

Reachability of linear hybrid systems described by the general model

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The reachability of standard and positive hybrid linear systems described by the general model is addressed. Necessary and sufficient conditions for the reachability of the standard general model are established. Sufficient condition is given for the reachability of positive hybrid system described by the general model. The considerations are illustrated by numerical examples.

Key words: reachability, standard, positive, general model, hybrid system, piecewise constant control

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1, 5].

The most popular models of two-dimensional (2D) linear systems are the discrete models introduced by Roesser [15], Fornasini and Marchesini [2, 3], and Kurek [14]. The models have been extended for positive systems. An overview of positive 2D system theory has been given in the monograph [4] and of positive 2D systems in [5].

Positive 2D hybrid linear systems have been introduced in [7, 8, 5] and positive fractional 2D hybrid linear systems in [9]. The general model of hybrid systems have been introduced in [11]. Comparison of different method of solution to 2D linear hybrid systems has been given in [13]. Realization problem for positive 2D hybrid systems

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has been addressed in [10]. The pointwise completeness and pointwise degeneracy of standard and positive hybrid linear systems have been investigated in [6].

In this paper the reachability of standard and positive hybrid linear systems described by the general model is addressed. Necessary and sufficient conditions for the reachability of the standard general model are established. Sufficient condition is given for the reachability of positive hybrid system described by the general model.

The structure of the paper is the following. In section 2 the general model of hybrid systems is presented and its solution and positivity theorem is recalled. Necessary and sufficient conditions for the reachability of the standard hybrid general model and sufficient condition for the reachability of positive hybrid systems are derived in section 3. The piecewise constant control and standard control of these systems are considered. Concluding remarks are given in section 4.

In the paper the following notation will be used. The set of $n \times m$ real matrices will be denoted by $\mathfrak{R}^{n \times m}$ and $\mathfrak{R}^n = \mathfrak{R}^{n \times 1}$. The set of $n \times m$ real matrices with nonnegative entries will be denoted by $\mathfrak{R}_+^{n \times m}$ and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$. The $n \times n$ identity matrix will be denoted by I_n and the transpose will be denoted by T .

2. General model of linear hybrid systems and its positivity

Consider the general model of linear hybrid systems described by the equations

$$\dot{x}(t, i+1) = A_0x(t, i) + A_1\dot{x}(t, i) + A_2x(t, i+1) + B_0u(t, i) + B_1\dot{u}(t, i) + B_2u(t, i+1) \quad (1a)$$

$$y(t, i) = Cx(t, i) + Du(t, i), \quad t \in \mathfrak{R}_+ = [0, +\infty], \quad i \in \mathbb{Z}_+ \quad (1b)$$

where $\dot{x}(t, i) = \frac{\partial x(t, i)}{\partial t}$, $x(t, i) \in \mathfrak{R}^n$, $u(t, i) \in \mathfrak{R}^m$, $y(t, i) \in \mathfrak{R}^p$ are the state, input and output vectors and $A_k \in \mathfrak{R}^{n \times n}$, $B_k \in \mathfrak{R}^{n \times m}$, $k = 0, 1, 2$, $C \in \mathfrak{R}^{p \times n}$, $D \in \mathfrak{R}^{p \times m}$.

Boundary conditions for (1a) are given by

$$x(0, i) = x_i, \quad i \in \mathbb{Z}_+ \quad \text{and} \quad x(t, 0) = x_{t0}, \quad \dot{x}(t, 0) = x_{t1}, \quad t \in \mathfrak{R}_+. \quad (2)$$

