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Pointwise completeness and pointwise degeneracy of standard and positive hybrid linear systems described by the general model

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Necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy of the standard and positive hybrid linear systems described by the general model are established. It is shown that the standard general model is always pointwise complete and it is not pointwise degenerated and the positive general model is pointwise complete if and only if its matrix A_2 is diagonal.

Key words: pointwise completeness, degeneracy, standard, positive, general model, hybrid system

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [4, 8].

The most popular models of two-dimensional (2D) linear systems are the discrete models introduced by Roesser [21], Fornasini and Marchesini [5, 6], and Kurek [18]. The models have been extended for positive systems. An overview of positive 2D system theory has been given in the monograph [8].

A dynamical system described by homogenous equation is called pointwise complete if every given final state of the system can be reached by suitable choice of its

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initial state. A system which is not pointwise complete, is called pointwise degenerated. The pointwise completeness and pointwise degeneracy belong to the basic concepts of the modern control theory of 2D linear systems and they play important role specially in positive 2D linear systems.

The pointwise completeness and pointwise degeneracy of linear continuous-time system with delays have been investigated in [19, 20], of discrete-time and continuous-time systems of fractional order in [1, 16] and of positive discrete-time systems with delays in [2, 3]. The pointwise completeness of linear discrete-time cone-systems with delays has been analyzed in [20]. The pointwise completeness and pointwise degeneracy of standard and positive linear systems with state-feedbacks have been investigated in [9, 10].

The pointwise completeness and pointwise degeneracy of 2D standard and positive Fornasini-Marchesini models have been addressed in [11]. Positive 2D hybrid linear systems have been introduced in [12, 13, 8] and positive fractional 2D hybrid linear systems in [14]. Comparison of different method of solution to 2D linear hybrid systems has been given in [17]. Realization problem for positive 2D hybrid systems has been addressed in [15].

In this paper the pointwise completeness and pointwise degeneracy of standard and positive hybrid linear systems described by the general model will be addressed. The structure of the paper is the following. In section 2 the pointwise completeness and the pointwise degeneracy of the standard general model is investigated. Necessary and sufficient conditions for the positivity and the pointwise completeness, pointwise degeneracy of the general model are established in section 3. Concluding remarks are given in section 4.

In the paper the following notation will be used. The set of $n \times m$ real matrices will be denoted by $\Re^{n\times m}$ and $\Re^n = \Re^{n\times 1}$. The set of $n \times m$ real matrices with nonnegative entries will be denoted by $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. The $n \times n$ identity matrix will be denoted by I_n In and the transpose will be denoted by T.

2. Pointwise completeness and pointwise degeneracy of standard general model

Consider the autonomous general model

$$\dot{x}(t,i+1) = A_0x(t,i) + A_1\dot{x}(t,i) + A_2x(t,i+1), \quad t \in \Re_+ = [0,+\infty], \quad i \in Z_+ = \{0,1,,...\} \tag{1}$$

where $\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t}$, $x(t,i) \in \Re^n$, $u(t,i) \in \Re^m$, $y(t,i) \in \Re^p$ are the state, input and output vectors.

Boundary conditions for (1) are given by

$$x(0,i) = x_i, \quad i \in Z_+ \quad \text{and} \quad x(t,0) = x_{t0}, \quad \dot{x}(t,0) = x_{t1}, \quad t \in \Re_+. \tag{2}$$

Definition 1 *The general model* (1) *is called pointwise complete at the point* (t_f,q) *if for every final state* $x_f \in \Re^n$ *there exist boundary conditions* (2) *such that* $x(t_f,q) = x_f$.

Theorem 1 *The general model* (1) *is always pointwise complete at the point* (t_f,q) *for any* $t_f > 0$ *and* $q = 1$.

