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Stabilization of autonomous linear time invariant fractional order derivative switched systems with different derivative in subsystems

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Abstract. In this paper, the stabilization problem of a autonomous linear time invariant fractional order (LTI-FO) switched system with different derivative order in subsystems is outlined. First, necessary and sufficient condition for stability of an LTI-FO switched system with different derivative order in subsystems based on the convex analysis and linear matrix inequality (LMI) for two subsystems is presented and proved. Also, sufficient condition for stability of an LTI-FO switched system with different derivative order in subsystems for more than two subsystems is proved. Then a sliding sector is designed for each subsystem of the LTI-FO switched system. Finally, a switching control law is designed to switch the LTI-FO switched system among subsystems to ensure the decrease of the norm of the switched system. Simulation results are given to show the effectiveness of the proposed variable structure controller.

Key words: linear time invariant fractional order derivative, switching systems, linear matrix inequality (LMI).

1. Introduction

In the last two decades, there has been increasing interest in the stability analysis and design methodology of switched systems due to their significance both in theory and applications [1–32]. A switched linear system is a hybrid system that comprises a collection of linear or nonlinear subsystems together with a switching rule that specifies the switching among the subsystems. It is well known that a different switching rule would produce different behavior of the system and hence lead to different system performances. However, the design of a switching strategy is generally very challenging. Some practical examples for switched systems are automated highway systems, automotive engine control system, networked control systems, chemical process, power systems and power electronics, robot manufacture, and stepper motors [1–4]. Some methods such as multi-Lyapunov functions and convex combination of vector fields have been used widely in the study of the integer order switching systems which is closely related to some investigations on differential inclusions [7–10]. For more details about background of the stability of integer order switched systems and related problems, see [18] and references therein. On the other hand, the stability problem of fractional derivative systems has been the focus of much attention in recent years [11–13]. This is due to the applicability of equations based on fractional derivatives in modeling various practical and engineering systems [19, 20, 21, 24]. Consequently, the study of fractional-order switching system instead of integer order seems indispensable.

First, we briefly review the advancements in variable structure control (VSC) because, the methodology of the paper is based on it. VSC system changes the structure or dynamics of the system by switching at precisely defined states to another member of a set of possible continuous functions of the state [26]. This technique provides a framework for definition of the appropriate control laws and the switching structure. The distinguishing feature of VSC is a sliding motion. In order to eliminate chattering in a variable structure control, the sliding sector was proposed to replace the sliding mode [28]. It has been shown that in any system, there is a subset of the state space in which some system norms are reduced without any control input, even if this set may contain only one element, i.e. the origin of the coordinate system. This is called the sliding sector, which can be designed using a Riccati equation [29, 30]. An extremum seeking control algorithm has been used to find the predetermined norm of the state, i.e. a Lyapunov function as in [30]. VSC that defined the sliding surface for linear time invariant fractional order systems (LTI-FOS) was introduced in [31]. In [27] the stabilization of a particular class of nonlinear systems of fractional order differential inclusions in fractional derivative chaos systems using variable structure control is considered. In [25] the control of a special class of single input single output (SISO) switched fractional order systems is considered. In [32], some stabilization issue for fractional order switching systems of fractional order linear systems has been addressed. In [17] the stabilization of linear time invariant systems with fractional derivatives using a limited number of available state feedback gains, using switching method is studied. Author in [5] has established the sufficient conditions for the asymptotic stability of positive fractional switched continuous-time linear systems for any switchings.

In this paper, a necessary and sufficient condition for stability of LTI-FO switched systems with different derivative in two subsystems and a sufficient condition in more than two subsystems is proved, and a switching law for continuous time LTI-FO switched systems is derived. Firstly, the existence of a system with integer order derivatives which has stability

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properties equivalent to the fractional system is proved. Then the extremum seeking control algorithm is used to find the Lyapunov function for the shadow LTI-FO switched system. Finally, a switching law based on the sliding sector is derived.

The paper is organized as follows. In Sec. 2, a review of fractional calculus and two important type of fractional order switching systems are presented. In Sec. 3, the problem formulation is given. In Sec. 4, we consider condition of stability of the autonomous LTI-FO switched system and the Lyapunov function for LTIFO switched system based on an equivalent integer order system from the point of view of stability. In Sec. 5, we present the conditions for stabilization of the LTI-FO switched system based on the extremum seeking method for computation of Lyapunov function and, the sliding sector is defined. In Sec. 6, the switching law based on the sliding sector is discussed. Finally, the results are illustrated using two examples.

2. Preliminaries

Given $0 < q < 1$, Riemann-Liouville definition of q-th order fractional derivative operator ${}_0D_t^q$ is given by

$$ {}_0D_t^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int\limits_0^t (t-\tau)^{-q} f(\tau) d\tau, $$

where $\Gamma(.)$ is the Gamma function generalizing factorial for non-integer arguments

$$ \Gamma(q) = \int\limits_0^\infty e^{-t} t^{q-1} dt. $$

For more details about other definitions of fractional derivative refer to [23]. In this paper Riemann-Liouville definition has been used.