Theorem 1 [11] *The solution of the equation (1a) with boundary conditions (2) has the form*

$$\begin{aligned}
 x(t, i) = & \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(T_{k, i-l-1} B_0 \int_0^t \frac{(t-\tau)^k}{k!} u(\tau, l) d\tau + T_{k, i-l} B_2 \int_0^t \frac{(t-\tau)^k}{k!} u(\tau, l) d\tau - T_{k, i-l-1} B_1 \frac{t^k}{k!} u(0, l) \right) \\
 & + \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \left(T_{k, i-l-1} B_1 \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} u(\tau, l) d\tau \right) + \sum_{l=0}^{\infty} T_{0, i-l-1} B_1 u(t, l) \\
 & - \sum_{k=0}^{\infty} \left(T_{k, i} B_2 \int_0^t \frac{(t-\tau)^k}{k!} u(\tau, 0) d\tau + T_{k, i} A_2 \int_0^t \frac{(t-\tau)^k}{k!} x(\tau, 0) d\tau + T_{k, i} \frac{t^k}{k!} x(0, 0) \right) \\
 & + \sum_{k=1}^{\infty} \left(T_{k, i} \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} x(\tau, 0) d\tau \right) + T_{0, i} x(t, 0)
 \end{aligned} \quad (3)$$

where

$$T_{i,j} = \begin{cases} I_n & \text{for } i = j = 0 \\ A_0 T_{i-1,j-1} + A_1 T_{i,j-1} + A_2 T_{i-1,j} & \text{for } i + j > 0; i, j \in \mathbb{Z}_+ \\ = T_{i-1,j-1} A_0 + T_{i,j-1} A_1 + T_{i-1,j} A_2 & \\ 0 & \text{for } k < 0 \text{ or } l < 0 \end{cases} \quad (4)$$

Definition 1 The general model (1) is called positive if $x(t, i) \in \mathfrak{R}_+^n$ and $y(t, i) \in \mathfrak{R}_+^p$, $t \in \mathfrak{R}_+$, $i \in \mathbb{Z}_+$ for any boundary conditions

$$x_{i0} \in \mathfrak{R}_+^n, \quad x_{i1} \in \mathfrak{R}_+^n, \quad t \in \mathfrak{R}_+, \quad x_i \in \mathfrak{R}_+^n, \quad i \in \mathbb{Z}_+ \quad (5)$$

and all inputs $u(t, i) \in \mathfrak{R}_+^m$, $\dot{u}(t, i) \in \mathfrak{R}_+^m$, $t \in \mathfrak{R}_+$, $i \in \mathbb{Z}_+$.

Theorem 2 [11] The general model (1) is positive if and only if

$$A_2 = M_n \quad (6a)$$

$$A_0, A_1 \in \mathfrak{R}_+^{n \times n}, \quad A = A_0 + A_1 A_2 \in \mathfrak{R}_+^{n \times n} \quad (6b)$$

$$B_k \in \mathfrak{R}_+^{n \times m}, \quad k = 0, 1, 2, \quad C \in \mathfrak{R}_+^{p \times n}, \quad D \in \mathfrak{R}_+^{p \times m} \quad (6c)$$

where M_n is the set of $n \times n$ Metzler matrices (with nonnegative off-diagonal entries).

3. Reachability

Definition 2 The model (1) is called reachable at the point (t_f, q) if for any given final state $x_f \in \mathfrak{R}^n$ there exists an input $u(t, i)$, $0 \leq t \leq t_f$, $0 \leq i \leq q$ which steers the system from zero boundary conditions to the state x_f , i.e. $x(t_f, q) = x_f$.

Theorem 3 The model is reachable at the point (t_f, q) for $t_f > 0$ and $q = 1$ if and only if one of the following conditions is satisfied:

$$1. \text{rank} [B_0, A_2 B_0, \dots, A_2^{n-1} B_0] = n \quad \Leftrightarrow \quad \text{rank} [I_n s - A_2, B_0] = n \quad \forall s \in \mathbb{C}$$

2. the rows of the matrix $e^{A_2 t} B_0$ are linearly independent over the field of complex numbers \mathbb{C} .