Proof From (1) for $i = 0$ we have

$$\dot{x}(t,1) = A_2 x(t,1) + F(t,0) \tag{3}$$

where

$$F(t,0) = A_0 x(t,0) + A_1 \dot{x}(t,0) = A_0 x_{t0} + A_1 x_{t1}. \tag{4}$$

Assuming $x_{t0} = 0$, $x_{t1} = 0$ we obtain $F(t,0) = 0$ and from (3)

$$x(t,1) = e^{A_2 t} x(0,1). \tag{5}$$

Substituting $t = t_f$ and $q = 1$ we obtain

$$x_f = e^{A_2 t_f} x(0,1) \tag{6}$$

and

$$x(0,1) = e^{-A_2 t_f} x_f. \tag{7}$$

Therefore, for any final state x_f there exist boundary conditions $x_{t0} = 0$, $x_{t1} = 0$ and $x_1 = e^{-A_2 t_f} x_f$ such that $x(t_f,1) = x_f$ since the matrix $e^{-A_2 t_f} x_f$ exists for any matrix A_2 and any $t_f > 0$. □

From theorem 2.1 we have the following corollaries.

Corollary 1 *Any general model* (1) *is pointwise complete at the point* $(t_f,1)$ *for arbitrary* $t_f > 0$.

Corollary 2 *The pointwise completeness of the general model at the point* $(t_f,1)$ *is independent of the matrices* A_0 *and* A_1 *of the model.*

Definition 2 *The general model* (1) *is called pointwise degenerated at the point* (t_f,q) *in the direction* v *if there exist a nonzero vector* $v \in \Re^n$ *such that for all boundary conditions* (2) *the solution of the model for* $t = t_f$, $i = q$ *satisfies the condition* $v^T x(t_f,q) = 0$.

Theorem 2 *The general model* (1) *is not pointwise degenerated at the point* $(t_f,1)$ *for any* $t_f > 0$.

Proof Using the solution of (3)

$$x(t,1)=e^{A_2t}x(0,1)+\int_0^t e^{A_2(t-\tau)}F(\tau,0)d\tau \tag{8a}$$

we obtain

$$v^Tx(t,1)=v^Te^{A_2t}x(0,1)+\int_0^t v^Te^{A_2(t-\tau)}F(\tau,0)d\tau \tag{8b}$$

where $F(t,0)$ is defined by (4). From (8b) it follows that does not exist a non-zero vector $v\in\Re^n$ such that for all boundary conditions (2) $v^T(t_f,1)=0$ since the matrix $e^{A_2t_f}$ is nonsingular for every matrix A_2 and t_f>0. □

Example 1 Consider the general model (1) with the matrices

$$A_0=\begin{bmatrix}1&2\\3&2\end{bmatrix},\quad A_1=\begin{bmatrix}1&1\\2&1\end{bmatrix},\quad A_2=\begin{bmatrix}-1&0\\1&-2\end{bmatrix}. \tag{9}$$

Find the boundary conditions (2) at the point $(t_f,q)=(1,1)$ for $x_f=\begin{bmatrix}2\\3\end{bmatrix}$.

Taking into account that the eigenvalues of A_2 are $\lambda_1=-1$, $\lambda_1=-2$ and using the Sylvester formula we obtain

$$e^{-A_2t}=\frac{A_2-\lambda_2I_n}{\lambda_1-\lambda_2}e^{-\lambda_1t}+\frac{A_2-\lambda_1I_n}{\lambda_2-\lambda_1}e^{-\lambda_2t}=\begin{bmatrix}e^t&0\\e^t-e^{2t}&e^{2t}\end{bmatrix}. \tag{10}$$

From (7) we have the desired boundary conditions

$$x(0,1)=e^{-A_2t_f}x_f=\begin{bmatrix}e^{t_f}&0\\e^{t_f}-e^{2t_f}&e^{2t_f}\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\Bigg|_{t_f=1}=\begin{bmatrix}2e\\2e+e^2\end{bmatrix} \tag{11}$$

and $x(t,0)=0$, $\dot{x}(t,0)=0$, $t\geqslant 0$.

