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}. \tag{3}$$

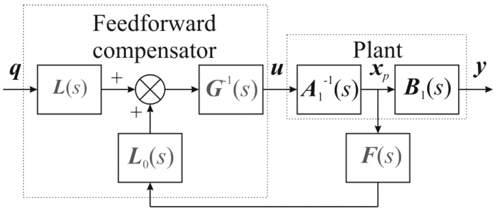

Fig. 1. Structure of the decoupled control system

Assuming dynamic block decoupling of the designed control system we group the output and the vector of exogenous signals into k blocks according to the partitions

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{y}_1(t) \\ \vdots \\ \boldsymbol{y}_i(t) \\ \vdots \\ \boldsymbol{y}_k(t) \end{bmatrix}, \qquad \boldsymbol{q}(t) = \begin{bmatrix} \boldsymbol{q}_1(t) \\ \vdots \\ \boldsymbol{q}_i(t) \\ \vdots \\ \boldsymbol{q}_k(t) \end{bmatrix}, \tag{4}$$

where $\boldsymbol{y}_i(t) \in R^{l_i}$, $\sum_{i=1}^{k} l_i = l$, $\boldsymbol{q}_i(t) \in R^{m_i}$, $\sum_{i=1}^{k} m_i = m$.

We want to design a decoupled system in which each part $i = 1, 2, ..., k$ of a system defined by pairs of vector signals $\boldsymbol{q}_i(t)$, $\boldsymbol{y}_i(t)$ could be controlled independently of other parts $j \neq i$. Moreover, each part of the block-decoupled system should be designed with individually supposed dynamic properties according to the set requirements.

Then the problem may be formulated as follows. Find an input output grouping (4) for system (1) to be dynamically block-decoupled so as to ensure that the decoupled system meets all synthesis goals, i.e. stability, lowest possible rank and exhibits the required performance. The problem is especially put for plants with more outputs than inputs after a failure.

3. Decoupling concept

The goal of decoupling the LTI dynamic system can be achieved in a control system structure presented in Fig. 1, which contains a dynamic feedforward compensator and a feedback matrix. It is one of the most common decoupling concept utilized in many papers, e.g. [13, 14, 16, 18, 19, 23]. The main decoupling problem is to find a method for block decoupling of the control system (between the signals $\boldsymbol{q}$ and $\boldsymbol{y}$) so as to obtain the stable transfer matrix $\boldsymbol{T}_{\boldsymbol{yq}}(s)$ free of cancellation of unstable hidden modes.

The feedback law, employed to decouple the system (the linear state variable feedback along with dynamic feedforward) is described by

$$\boldsymbol{u}(s) = \boldsymbol{G}^{-1}(s)\boldsymbol{L}_0(s)\boldsymbol{f}(s) + \boldsymbol{G}^{-1}(s)\boldsymbol{L}(s)\boldsymbol{q}(s), \tag{5}$$

where

$$\boldsymbol{f}(s) = \boldsymbol{F}(s)\boldsymbol{x}_{\boldsymbol{p}}(s) \triangleq \boldsymbol{F}\boldsymbol{x}(t), \tag{6}$$

$\boldsymbol{x}_{\boldsymbol{p}}(s)$ is a partial state vector of the plant, $\boldsymbol{G}(s) \in R[s]^{m\times m}$, $\boldsymbol{L}(s) \in R[s]^{m\times l}$, $\boldsymbol{L}_0(s) \in R[s]^{m\times m}$, $\boldsymbol{F}(s) \in R[s]^{m\times m}$ are polynomial matrices such that $\boldsymbol{G}^{-1}(s)\boldsymbol{L}_0(s)$

and $\boldsymbol{G}^{-1}(s)\boldsymbol{L}(s)$ are proper and $\boldsymbol{F}(s)\boldsymbol{A}_1^{-1}(s)$ is strictly proper. Without any loss of generality the matrix $\boldsymbol{L}_0(s)$ may be taken as $\boldsymbol{L}_0(s) = \boldsymbol{I}_m$.