It has been shown that the system ${}_0D_t^q X(t) = AX(t)$ is asymptotically stable if the following condition is satisfied [22]

$$ |\arg(\lambda(A))| > \frac{q\pi}{2}, $$

where $0 < q < 2$, and $\lambda(A)$ are eigenvalues of the matrix A. The stable and unstable regions for $0 < q < 1$ are shown in Fig. 1.

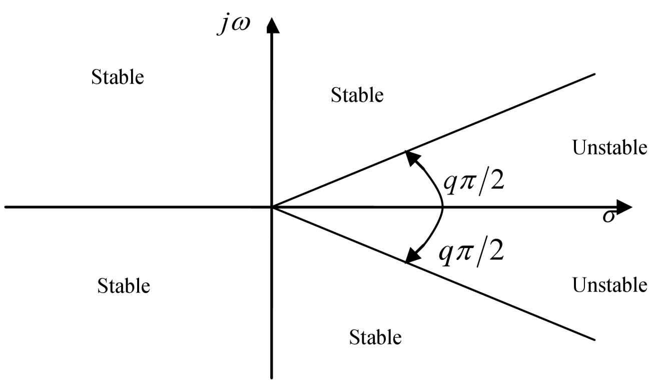


Fig. 1. Stability region of LTI-FOS with order $0 < q < 1$

Given a family of linear fractional derivative systems

$$ \begin{cases} {}_0D_t^q X(t) = A_i X(t), \quad 0 < q < 1, \quad i = 1, 2, \ldots, N \\ {}_0D_t^{q-1} X(t)\,|_{t=0} = X_0 \end{cases} \tag{1} $$

where $X(t) \in R^n$ is the continuous state variable, and ${}_0D_t^{q-1} X(t)\,|_{t=0} = X_0$ is initial condition of derivative. Fractional differential equations in terms of the Riemann-Liouville derivatives require initial conditions expressed in terms of initial values of fractional derivatives of the unknown function [14]. For more details about how to impose physically coherent initial conditions to a fractional system, see [15].

The following two types of switching systems can be defined.

Type I switching systems: Switching logic among the systems is unknown. At each time instant, it is only known that

$$ {}_0D_t^q X(t) \in \{A_i X(t) : i = 1, 2, \ldots, N\}. \tag{2} $$

To analyze this linear differential inclusion (LDI) we assume that the switch is arbitrary.

Type II switching systems: The switch is orchestrated by the controller/supervisor that can choose one of the systems at each instant based on time, the measurement of the states or a certain output. It is assumed that the state X is available for measurement. For this case, the switching strategy can be optimized for the best performances. The system can be written as

$$ {}_0D_t^q X(t) = A_{\sigma(x)} X(t), \quad 0 < q < 1, \tag{3} $$

where $\sigma(x) = i$ for $X \in \Omega_i$ and $\cup \bigcup\limits_{i=1}^{N} \Omega_i = R^n$. The design problem boils down to the construction of the sets Ω_i with a well-designed switching law. To differentiate the above two switching types, we simply call the system (2) the fractional order LDI and the system (3) the switched system.

3. Problem description

Consider a LTI-FO switched system described by the pseudo-state space equation as follows

$$ \begin{cases} {}_0D_t^{q_\sigma} X(t) = A_\sigma X(t), \quad 0 < q_\sigma < 1 \\ {}_0D_t^{q_\sigma - 1} X(t)\,|_{t=0} = X_0 \end{cases} \tag{4} $$

where $X(t) \in R^n$ is the continuous state variable, σ denote a switching signal taking values as $\sigma = 1, 2, \ldots, N$ and a finite set of matrices $\cap A := \{A_\sigma : \sigma = 1, 2, \ldots, N\}$ is given, and ${}_0D_t^{q_\sigma} X(t)$ is the Riemann-Liouville derivative of order q_σ, $0 < q_\sigma < 1$, of $X(t)$ relative to time, and ${}_0D_t^{q_\sigma - 1} X(t)\,|_{t=0} = X_0$ is initial condition of derivative. This paper presents conditions for stabilization of LTI-FO switched systems and extracts stabilizer switching control based on sliding sector.

4. Stability conditions of LTI-FO switched system with different derivative

The objective of this section is to construct a continuous Lyapunov function whose derivative along any state trajectory is negative for each subsystem, within some regions in the state-space. Moreover, these regions must cover the entire state-space. A necessary and sufficient condition for stabilization in the case of two subsystems and a sufficient condition in the general case with different derivative order will be presented. The following theorem has been proved in LMI studies for LTIFOS based on the mapping in Ω plane.

Theorem 1. [11]: The system ${}_0D_t^q X(t) = AX(t)$, where $0 < q < 1$, is t^{-q} – stable if and only if there is a symmetric positive definite matrix P such that

$$\left(-\left(-A\right)^{\frac{1}{2-q}}\right)^T P + P\left(-\left(-A\right)^{\frac{1}{2-q}}\right) < 0 \tag{5}$$

where $(-A)^{\frac{1}{2-q}}$ is defined as $e^{(1/(2-q))\log(-A)}$.

Theorem 2. Given the switched system (4) with $N = 2$, the point $X = 0$ is a stabilized switched equilibrium if there exists $\alpha \in (0, 1)$ such that

$$A = \alpha\left(-(-A_1)^{\frac{1}{2-q_1}}\right) + (1-\alpha)\left(-(-A_2)^{\frac{1}{2-q_2}}\right) \tag{6}$$

and A is Hurwitz i.e. all of eigenvalues of matrix A are in the left half plane.