The above conditions can be extended as follows.

From (1) for $i=1$ we have

$$\dot{x}(t,2)=A_2x(t,2)+F(t,1) \tag{12}$$

where

$$F(t,1)=A_0x(t,1)+A_1\dot{x}(t,1). \tag{13}$$

Substitution of (3) and (5) for $F(t,0)=0$ into (13) yields

$$F(t,1)=(A_0+A_1A_2)x(t,1)=(A_0+A_1A_2)e^{A_2t}x(0,1). \tag{14}$$

Assuming $x(0,1)=0$ we obtain $F(t,1)=0$ and from (12)

$$x(t,2)=e^{A_2 t}x(0,2). \tag{15}$$

Continuing this procedure for $i=2,\dots,q-1$ we obtain the following theorem, which is an extension of Theorem 1.

Theorem 3 *The general model* (1) *is always pointwise complete at the point* (t_f,q)*,* $t_f>0$*,* $q\in N=\{1,2\dots\}$ *for any matrices* A_k*,* $k=0,1,2$*.*

Theorem 2 can be also extended for any point (t_f,q).

3. Pointwise degeneracy and pointwise degeneracy of the positive general model

Definition 3 *The model* (1) *is called positive if* $x(t,i)\in\Re_+^n$*,* $t\in\Re_+$*,* $i\in Z_+$ *for any boundary conditions*

$$x_{t0}\in\Re_+^n,\quad x_{t1}\in\Re_+^n,\quad t\in\Re_+,\quad x_i\in\Re_+^n,\quad i\in Z_+. \tag{16}$$

Theorem 4 *The general model* (1) *is positive if and only if*

$$A_2\in M_n \tag{17a}$$

$$A_0,A_1\in\Re_+^{n\times n},\quad A=A_0+A_1A_2\in\Re_+^{n\times n} \tag{17b}$$

where M_n *is the set of* $n\times n$ *Metzler matrices (with nonnegative off-diagonal entries).*

Proof Necessity. Necessity of $A_0\in\Re_+^{n\times n}$ and $A_1\in\Re_+^{n\times n}$ follows immediately from (4) since $F(t,0)\in\Re_+^n$, $t\in\Re_+$ and x_{t0}, x_{t1} are arbitrary. From (5) it follows that $A_2\in M_n$ since $e^{A_2 t}\in\Re_+^{n\times n}$ only if A_2 is a Metzler matrix, $x(t,1)\in\Re_+^n$, $t\in\Re_+$ and $x(0,1)$ is arbitrary. From (12) it follows that $F(t,1)\in\Re_+^n$, $t\in\Re_+$ for any $x(0,1)\in\Re_+^n$ only if $A=A_0+A_1A_2\in\Re_+^{n\times n}$. The proof of sufficiency is similar to the one given in [8 p.255]. □

Definition 4 *The positive general model* (1) *is called pointwise complete at the point* (t_f,q) *if for every final state* $x_f\in\Re_+^n$ *there exist boundary conditions* (16) *such that*

$$x_{t_f,q}=x_f,\quad t_f>0,\quad q\in N=\{1,2,\dots\}. \tag{18}$$

It is assumed that $x_{t0}=0$ *and* $x_{t1}=0$ *for* $t\in\Re_+$*.*

Theorem 5 *The positive general model* (1) *is pointwise complete at the point* $(t_f,1)$ *if and only if the matrix* A_2 *is diagonal.*

Proof In a similar way as in proof of Theorem 1 we may obtain the equation (7). It is well-known [8] that $e^{A_2t} \in \Re_+^{n\times n}$, $t \in \Re_+$ if and only if A_2 is a Metzler matrix. Hence $e^{-A_2t} \in \Re_+^{n\times n}$ if and only if A_2 is a diagonal matrix. In this case for arbitrary $x_f \in \Re_+^n$ if and only if $x(0,1) \in \Re_+^n$. □

In a similar way as for standard general model we can prove the following theorem.