According to this scheme the considered decoupling system is defined in s-domain by proper and possible low-order transfer matrix $\boldsymbol{G}^{-1}(s)\boldsymbol{L}(s)$ for the dynamic feedforward compensator along with a feedback matrix $\boldsymbol{F}(s)$. For the applied decoupling law the transfer matrix $\boldsymbol{T}_{\boldsymbol{yq}}(s)$ takes the form

$$\begin{aligned}\boldsymbol{T}_{\boldsymbol{yq}}(s) &= \boldsymbol{B}_1(s)\left[\boldsymbol{G}(s)\boldsymbol{A}_1(s) - \boldsymbol{F}(s)\right]^{-1}\boldsymbol{L}(s)\\ &= \boldsymbol{N}(s)\boldsymbol{D}^{-1}(s)\end{aligned} \tag{7}$$

with $\boldsymbol{N}(s) = \text{block diag}[\boldsymbol{N}_{ij}(s)] \in R[s]^{l\times m}$ and $\boldsymbol{D}(s) = \text{block diag}[\boldsymbol{D}_{ii}(s)] \in R[s]^{m\times m}$ where $i = 1, 2, ..., k$ and $j = 1, 2, ..., k$ according to the partition (4).

The decoupling procedure starts with determination of the numerator matrix of the system. It is taken as a block diagonal matrix $\boldsymbol{N}(s) = \text{block diag}[\boldsymbol{N}_{ii}(s),\ i = 1, 2, ..., k]$, where particular blocks $\boldsymbol{N}_{ii}(s)$ are the greatest common left divisors (*g.c.l.d.*) of columns of i-th row-block $\boldsymbol{B}_{1i}(s)$ of $\boldsymbol{B}_1(s)$ caused by the partition (4). Then $\boldsymbol{B}_1(s)$ takes the form

$$\boldsymbol{B}_1(s) = \boldsymbol{N}(s)\boldsymbol{B}(s). \tag{8}$$

In general, the decoupled system does not have to be stable but it should be free of any unstable cancellations, unobservable and/or uncontrollable, unstable poles. However, if the polynomial matrix $\widetilde{\boldsymbol{G}}(s) \in R[s]^{l\times l}$, which is a *g.c.l.d.* of all columns $\boldsymbol{B}(s)$ defined by the relation

$$\boldsymbol{B}(s) = \widetilde{\boldsymbol{G}}(s)\widetilde{\boldsymbol{B}}(s) \tag{9}$$

is not unimodular and if its zeros lie in the unstable region of the complex plane, the (unobservable) poles of decoupled system corresponding to these zeros are fixed and unstable [19]. These so called 'interconnection' transmission zeros cannot be eliminated by a feedforward compensator of zero order. So, in such a case a dynamic compensator have to be used. To remove these unobservable poles we can use the compensation scheme together with an additional dynamic feedforward compensator obtained by augmenting the plant model with a serial dynamic element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$. This element has to be connected to the input of the original plant presented in Fig. 2 and finally shifted into the structure of dynamic feedforward compensator [19].

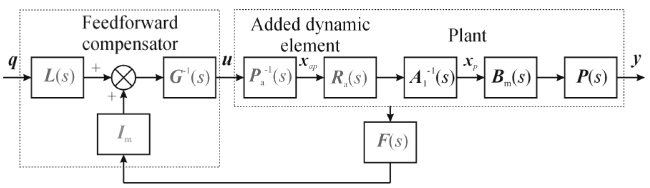

Fig. 2. Structure of the decoupled system for the augmented plant

After calculations of the element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$ the standard procedure with an augmented plant can be used and a decoupled system $\boldsymbol{T}_{\boldsymbol{yq}}(s)$ without fixed poles caused by $\widetilde{\boldsymbol{G}}(s)$ is automatically obtained. An algorithm which may be used to calculate this additional dynamics was analyzed in [19] and has been modified in [24] to make it more reliable and efficient.

Analysis of (8), (9) shows that the presence of the interconnection transmission zeros may depend on the grouping (4) of plants inputs and outputs. So in many cases it may be reasonable to group the inputs and outputs in a way the control goals are achieved, but with reduction of the control system (synthesis) complicity, its size and sensitivity to the model accuracy. Thus input output pairing procedures should be completed by the interconnection transmission zeros checking procedure, especially for nonsquare (left invertible) plants, which for the purpose of control system synthesis, are squared down.

4. Decoupling of left-invertible plants

It is not possible to synthesize the decoupler for diagonal (row-by-row) decoupling of a non-square plant with $l > m$ but as it was shown in [16] there is a chance to synthesize a block-decoupled control system. In order to do it we adopt the following lemmas and theorems given in [16, 19].

Theorem 1 [16]. A left-invertible plant with the transfer matrix (7) of rank m can be block decoupled according to the partition (4) by use of a linear state variable feedback and a dynamic feedforward if and only if $rank\boldsymbol{B}_{1i}(s) = m_i$, $i = 1, 2, ..., k$.