Proof. (*Sufficiency*) if the convex combination A is stable, there exist two positive definite symmetric matrices P, Q such that according to LMI problem

$$A^T P + PA = -Q. \tag{7}$$

Using (6) and state vector of the system according to solution method of the LMI problems in order to construction of quadratic function, we can rewrite (7) as

$$\begin{aligned}&\alpha X^T\left[\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P + P\left(-(-A_1)^{\frac{1}{2-q_1}}\right)\right]X\\&(1-\alpha)X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P + P\left(-(-A_2)^{\frac{1}{2-q_2}}\right)\right]X\\&= -X^T QX.\end{aligned} \tag{8}$$

Let $\lambda_{\min}$ be smallest (positive real) eigenvalue of Q. Given $0 < \varepsilon \le \lambda_{\min}$, we have

$$-X^T QX \le -\varepsilon X^T X \tag{9}$$

so that (8) can be rewritten as

$$\begin{aligned}&\alpha X^T\left[\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P + P\left(-(-A_1)^{\frac{1}{2-q_1}}\right)\right]X\\&(1-\alpha)X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P\right.\\&\left.P\left(-(-A_2)^{\frac{1}{2-q_2}}\right)\right]X \le -\varepsilon X^T X\end{aligned} \tag{10}$$

or equivalently

$$\begin{aligned}&\alpha\left(X^T\left[\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P + P\left(-(-A_1)^{\frac{1}{2-q_1}}\right)\right]X + \varepsilon X^T X\right)\\&+(1-\alpha)\left(X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P\right.\right.\\&\left.\left.+P\left(-(-A_2)^{\frac{1}{2-q_2}}\right)\right]X + \varepsilon X^T X\right) \le 0.\end{aligned} \tag{11}$$

This means that for every nonzero X we have either

$$X^T\left[\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P + P\left(-(-A_1)^{\frac{1}{2-q_1}}\right)\right]X + \varepsilon X^T X \le 0$$

or

$$X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P + P\left(-(-A_2)^{\frac{1}{2-q_2}}\right)\right]X + \varepsilon X^T X \le 0$$

or equivalently we have either

$$X^T\left[\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P + P\left(-(-A_1)^{\frac{1}{2-q_1}}\right)\right]X \le -\varepsilon X^T X \tag{12}$$

or

$$X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P + P\left(-(-A_2)^{\frac{1}{2-q_2}}\right)\right]X \le -\varepsilon X^T X. \tag{13}$$

Now, define two regions

$$\begin{aligned}\Omega_i = &\left\{X^T\left[\left(-(-A_i)^{\frac{1}{2-q_i}}\right)^T P + P\left(-(-A_i)^{\frac{1}{2-q_i}}\right)\right]X\right.\\&\left.\le -\varepsilon X^T X\right\}, \qquad i \in \{1, 2\}.\end{aligned} \tag{14}$$

There are two closed regions which overlap. It is easy to show that any strategy where the system $\sum_i$ is active in region Ω_i assures stability, using the Lyapunov function $V(X) = X^T PX$ (with P given by Eq. (7)). In fact, within the region Ω_i

$$\begin{aligned}\dot V(X) &= \dot V_i(X)\\&= X^T\left[\left(-(-A_i)^{\frac{1}{2-q_i}}\right)^T P + P\left(-(-A_i)^{\frac{1}{2-q_i}}\right)\right]X\\&\le -\varepsilon X^T X,\end{aligned}$$

while at the switching points (which are interior to the region $\Omega_1 \cap \Omega_2$)

$$\begin{aligned}\dot V(X) &= \sup_{\gamma\in[0,1]}\left\{\gamma \dot V_1(X) + (1-\gamma)\dot V_2(X)\right\}\\&\le \max_{i=1,2}\left\{\dot V_i(X)\right\} \le -\varepsilon X^T X.\end{aligned}$$

(*Necessity*) If the switched equilibrium is stable, for every $X \ne 0$ one of the conditions (12) or (13) must be satisfied, or stated otherwise it is necessary that

$$X^T\left[-\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P - P\left(-(-A_1)^{\frac{1}{2-q_1}}\right) - \varepsilon I\right]X \ge 0$$

when

$$X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P + P\left(-(-A_2)^{\frac{1}{2-q_2}}\right) + \varepsilon I\right]X \ge 0 \tag{15}$$

and

$$X^T\left[-\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P - P\left(-(-A_2)^{\frac{1}{2-q_2}}\right) - \varepsilon I\right]X \ge 0$$

when

$$X^T\left[\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P + P\left(-(-A_1)^{\frac{1}{2-q_1}}\right) + \varepsilon I\right]X \ge 0. \tag{16}$$