Theorem 6 *The positive general model* (1) *is pointwise complete at the point* (t_f,q) $t_f > 0$, $q \in N = \{1,2\dots\}$ *if and only if the matrix* A_2 *is diagonal.*

From Theorem 6 we have the following corollary.

Corollary 3 . *The pointwise completeness of the positive general model* (1) *is independent of the matrices* A_0 *and* A_1 *of the model.*

Definition 5 *The positive general model* (1) *is called pointwise degenerated at the point* (t_f,q) *if there exists at least one final state* $x_f \in \Re_+^n$ *such that* $x(t_f,q) \neq x_f$ *for all* $x(0,i) \in \Re_+^n$ *and* $x(t,0) = 0$, $\dot{x}(t,0) = 0$, $t \in \Re_+$.

Theorem 7 *The positive general model* (1) *is pointwise degenerated at the point* (t_f,q) *if the matrix* $A_2 \in M_n$ *is not diagonal.*

Proof In a similar way as in proof of Theorem 1 we may obtain the equality (7) which can be satisfied for $x_f \in \Re_+^n$ and $x(0,i) \in \Re_+^n$ if and only if the Metzler matrix A_2 is diagonal. The proof for $q > 1$ is similar. □

These considerations can be easily extended for $x(0,i) \in \Re_+^n$, $x(t,0) \in \Re_+^n$ and $\dot{x}(t,0) \in \Re_+^n$, $t \in \Re_+$.

Example 2 Consider the general model (1) with the matrices

$$A_0 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}. \tag{19}$$

The model is positive since the matrices A_0 and A_1 have nonnegative entries and

$$A = A_0 + A_1A_2 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \Re_+^{2\times 2}. \tag{20}$$

The matrix A_2 is diagonal and the positive model with (19) by Theorem 5 is pointwise complete at the point $(t_f,1)$, $t_f > 0$. Using (7) we obtain

$$x(0,1) = e^{-A_2t_f}x_f = \begin{bmatrix} e^{t_f} & 0 \\ 0 & e^{2t_f} \end{bmatrix} x_f \in \Re_+^2 \tag{21}$$

for any $x_f \in \Re_+^2$ and $t_f \in \Re_+$.

Example 3 Consider the general model (1) with the matrices (9). The model is positive since A_0 and A_1 have nonnegative entries, $A_2 \in M_n$ and

$$A = A_0 + A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \in \Re_+^{2\times 2} \tag{22}$$

Let $x_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Using (10) and (7) we obtain the vector

$$x(0,1) = e^{-A_2 t_f} x_f = \begin{bmatrix} e^{t_f} & 0 \\ e^{t_f} - e^{2t_f} & e^{2t_f} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{t_f} \\ e^{t_f} - e^{2t_f} \end{bmatrix} \tag{23}$$

with negative second component for $t_f > 0$. Therefore, the model is pointwise degenerated at the point $(t_f, 1)$. The same result follows from Theorem 7 since the matrix A_2 is not diagonal. Note that the vector $x(0,1)$ given by (11) for $x_f = [2 \;\; 3]^T$ has positive components.

4. Concluding remarks

The pointwise completeness and pointwise degeneracy of the standard and positive hybrid linear systems described by the general model have been addressed. Necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy have been established. It has been shown that the standard general model (1) is always pointwise complete at the point (t_f, q) for any t_f and $q \geqslant 1$ and it is not pointwise degenerated at any point. Necessary and sufficient conditions for the positivity of general model (1) have been established. The positive general model is pointwise complete at the point (t_f, q) for $t > 0$, $q \geqslant 1$ if and only if the Metzler matrix A_2 of the model is diagonal. The considerations have been illustrated by numerical examples. These considerations can be extended for linear hybrid systems with delays. An extension of these considerations for fractional hybrid systems is an open problem.

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