If assumption of Theorem 1 is satisfied, then there exist k unimodular matrices $\boldsymbol{U}_i(s) \in R[s]^{l_i\times l_i}$, $i = 1, 2, ..., k$ such that

$$\boldsymbol{U}_i(s)\boldsymbol{B}_{1i}(s) = \begin{bmatrix}\boldsymbol{B}_{mi}(s)\\ \boldsymbol{0}\end{bmatrix} \tag{10}$$

with $\boldsymbol{B}_{mi}(s)$ of full rank. Parting $\boldsymbol{U}_i^{-1}(s) = \begin{bmatrix}\boldsymbol{P}_i(s) & \boldsymbol{R}_i(s)\end{bmatrix}$ one can define

$$\boldsymbol{B}_{1i}(s) = \begin{bmatrix}\boldsymbol{P}_i(s) & \boldsymbol{R}_i(s)\end{bmatrix}\begin{bmatrix}\boldsymbol{B}_{mi}(s)\\ \boldsymbol{0}\end{bmatrix} = \boldsymbol{P}_i(s)\boldsymbol{B}_{mi}(s). \tag{11}$$

Then after defining

$$\boldsymbol{P}(s) = \begin{bmatrix}\boldsymbol{P}_1(s) & & \\ & \dots & \\ & & \boldsymbol{P}_k(s)\end{bmatrix} \in R[s]^{l\times m} \tag{12}$$

and

$$\boldsymbol{B}_m(s) = \begin{bmatrix}\boldsymbol{B}_{m1}(s) & & \\ & \dots & \\ & & \boldsymbol{B}_{mk}(s)\end{bmatrix} \in R[s]^{m\times m} \tag{13}$$

the transfer matrix (2) takes the form

$$\boldsymbol{T}(s) = \boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s) = \boldsymbol{P}(s)\boldsymbol{B}_m(s)\boldsymbol{A}_1^{-1}(s) \tag{14}$$

the inner square part $\boldsymbol{T}_m(s) = \boldsymbol{B}_m(s)\boldsymbol{A}_1^{-1}(s)$ of which may be decoupled by using any known decoupling method.

Then the transfer matrix (7) of the decoupled system takes the form

$$\begin{aligned} \boldsymbol{T}_{yq}(s) &= \boldsymbol{P}(s)\boldsymbol{B}_m(s)\left[\boldsymbol{G}(s)\boldsymbol{A}_1(s) - \boldsymbol{F}(s)\right]^{-1}\boldsymbol{L}(s) \\ &= \boldsymbol{P}(s)\boldsymbol{N}_m(s)\boldsymbol{D}^{-1}(s) \end{aligned} \tag{15}$$

with

$$\boldsymbol{N}_m(s) = \text{block diag}[\boldsymbol{N}_{ii}(s), i = 1, ..., k] \in R[s]^{m\times m}. \tag{16}$$

The stability of the decoupled system is described by the following lemmas and theorem.

Lemma 1 [19]. The block diagonal matrix $\boldsymbol{D}(s) \in R[s]^{l\times l}$ that satisfies the relation (7) exists if there exist polynomial matrices $\overline{\boldsymbol{L}}(s) \in R[s]^{m\times(m-l)}$ and $\overline{\boldsymbol{B}}(s) \in R[s]^{m\times(m-l)}$ of full rank such that $\boldsymbol{G}(s)\boldsymbol{A}_1(s) - \boldsymbol{F}(s) - \boldsymbol{L}(s)\boldsymbol{D}(s)\boldsymbol{B}(s) = \overline{\boldsymbol{L}}(s)\overline{\boldsymbol{B}}(s)$.

Theorem 2 [19]. The closed-loop poles of the decoupled system $\boldsymbol{T}_{\boldsymbol{yq}}(s)$ realized by linear state variable feedback (*l.s.v.f.*) with dynamic feedforward consist of the zeros of $\left|[\boldsymbol{L}(s), \overline{\boldsymbol{L}}(s)]\right|$, which are uncontrollable, the zeros of $\left|[\boldsymbol{B}^{\mathrm{T}}(s), \overline{\boldsymbol{B}}^{\mathrm{T}}(s)]^{\mathrm{T}}\right|$, which are unobservable and the zeros of $|\boldsymbol{D}(s)|$, which are controllable and observable.

Lemma 2 [16]. The invariant zeros of $\boldsymbol{B}_m(s)\boldsymbol{A}_1^{-1}(s)$ are precisely those of $\boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s)$.

By applying the above method for preparing the plant model to the standard decoupling procedure [13, 24, 25] we obtain the design algorithm for the considered block-decoupled control system. The detailed description of all steps of the standard decoupling procedure may be found in [18]. However, problems arise if the conditions of the Theorem 1 are not satisfied and the plant may not be decoupled without a squaring down procedure, which boils down to the arbitrary crossing out some plant outputs (rows of the plant transfer matrix). It may result in some additional problems in the controller synthesis.