We do not consider the case in which A_i is stable for some i. In this case the condition is trivially true because one of the two inequalities is always satisfied. Using of the S-procedure in [7] to one of the previous conditions (e.g. to (15)), we conclude that for some $\eta \geq 0$ the following relation holds

$$X^T\left[-\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P - P\left(-(-A_1)^{\frac{1}{2-q_1}}\right) - \varepsilon I\right]X$$
$$-\eta X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P + P\left(-(-A_2)^{\frac{1}{2-q_2}}\right) + \varepsilon I\right]X \geq 0 \tag{17}$$

or equivalently

$$\begin{aligned} X^T &\left[\left(\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T + \eta\left(-(-A_2)^{\frac{1}{2-q_1}}\right)^T\right)P\right. \\ &\left.+P\left(\left(-(-A_1)^{\frac{1}{2-q_2}}\right) + \eta\left(-(-A_2 t)^{\frac{1}{2-q_2}}\right)\right)\right]X \\ &\leq -\varepsilon(1+\eta)X^T X. \end{aligned} \tag{18}$$

We can rewrite (18) in terms of a convex combination of A_i as follows

$$\begin{aligned} X^T &\left[\frac{\left(\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T + \eta\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T\right)}{(1+\eta)}P\right. \\ &\left.+P\frac{\left(\left(-(-A_1)^{\frac{1}{2-q_1}}\right) + \eta\left(-(-A_2)^{\frac{1}{2-q_2}}\right)\right)}{(1+\eta)}\right]X \\ &\leq -\varepsilon X^T X. \end{aligned} \tag{19}$$

This means that $X = 0$ must be stable equilibrium for the average system in Eq. (6) where $\alpha = \dfrac{1}{1+\eta}$ and $1-\alpha = \dfrac{\eta}{1+\eta}$. Thus the condition is also necessary.

When there are more than two subsystems with different derivative order, it is possible to search for a pair of subsystems satisfying in Theorem (2). Moreover, Theorem (2) can be generalized to the case of N subsystems with different derivative order as a sufficient condition only.

Theorem 3. Given the switched system (4), if there exist $\alpha_i \in (0,1), i = 1, \ldots N$ such that

$$\sum_{i=1}^{N} \alpha_i = 1, \tag{20}$$

$$A = \sum_{i=1}^{N} \alpha_i \left(-(-A_i)^{\frac{1}{2-q_i}}\right) \tag{21}$$

and matrix A is Hurwitz, then the point $X = 0$ is a stabilized switched equilibrium under switching law.

Proof. The proof immediately follows from the proof of sufficiency for Theorem (2) with minor modifications.

The following lemma can be deducted from Theorem (1), for definition of the integer order system for each LTI-FO switched system in point of view of stability.

Lemma 1. The system ${}_0D_t^{q_\sigma} X(t) = AX(t)$, where $0 < q_\sigma < 1$, is stable if and only if the following integer-order system is stable:

$$\dot{X}(t) = \left(-(-A)^{\frac{1}{2-q_\sigma}}\right) X(t). \tag{22}$$

Proof. Assume that system (22) is stable. Consider the Lyapunov function below,

$$\begin{aligned} V(t) = \|X\|_p^2 = X^T(t)PX(t) > 0, \\ \forall X \in R^n, \qquad X \neq 0, \end{aligned} \tag{23}$$

where P is a symmetric positive definite matrix. According to system (22), we have

$$\dot{V}(t) = X^T(t)\left(\left(-(-A)^{\frac{1}{2-q_\sigma}}\right)^T P + P\left(-(-A)^{\frac{1}{2-q_\sigma}}\right)\right)X(t).$$

Since system (22) is stable, we will have $\dot{V}(t) < 0$. Therefore, Eq. (5) holds. The proof for the reverse case is obvious.

Lemma (1) and Theorem (3) shows the relationship of the LMI inequality in Eq. (5) with an integer-order linear system which ensures the stability of the LTIFO switched system under consideration in Eq. (4). Therefore, system (22) is a shadow (equivalent) system of LTI-FO system from the point of view of stability. Therefore, according to Theorem (3) and Lemma (1) the sufficient condition for stability of LTI-FO switched system in Eq. (4) is obtained, i.e. the following relationship holds after choosing the Lyapunov function:

$$\begin{aligned} \dot{V}(t) = X^T(t)\left(A^T P + PA\right)X(t) \\ < -X^T RX, \qquad \forall X \in R^n, \end{aligned} \tag{24}$$

where A is given in Eq. (21), P is a symmetric positive definite matrix, and R is a symmetric positive semi-definite matrix.

5. Extremum seeking algorithm and definition of sliding sector

In this section we propose a method based on extremum seeking algorithm for finding the Lyapunov function Eq. (23) and its derivative in Eq. (24). The extremum seeking algorithm has been used in [6, 16, 28] for determination of the sliding sectors in systems with integer derivatives. Here we need to make some modifications to details of this method, in implementation.