5. Squaring down and grouping of the nonsquare plant I/O

Due to the difficulties raised with the analysis and synthesis of the control system for nonsquare plants, squaring down the nonsquare MIMO plant is a common practice in practical control system design. Such a practice is also known as a selection of secondary measurements.

Let us make partition of a nonsquare transfer matrix of the plant into the form

$$\boldsymbol{T}(s) = \begin{bmatrix} \boldsymbol{T}_S(s) \\ \boldsymbol{T}_R(s) \end{bmatrix} \tag{17}$$

with $\boldsymbol{T}_S(s) \in R(s)^{m\times m}$ and $\boldsymbol{T}_R(s) \in R(s)^{(l-m)\times m}$. The goal of the squaring down procedure is to select such m from among l plant outputs. As the number of possible combination rises quickly with the difference between l and m, it is necessary to utilize devices such NSRGA to analyze the plant model. For the given transfer matrix $\boldsymbol{T}(s)$ determining the NSRGA requires calculation of (MATLAB command)

$$\mathit{NSRGA}(0) = \boldsymbol{T}(0).*\mathit{pinv}(\boldsymbol{T}^T(0)). \tag{18}$$

The result of the plant NSRGA analysis may be used to determine a sum squared error (SSE) which determines the possible levels of the plant outputs steady state errors. However, as it was shown in [4, 6], the exact analytical relationship between the SSE and the NSRGA is available only for the case when $l - m = 1$. In such a case, to select a square subsystem, one eliminates the output with the smallest row sum in the NSRGA. In any other case the NSRGA analysis may result in suboptimal solutions. That is why such an analysis should be complemented by heuristics and any other additional conditions.

Apart from the squaring down proposals, analysis of the plant RGA matrix allows one to determine an input-output pairing. A general pairing rule is that such input-output pair should be chosen so that its corresponding RGA element be close to one [1], but also the elements should be positive and not large [6]. Despite such clear rules in many cases the RGA analysis allows one only to determine a subset of possible squarings, pairings and/or groupings (for block decoupling) and the final decision has to be made according to some other conditions. That is especially when the grouping decision may vary the synthesis procedure of the controller, which makes possible reconfiguration more difficult, and may result in different controller features, such like its rank.

Usually, it is undesirable to impose restrictions on the controller design method, but if restrictions on the controller design method or the maximum controller order do play a role, a controller dependent selection method may be advantageous. For efficiency reasons, input output selection should not involve complete controller design [26].

As it was shown in the previous section if zeros of the matrix $\widetilde{\boldsymbol{G}}(s)$ from Eq. (9) lie in the unstable region of the complex plane, the (unobservable) poles of the decoupled system corresponding to these zeros are fixed and unstable. Although the presence of zeros for the nonsquare plant is rare, the presence of zeros of the squared down plant is most common. It is most important that such "squaring down zeros" be completely virtual and be not connected with plant physics.

Let us denote $\boldsymbol{B}_{ms}(s) \in R[s]^{m\times m}$ as the numerator matrix of the squared down part of the transfer matrix $\boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s)$. Then the decoupling algorithm uses the substituted transfer matrix $\boldsymbol{B}_{ms}(s)\boldsymbol{A}_1^{-1}(s)$. It means that after calculating the numerator matrix of the system $\boldsymbol{N}_s(s)$ as in Eq. (8) we obtain

$$\boldsymbol{B}_{ms}(s) = \boldsymbol{N}_s(s)\boldsymbol{B}_s(s) \tag{19}$$

and then the polynomial matrix $\widetilde{\boldsymbol{G}}_s(s)$, determined as a *g.c.l.d.* of all columns of the matrix $\boldsymbol{B}_s(s)$

$$\boldsymbol{B}_s(s) = \widetilde{\boldsymbol{G}}_s(s)\widetilde{\boldsymbol{B}}_s(s). \tag{20}$$

Zeros of the determinant of matrix $\widetilde{\boldsymbol{G}}_s(s)$ may be called "interconnection squaring down zeros" and their presence and values determine the structure of the decoupler (change its rank) and necessary synthesis methods.

Theorem 3. If any of zeros of the determinant of matrix $\widetilde{\boldsymbol{G}}_s(s)$ lie in the unstable region of the complex plane, the (unobservable) poles of decoupled system corresponding to these zeros are fixed and unstable.

Proof of Theorem 3 goes directly from the proof of the "interconnecting" transmission zeros given in [19].

Influence of the 'squaring down interconnection' transmission zeros can be eliminated only by a dynamic feedforward compensator. When it is necessary, that is when any of such zeros lie in the right part of the complex plane, then, as it was described in third section, in the decoupling algorithm the plant model has to be augmented with a serial dynamic element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$, finally shifted into the structure of the dynamic feedforward compensator.

Taking all the above into account the squaring down procedure of the left invertible plant to be decoupled should consist of the following steps:

Step 1. Check conditions of Theorem 1 to establish whether a block decoupling without crossing out any output is possible at all. If yes, do calculations (10)–(13), and synthesize the decoupler for the transfer matrix (14) with numerator matrix substituted by (13). If not, go to Step 2.

Step 2. Calculate and analyze the NSRGA. Designate row(s) to be crossed out and grouping strategy for other inputs and outputs.

Step 3. Do calculations (19), (20) and check existence of the "interconnection squaring down zeros". If they do not exist, do calculations of the dynamic decoupling algorithm for the assumed input output grouping. If not, go to Step 4.

Step 4. If any of "interconnection squaring down zeros" lie in the left part of the complex plane: check other grouping combinations or calculate a dynamic element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$ and if the rank of the controller is acceptable continue calculations of the dynamic decoupling algorithm.

The presented above sketch of the algorithm does not allow one to automate fully the input-output grouping selection for dynamic decoupling purposes. However, it is an crucial tool in the plant analysis, especially important in the adaptation and reconfigurable control systems (with actuators faults).

6. Example

In order to illustrate the theoretical considerations an example of a decoupling of the reconfigured control system is presented. Let assume a plant (of $n = 5$ order with $m = 3$ inputs and $l = 3$ outputs) defined by the following matrices of the state and output Eq. (1)

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{D} = \mathbf{0}. \tag{21}$$

It has the poles located at $s_1 = 2$, $s_{2,3} = -0.2150 \pm i1.3071$, $s_4 = -1$, $s_5 = -0.5698$ so, it is unstable. The value of the plant transmission zero depends on e.g. value of element b_{32} of matrix $\boldsymbol{B}$. For $b_{32} > -1$ the real part of the transmission zero is negative, while for $b_{32} < -1$ is positive. The plant given has transmission zero $s_1^o = -2$, then it is a minimum phase. Its transfer matrix can be described in the *r.r.p.* matrix fraction as follows

$$\boldsymbol{B}_1(s) = \begin{bmatrix} s-2 & s-8 & 4 \\ 1 & s+4 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$
$$\boldsymbol{A}_1(s) = \begin{bmatrix} s^2-2s & -8s-1 & 4s \\ s-2 & s^2+s-6 & -s+3 \\ 0 & 0 & s+1 \end{bmatrix}. \tag{22}$$

Before starting the design procedure we have assumed that the control system will be diagonally decoupled. As the calculated interconnection transmission zero of the plant is minimum phase then it is not necessary to synthesize an additional dynamic element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$. The system after decoupling will have an unobservable but stable pole $s_{uo} = -2$.

Adopting the following values of poles: $s_1 = -0.5$ for the first, $s_2 = -0.4$ for the second and $s_3 = -0.6$, $s_4 = -0.4$ for the third block we set matrix $\boldsymbol{D}(s)$ as

$$\boldsymbol{D}(s) = \begin{bmatrix} s+0.5 & 0 & 0 \\ 0 & s+0.4 & 0 \\ 0 & 0 & s^2+s+0.24 \end{bmatrix}$$

which with $\boldsymbol{N}(s) = \boldsymbol{I}_3$ allows one to calculate a dynamic feedforward compensator $\boldsymbol{G}^{-1}(s)\boldsymbol{L}(s)$ and the feedback matrix $\boldsymbol{F}$.

As it is shown in Fig. 3, according to our assumptions, there are not any interactions between signals $y_1(t)$, $y_2(t)$, and $y_3(t)$. Change in the value of the first input $q_1(t)$ at $t = 10s$ influences only the first output $y_1(t)$. Similarly, input $q_2(t)$ does not influence any other outputs, but $y_2(t)$ and $y_3(t)$ depend only on input $q_3(t)$. So, the system is completely decoupled and all of the assumed design objectives are achieved.