5.1. Lyapunov function found by extremum seeking algorithm.

Define a cost function J as

$$J = \underset{1 \leq j \leq n}{Max}\left(Real\left(\lambda_j(A)\right)\right), \tag{25}$$

where A is the defined matrix in Eq. (21). By minimizing the cost function (25) subject to the convex combination in Eq. (20), if the value of the minimized objective function is negative, we can guarantee the existence of the stabilizing control for system (4) using a convex combination of the subsystems, and consequently, we can guarantee the presence of α_i^* and define its value. This is because the convex combination ensures that system states converge by the obtained switching between subsystems. It is assumed that there is a set of points $\alpha_1^*, \ldots, \alpha_N^*$ that minimizes the cost function J with constraints α_i to the minimum value $J^*(\alpha_1^*, \alpha_2^*, \ldots, \alpha_N^*)$, i.e.

$$J(\alpha_1, \alpha_2, \ldots, \alpha_N) \geq J^*(\alpha_1^*, \alpha_2^*, \ldots, \alpha_N^*)$$
$$\forall \alpha_i \in (0,1), \quad (i = 1, 2, \ldots, N); \quad \sum_{i=1}^{N} \alpha_i = 1, \tag{26}$$

$$J^*(\alpha_1^*, \alpha_2^*, \ldots, \alpha_N^*) = \underset{1 \leq j \leq n}{Max} \left(Real \left(\lambda_j \left(A^* \right) \right) \right) \leq 0, \tag{27}$$

where A^* is determined by

$$A^* = \sum_{i=1}^{N} \alpha_i^* \left(- \left(-A_i \right)^{\frac{1}{2-q_i}} \right). \tag{28}$$

Note: If the minimum value of the cost function (25) subject to constraints in Theorem (3) i.e. $\alpha_i \in (0,1), i = 1, \ldots N$ and $\sum\limits_{i=1}^{N} \alpha_i = 1$ turns out to be negative, it means that, we can place the eigenvalues of the LTIFO switched system in the stable area by switching between subsystems.

Now, consider a system described by the following state equation.

$$\dot{X}(t) = A^* X(t), \tag{29}$$

where $X(t) \in R^n$ is the stable variable and A^* is the system matrix determined by the extremum seeking method algorithm in Eq. (28). It is clear that the system in (29) is stable as all eigenvalues of A^* are in the stable region of complex plane. Thus there exists a positive definite symmetric matrix P and a positive semi-definite symmetric matrix R according to Lemma (1) such that

$$\dot{V}(t) = X^T(t) \left(\left(A^* \right)^T P + P A^* \right) X(t) < -X^T R X,$$
$$\forall X \in R^n,$$

where $P \in R^{n \times n}$, $R \in R^{n \times n}$, and $V(t)$ is the Lyapunov function candidate defined as Eq. (23) which is used to design a sliding sector in the next section.

5.2. Sliding sector for LTI-FO switched systems with different derivative order.

For each subsystem

$$_0D_t^{q_\sigma} X(t) = A_\sigma X(t) \tag{30}$$

the inequality

$$\dot{V}(t) = X^T(t) \left(\left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right)^T P + P \left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right) \right) X(t)$$
$$< -X^T R X, \forall X \in R^n$$

may not hold especially when the subsystem is unstable. It is possible to decompose the state space for each subsystem into two parts such that one part satisfies the condition

$$\dot{V}(t) = X^T(t) \left(\left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right)^T P + P \left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right) \right) X(t)$$
$$> -X^T R X$$

for some element $X \in R^n$, and the other part satisfies the condition

$$\dot{V}(t) = X^T(t) \left(\left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right)^T P + P \left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right) \right) X(t)$$
$$\leq -X^T R X$$

for some other element $X \in R^n$.

Definition 1. The sliding sector defined in the state space R^n for the system given by Eq. (4) is as

$$S_\sigma = \left\{ X | X^T(t) \left(\left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right)^T P \right. \right.$$
$$\left. \left. + P \left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right) \right) X(t) \leq -X^T R X, \; X \in R^n \right\}, \tag{31}$$

where P and R are matrices used in the Lyapunov function in Eqs. (23) and (24). Inside the mentioned sliding sector the norm of the LTI-FO switched system decreases.

The presence of such a sliding sector for systems with integer derivatives is proved in [28], and based on Lemma (1), this result is also valid for LTIFO switched systems.

6. Switching law

Based on the sliding sectors defined in the last section, a switching law is designed in the following theorem.

Theorem 4. The variable structure control to stabilize the switched system given by Eq. (4) is

$$u(t) = \sigma(t), \tag{32}$$

where $\sigma(t)$ is a switching function of t taking values from a finite set $\sum := \{\sigma : \sigma = 1, 2, \ldots, N\}$ and specified by the VSC rule $\sigma(t) = j, \quad j \in \{1, 2 \ldots N\}$ if

$$X(t) \in S_j$$

and

$$\dot{V}(t) = X^T(t) \left(\left(-(-A_j)^{\frac{1}{2-q_j}} \right)^T P + P \left(-(-A_j)^{\frac{1}{2-q_j}} \right) \right) X(t)$$
$$\leq X^T(t) \left(\left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right)^T P + P \left(-(-A_\sigma)^{\frac{1}{2-q_\sigma}} \right) \right) X(t),$$
$$\forall \sigma = 1, 2, \ldots, N \tag{33}$$

in which S_j is the sliding sector defined in Eq. (31).