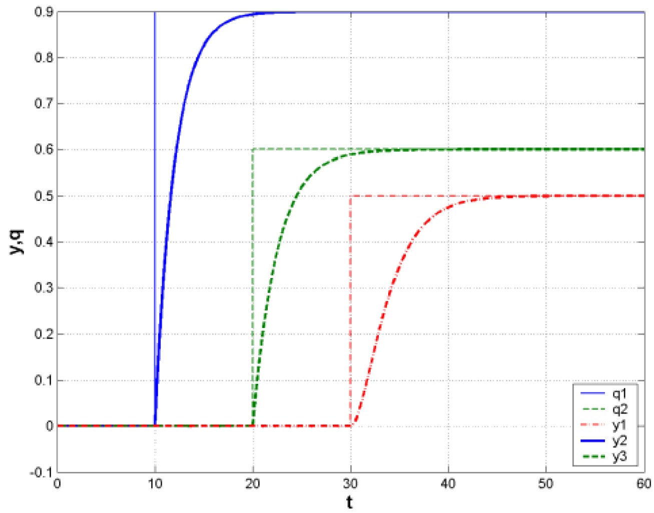

Fig. 3. Results of simulation of the diagonal decoupled control system

Parameter changing. However, if element b_{32} of the matrix $\boldsymbol{B}$ in (21) is changed to $b_{32} = -2$ the transmission zero shifts to $s_1^o = 1$ and the plant becomes nonminimum phase one. This plant can be described in the *r.r.p.* matrix fraction as follows

$$\boldsymbol{B}_1(s) = \begin{bmatrix} s-2 & -2s+16 & 1 \\ -0.5 & s+1 & 0.5 \\ 0 & -2 & 0 \end{bmatrix}, \quad \boldsymbol{A}_1(s) = \begin{bmatrix} s^2-2s & 16s+2 & s \\ -0.5s+1 & s^2+s-6 & 0.5s \\ 0 & 0 & s+1 \end{bmatrix}. \tag{23}$$

Such system is not possible to diagonalize without an additional dynamic element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$, thus without increasing its order to 6. Omitting this additional element results in system instability. What is worse, the results of synthesis of the $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$ are very sensitive to the value of the zero which makes the system impractical in real applications. However, such a system may be successfully used in control systems which use a plants model, e.g. ([27–31]).

Fault of the first input. When the first input in model (21) fails, then the first column of matrix $\boldsymbol{B}$ in the plant description (21) is crossed out and the transfer function of such system takes the form

$$\boldsymbol{B}_1(s) = \begin{bmatrix} s^2+1 & 0 \\ s^2+4s+3.75 & -s-2 \\ s & 0 \end{bmatrix}, \quad \boldsymbol{A}_1(s) = \begin{bmatrix} s^3+s^2-2.25s-3.25 & -s^2+s+2 \\ 4.25s+4.25 & s^2-s-2 \end{bmatrix}. \tag{24}$$

Analysis of the RGA for this plant does not allow one to decide on proper input-output pairing as

$$RGA(0) = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \tag{25}$$

so we have to seek for combinations which satisfy conditions of Theorem 1. It is possible with pairing ($q_1 \to y_2$; $q_2 \to y_1, y_3$). According to the assumed partition the transfer matrix (2) takes the form (14) with matrices

$$\boldsymbol{B}_m(s) = \begin{bmatrix} s^2+4s+3.75 & -s-2 \\ 1 & 0 \end{bmatrix},$$

$$\boldsymbol{P}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s^2+1 & s \\ 0 & s & 1 \end{bmatrix}.$$

Then calculations are continued for a square plant with matrix $\boldsymbol{B}_1(s) := \boldsymbol{B}_m(s)$.

As the calculated “squared down interconnection transmission zero” of the plant is stable, then it is not necessary to synthesize an additional dynamic element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$. The system after decoupling will have an unobservable but stable pole $s_{uo} = -2$.

Adopting the following values of poles: $s_1 = -0.5$ for the first block and $s_2 = -0.4$, $s_3 = -0.6$, $s_4 = -0.4$ for the second block we set matrix $\boldsymbol{D}(s)$ as

$$\boldsymbol{D}(s) = \begin{bmatrix} s+0.5 & 0 \\ 0 & s^3+1.4s^2+0.64s+0.096 \end{bmatrix}$$

and obtain the rest of the elements of the decoupled system, the rank of which in this case is 4.

As it is shown in Fig. 4 according to our assumptions there is no interaction between signals $y_1(t)$ and $y_2(t)$, $y_3(t)$. Change in the value of the first input $q_1(t)$ at $t = 10s$ influences only the second output $y_2(t)$. Similarly, reference input $q_2(t)$ does not influence any other output but $y_1(t)$ and $y_3(t)$. So, the system is decoupled and all of the assumed design objectives are achieved.