Proof. Consider the Lyapunov function defined in (23), i.e.

$$V(t) = \|X\|_p^2 = X(t) P X(t), \quad \forall X \in R^n, X \neq 0.$$

Its derivative for the autonomous system in Eq. (29) is

$$\dot{V}(t) = X^T(t) \left(\left(A^* \right)^T P + P A^* \right) X(t) \leq -X^T R X$$

using A^* from Eq. (28

$$\dot{V}(t) = X^T(t) \left[\sum_{i=1}^{N} \alpha_i^* \left(-(-A_i)^{\frac{1}{2-q_i}} \right)^T P \right.$$
$$\left. + P \left(\sum_{i=1}^{N} \alpha_i^* \left(-(-A_i)^{\frac{1}{2-q_i}} \right) \right) \right] X(t) \leq -X^T R X. \tag{34}$$

We can rewrite (34) as

$$\alpha_1^* X^T\left[\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P + P\left(-(-A_1 t)^{\frac{1}{2-q_1}}\right)\right]X$$
$$+\alpha_2^* X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P + P\left(-(-A_2)^{\frac{1}{2-q_2}}\right)\right]X$$
$$\vdots$$
$$+\alpha_N^* X^T\left[\left(-(-A_N)^{\frac{1}{2-q_N}}\right)^T P + P\left(-(-A_N)^{\frac{1}{2-q_N}}\right)\right]X$$
$$\leq -X^T R X \tag{35}$$

or equivalently using $\sum_{i=1}^{N} \alpha_i^* X^T R X = X^T R X$

$$\alpha_1^*\left(X^T\left[\left(-(-A_1)^{\frac{1}{2-q_1}}\right)^T P + P\left(-(-A_1)^{\frac{1}{2-q_1}}\right)\right]X + X^T R X\right) + \alpha_2^*\left(X^T\left[\left(-(-A_2)^{\frac{1}{2-q_2}}\right)^T P + P\left(-(-A_2)^{\frac{1}{2-q_2}}\right)\right]X + X^T R X\right) + \cdots + \alpha_N^*\left(X^T\left[\left(-(-A_N)^{\frac{1}{2-q_N}}\right)^T P + P\left(-(-A_N)^{\frac{1}{2-q_N}}\right)\right]X + X^T R X\right) \leq 0 \tag{36}$$

there exists a $\sigma = j$ such that

$$\dot{V}(t) = X^T(t)\left(\left(-(-A_j)^{\frac{1}{2-q_j}}\right)^T P + P\left(-(-A_j)^{\frac{1}{2-q_j}}\right)\right)X(t) \leq -X^T R X.$$

Therefore we choose the discrete state σ to be equal to j, then the derivative function of the Lyapunov function in Eq. (23) for the LTI-FO switched system with different derivative order in Eq. (4) satisfies the following inequality:

$$\dot{V}(t)=X^T(t)\left(\left(-(-A_j)^{\frac{1}{2-q_j}}\right)^T P+P\left(-(-A_j)^{\frac{1}{2-q_j}}\right)\right)X(t) \leq -X^T R X, \forall X \in R^n$$

which means the proposed switching law in (32) stabilizes Eq. (4).

Remark 1. According to the VSC proposed in Theorem (4), the current subsystem A_σ should be chosen as A_j. If there is only one sliding sector for the current state $X(t)$, then the derivative of the Lyapunov function in Eq. (23) for the system (4) satisfies the following inequality:

$$\dot{V}(t)=X^T(t)\left(\left(-(-A_j)^{\frac{1}{2-q_j}}\right)^T P+P\left(-(-A_j)^{\frac{1}{2-q_j}}\right)\right)X(t) < -X^T(t) R X(t), \forall X \in R^n.$$

If there is more than one sliding sector for the current state, the condition in (33) given in the theorem ensure that the Lyapunov function (23) decreases with quickest speed over all possible switching laws.

Remark 2. Theorem (4) gives a VSC law for a continuous-time system. The VSC law switches the control input among the subsystems such that a Lyapunov function continues to decrease inside a sliding sector associated with the control law.

Remark 3. Modern controller design schemes may yield a controller for two (or more) plants (see e.g., [1, 2]). The resulting configuration, depicted in Fig. 2, can be described mathematically using a switched system, and its stability analysis is clearly of great importance. The formulation of the closed loop system is as follow

$$\begin{cases} {}_0D_t^{q_\sigma} X(t) = A_{p_\sigma} X(t) + BU(t),\ 0 < q_\sigma < 1, \\ {}_0D_t^{q_\sigma - 1} X(t)\,|_{t=0} = X_0 \sigma = 1, 2, \ldots, N \end{cases}$$

where $U(t) = KX(t)$ are input signal and $K \in R^{1\times n}$ are state feedback gain. According to result of the Theorems 2, 3 and 4, this problem can be analyzed. In this case, the switched system is defined by

$$A_1 = A_{p_1} + BK, \quad A_2 = A_{p_2} + BK, \ldots,$$
$$A_N = A_{p_N} + BK.$$

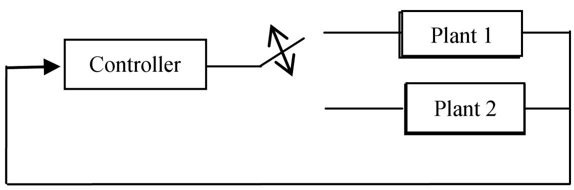


Fig. 2. Switching between two controllers

7. Simulation results

Example 1. Consider the system given by Eq. (4) with the following subsystems

$$Sys1: {}_0D_t^{0.7} X(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -1 \end{bmatrix} X(t),$$

$$Sys2: {}_0D_t^{0.5} X(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -4 \end{bmatrix} X(t),$$

$$Sys3: {}_0D_t^{0.95} X(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 1 & 2 \end{bmatrix} X(t);$$

It is desired to stabilize the above system by switching between the subsystems given by $Sys1$, $Sys2$, $Sys3$. Note that eigenvalues of three subsystems defined by A_1, A_2, A_3, are as follow

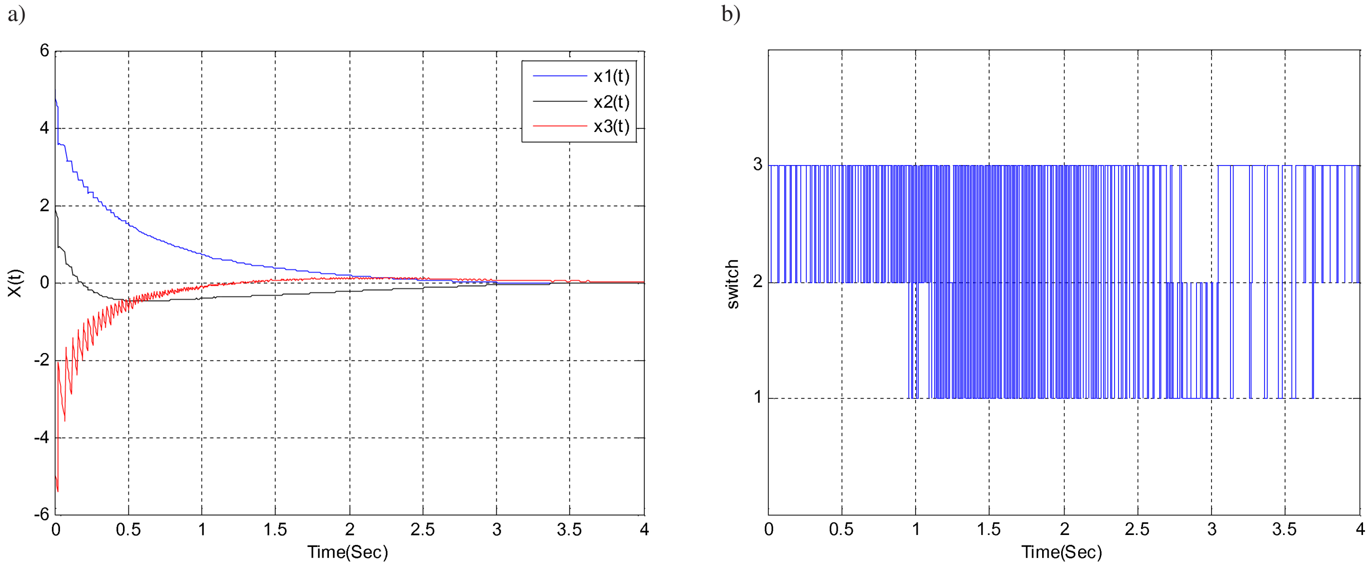


Fig. 3. VSC for LTI-FO switched system (sampling interval, $h = 0.005$ second): a) continuous state $X(t)$, b) switching function $\sigma(t)$ between subsystems

$$\lambda(A_1) = \left\{ -0.6478 \pm j1.7214, \quad 0.2956 \right\} \Rightarrow$$
$$\arg(\lambda) = \left\{ 0°, \quad 110.622°, \quad 249.378° \right\}$$

$$\lambda(A_2) = \left\{ -4.1819, \quad -0.4064, \quad 0.5884 \right\} \Rightarrow$$
$$\arg(\lambda) = \left\{ 180°, \quad 180°, \quad 0° \right\}$$

$$\lambda(A_3) = \left\{ -1.4675, \quad 1.7338 \pm j1.0405 \right\} \Rightarrow$$
$$\arg(\lambda) = \left\{ 30.9703°, \quad -30.9703°, \quad 180° \right\}.$$

With the extremum seeking method, the positive coefficients of the convex combination are found as $\alpha_1* = 0.5186$, $\alpha_2* = 0.3508$, $\alpha_3* = 0.1306$ which gives A^* in Eq. (28). Choose the positive definite matrix as the identity matrix, i.e.

$$R = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and solve $(A^*)^T P + PA^* < -R$. This gives

$$P = \begin{bmatrix} 3.2269 & 3.0486 & 1.8602 \\ 3.0486 & 5.5730 & 2.4510 \\ 1.8602 & 2.4510 & 2.5768 \end{bmatrix}.$$

The simulation result with the proposed VSC, the above parameters and the initial condition $X_0 = [\ 5 \quad 2 \quad -5\ ]^T$ and $h = 0.005$ are as shown in Fig. 3.