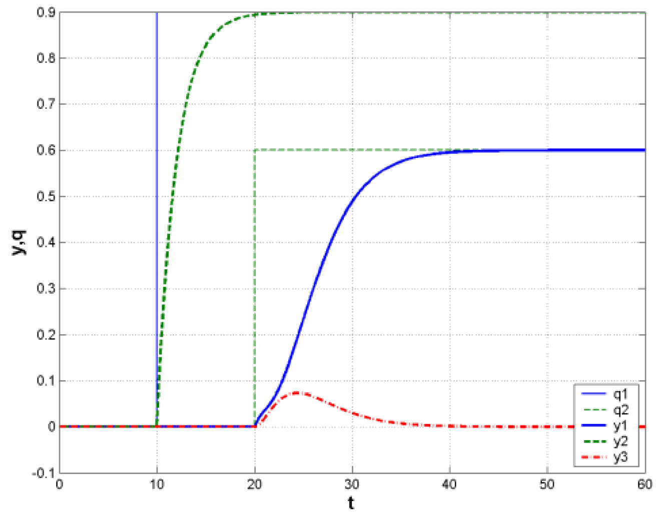

Fig. 4. Results of simulation of the block decoupled control system (fault of the first input)

Fault of the second input. When the second input in model (21) fails then the transfer function takes the form

$$B_1(s) = \begin{bmatrix} s^2+2 & -s-4 \\ -1 & 0 \\ -1 & 1 \end{bmatrix},$$
$$A_1(s) = \begin{bmatrix} s^3+s^2+2s+1 & -2s^2-4s-1 \\ 0 & s^2-s-2 \end{bmatrix}. \quad (26)$$

Matrix $B_1(s)$ has the form, which does not allow one to assume any partition satisfying conditions of Theorem 1. So to obtain a square $B_m(s)$ one has to omit one row of $B_1(s)$ form (26). Looking at $B_1(s)$ it seems reasonable to omit the second row but checking RGA, which for the system (26) takes the form

$$RGA(0) = \begin{bmatrix} 0.38 & 0.57 \\ 0.38 & 0.43 \\ 0.24 & 0 \end{bmatrix} \quad (27)$$

suggest omitting the third one. However, such two systems differ considerably as two of them have interconnection transmission zeros, in the first case minimumphase, in the second one nonminimumphase.

a) Omitting the third row of the matrix $B_1(s)$ (26) gives

$$B_m(s) = \begin{bmatrix} s^2+2 & -s-4 \\ -1 & 0 \end{bmatrix} \quad (28)$$

with one "interconnection squaring down zero" $s_1^o = -4$. The so squared down plant RGA is then

$$RGA(0) = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \quad (29)$$

which does not allow one to decide on proper input-output pairing. Thus the assumed pairing ($q_1 \to y_1$; $q_2 \to y_2$). As the calculated interconnection transmission zero of the plant is stable then it is not necessary to synthesize an additional dynamic element. The system after decoupling will have an unobservable but stable pole $s_{uo} = -4$.

Assuming the following values of poles: $s_1 = -0.5$ for the first block, $s_2 = -0.4$, $s_3 = -0.6$, $s_4 = -0.4$ for the second block we set matrix $D(s)$ as

$$D(s) = \begin{bmatrix} s+0.5 & 0 \\ 0 & s^3+1.4s^2+0.64s+0.096 \end{bmatrix}.$$

As it is shown in Fig. 5, according to our assumptions, change in the value of the first input $q_1(t)$ at $t = 10s$ influences the first $y_1(t)$ and the third (omitted in calculation) $y_3(t)$ output. Similarly, reference input $q_2(t)$ does not influence the output $y_1(t)$ but $y_2(t)$ and $y_3(t)$ only. So, the system is decoupled and all of the assumed design objectives are achieved.

b) If for any reason the second output has been omitted, then $B_1(s)$ (26) takes the form

$$B_m(s) = \begin{bmatrix} s^2+2 & -s-4 \\ -1 & 1 \end{bmatrix} \quad (30)$$

with two virtual "interconnection squaring down zeros" $s_1^o = -1$ and $s_2^o = 2$. As the calculated interconnection transmission zero of the plant s_2^o has its real part in the right part of the complex plane, then it is necessary to synthesize an additional dynamic element $R_a(s)P_a^{-1}(s)$. It may be taken as

$$R_a(s) = \begin{bmatrix} -1.549s-1.549 & -0.034s-0.034 \\ 0.934s-1.154 & 0.015 \end{bmatrix},$$
$$P_a(s) = \begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix}. \quad (31)$$

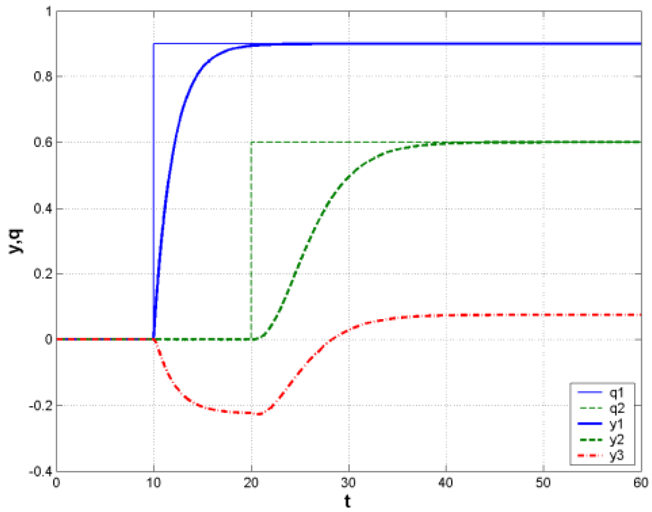

Fig. 5. Results of simulation of the block decoupled control system (fault of the second input, third output omitted)

For the assumed pairing ($q_1 \to y_1$; $q_2 \to y_3$) we obtain a numerator matrix of the decoupled system as

$$N(s) = \begin{bmatrix} s^2-s-2 & 0 \\ 0 & s^2-s-2 \end{bmatrix} \quad (32)$$

and for the following values of poles: $s_1 = -0.5$, $s_2 = -0.4$, $s_3 = -0.6$ for the first and $s_4 = -0.4$, $s_5 = -1.5$, $s_6 = -1.3$, $s_7 = -1$ for the second block we set matrix $D(s)$ as

$$D(s) = \begin{bmatrix} a^* & 0 \\ 0 & b^* \end{bmatrix}$$

where

$$a^* = s^3 + 1.5s^2 + 0.74s + 0.12$$

$$b^* = s^4 + 4.2s^3 + 6.27s^2 + 3.85s + 0.78$$

so the decoupled system is of rank 7.

As it is shown in Fig. 6., also in this case all of the assumed design objectives are achieved. Change in the value of the first input $q_1(t)$ at $t = 10s$ influences the first $y_1(t)$ and the second (omitted in calculation) $y_2(t)$ output. Similarly input $q_2(t)$ influences outputs $y_2(t)$ and $y_3(t)$ only.

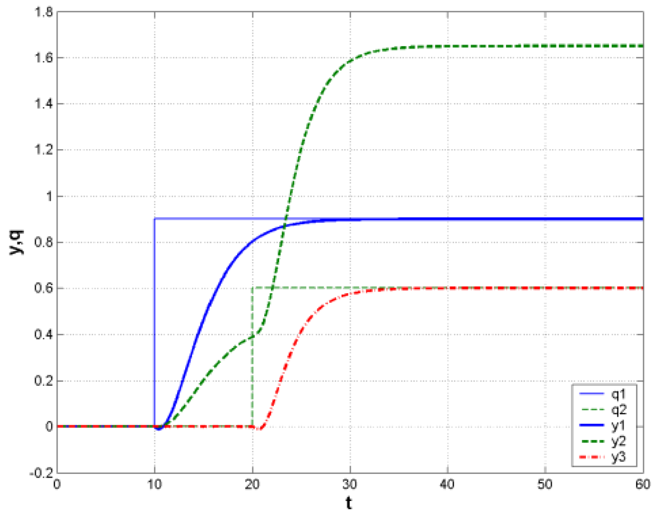

Fig. 6. Results of simulation of the block decoupled control system after using squaring down technique (fault of the second input, second output omitted)

Fault of the third input. When the third input fails the system becomes uncontrollable and it is not possible to decouple the plant.

As it was shown in the above example there may be a lot of different strategies of the input-output grouping of the considered plant after its input failure. The control objectives have been met in any case but e.g. the rank of the decoupled system changed from four (in most cases) to seven when the virtual "interconnection squaring down zeros" were not avoidable.

7. Conclusions and final remarks

In the paper an analysis of the synthesis procedure of the dynamic block decoupling system for the dynamic plants with the number of inputs being less than the number of outputs using squaring down technique has been presented. The theorems given in the paper allow one to check the possibility of decoupling the plant together with checking the possibility of minimizing the decoupler rank and calculations reliability. The procedure proposed in the paper helps in synthesis of a dynamically decoupled control system and its application ensures that the system will be stable and will meet all designing goals for any MIMO, in general unstable, non-minimumphase plants. The presented example confirms the correctness of the analysis. Such a dynamic decoupling procedure provides a possibility to its application to build e.g. adaptive decoupled, reconfigurable, fault tolerant MIMO systems.

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