Example 2. Consider the system given by Eq. (4) with following subsystems,

$$Sys1 : {}_0D_t^{0.4} X(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix} X(t),$$

$$Sys2 : {}_0D_t^{0.6} X(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.4 & -2 & 0 \end{bmatrix} X(t),$$

$$Sys3 : {}_0D_t^{0.5} X(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 2 & -0.8 & 2 \end{bmatrix} X(t),$$

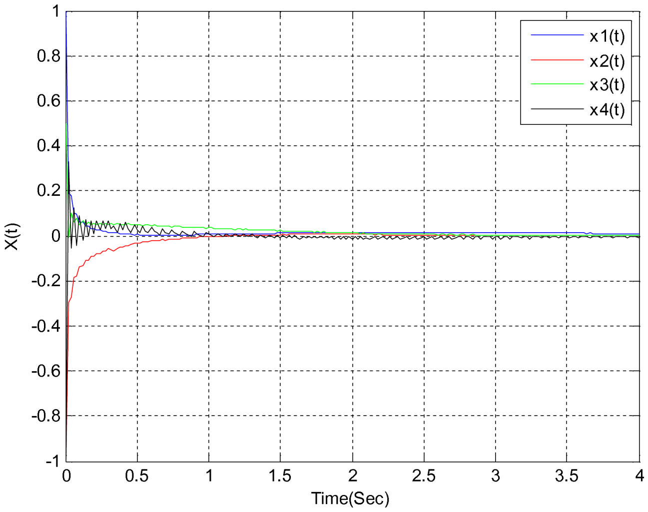
$$Sys4 : {}_0D_t^{0.3} X(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -3 & -1 & -0.8 & 6 \end{bmatrix} X(t).$$

It is desired to stabilize the above system by switching between the subsystems given by $Sys1$, $Sys2$, $Sys3$, $Sys4$. Note that eigenvalues of subsystems defined by A_1, A_2, A_3, A_4 are as follow

$$\lambda(A_1) = \left\{ 0.5 \pm j0.866, \quad -1.6180, \quad 0.618 \right\} \Rightarrow$$
$$\arg(\lambda) = \left\{ 180°, \quad 60°, \quad -60°, \quad 0° \right\},$$

$$\lambda(A_2) = \left\{ -0.2974 \pm j1.5051, \quad 0.5948, \quad 0 \right\} \Rightarrow$$
$$\arg(\lambda) = \left\{ 0°, \quad 0°, \quad 101.1772°, \quad 258.8228° \right\},$$

a) b)



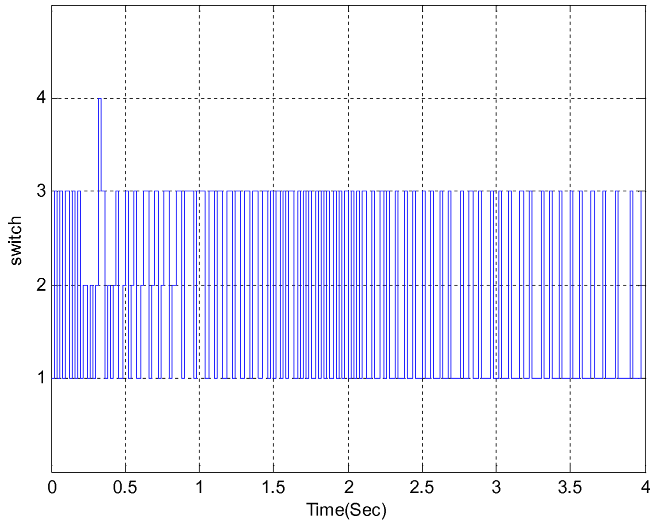


Fig. 4. VSC for LTI-FO switched system (sampling interval, $h = 0.01$ second) a) continuous state $X(t)$, b) switching function $\sigma(t)$ between subsystems

$$\lambda(A_3)=\left\{1.5403 \pm j0.4511, \quad -0.5403 \pm j1.1229\right\} \Rightarrow$$

$$\arg(\lambda) = \left\{16.3238°, \quad 16.3238°, \quad 115.6947°, \quad 244.3053°\right\},$$

$$\lambda(A_4) = \left\{-0.4755 \pm j0.5888, \quad 0.8599, \quad 6.0911\right\} \Rightarrow$$

$$\arg(\lambda) = \left\{128.9255°, \quad 231.0745°, \quad 0°, \quad 0°\right\}.$$

With the extremum seeking method, the positive coefficients of the convex combination are found as $\alpha_1* = 0.3759$, $\alpha_2* = 0.5445$, $\alpha_3* = 0.0597$, $\alpha_4* = 0.0199$, which gives A^* in Eq. (28). Choose the positive definite matrix as the identity matrix, i.e $R = I_4$ and, solve $(A^*)^T P + PA^* < -R$. This gives

$$P = \begin{bmatrix} 3.0237 & 0.7343 & 2.1162 & 0.3559 \\ 0.7343 & 7.4217 & -0.1945 & 9.1760 \\ 2.1162 & -0.1945 & 4.0334 & -0.9892 \\ 0.3559 & 9.1760 & -0.9892 & 15.1100 \end{bmatrix}.$$

The simulation result with the proposed VSC, the above parameters and the initial condition $X_0 = [\ 1 \quad -1 \quad 0.5 \quad -1\ ]^T$ are as shown in Fig. 4.

8. Conclusions

In this paper, a stability condition of the LTI-FO switched systems with a different derivative order in subsystems is proved. Then based on the proved sufficient condition, the Lyapunov functions are found using the extremum seeking method. Also the definition of a sliding sector in LTI-FO switched system with a different derivative order in subsystems is presented. Finally, a variable structure controller with a sliding sector is designed for the switched system. Simulation results are used to show the main points of the paper.

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