Proof Let $B_1 = B_2 = 0$. For $i = 0$ and zero boundary conditions from (1a) we have

$$\dot{x}(t, 1) = A_2 x(t, 1) + B_0 u(t, 0)$$

and

$$x_f = x(t_f, 1) = \int_0^{t_f} e^{A_2(t_f-\tau)} B_0 u(\tau, 0) d\tau \quad (7)$$

since $x(0, 1) = 0$. From Sylvester formula we have

$$e^{A_2 t_f} = \sum_{k=0}^{n-1} A_2^k c_k(t_f). \quad (8)$$

Substitution of (8) into (7) yields

$$x_f = \sum_{k=0}^{n-1} A_2^k B_0 \int_0^{t_f} c_k(t_f - \tau) u(\tau, 0) d\tau = [B_0, A_2 B_0, \dots, A_2^{n-1} B_0] \begin{bmatrix} v_0(t_f) \\ v_1(t_f) \\ \vdots \\ v_{n-1}(t_f) \end{bmatrix} \quad (9)$$

where

$$v_k(t_f) = \int_0^{t_f} c_k(t_f - \tau) u(\tau, 0) d\tau. \quad (10)$$

The equation (9) has a solution $v_k(t_f)$ for $k = 0, 1, \dots, n-1$ and any given x_f if and only if the condition 1. is satisfied.

The equivalence of the condition 1. and 2. are known (see [12] pp. 131). □

Theorem 4 *The model (1) is reachable at the point $(t_f, 1)$ if and only if the matrix*

$$R_f = \int_0^{t_f} e^{A_2 \tau} B_0 B_0^T e^{A_2^T \tau} d\tau, \quad t_f > 0 \quad (11)$$

is positive defined (nonsingular), (see [12] pp. 130). Moreover, the input which steers the system from zero boundary conditions to x_f is given by

$$u(t, 0) = B_0^T e^{A_2^T(t_f-\tau)} R_f^{-1} x_f \quad (12)$$

Proof If the matrix R_f is invertible (nonsingular) then (12) is well defined. We shall show that the input (12) steers the system from zero boundary conditions to x_f . Substituting (12) into (7) we obtain

$$x_f = x(t_f, 1) = \int_0^{t_f} e^{A_2(t_f-\tau)} B_0 B_0^T e^{A_2^T(t_f-\tau)} d\tau R_f^{-1} x_f = x_f \quad (13)$$

since

$$\int_0^{t_f} e^{A_2(t_f-\tau)} B_0 B_0^T e^{A_2^T(t_f-\tau)} d\tau = R_f.$$

□

Remark 1 Reachability is independent of the matrices A_0, A_1, B_1, B_2 .

Remark 2 To simplify the calculation we may assume that $u(t, 0)$ is piecewise constant (is the step function).

Example 1 Consider the general model (1) with the matrices

$$A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (14)$$

and arbitrary remaining matrices of the system.

Applying the condition 1. of Theorem 3 we obtain

$$\text{rank}[B_0, A_2 B_0] = \text{rank} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 2 \quad (15a)$$

and

$$\text{rank}[I_n s - A_2, B_0] = \text{rank} \begin{bmatrix} s-1 & 0 & 1 \\ -1 & s-2 & 0 \end{bmatrix} = 2 \quad \forall s \in \mathbb{C}. \quad (15b)$$

Therefore, the system (1) with matrices (14) is reachable for $q = 1$ and $t_f > 0$. Assuming $t_f = 2$ and

$$x_f = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (16)$$

from (12) and (11) we may find the input that steers the system from zero boundary conditions to the desired state (16)

$$u(t, 0) = B_0^T e^{A_2^T(t_f-\tau)} R_f^{-1} x_f = 0.5519e^{2-t} - 0.0953e^{4-2t}.$$

The plots of state variables for $q = 1, t_f \in [0, 2]$ and input for $q = 0$ and $t_f \in [0, 2]$ are shown in Fig. 1.

Let us assume, that input of the system is piecewise constant, i.e.

$$u(t, 0) = \begin{cases} u_1 & \text{for } 0 \leq t < t_1 \\ u_2 & \text{for } t_1 \leq t \leq t_f \end{cases} \quad (17)$$

where u_1, u_2 are constant values.

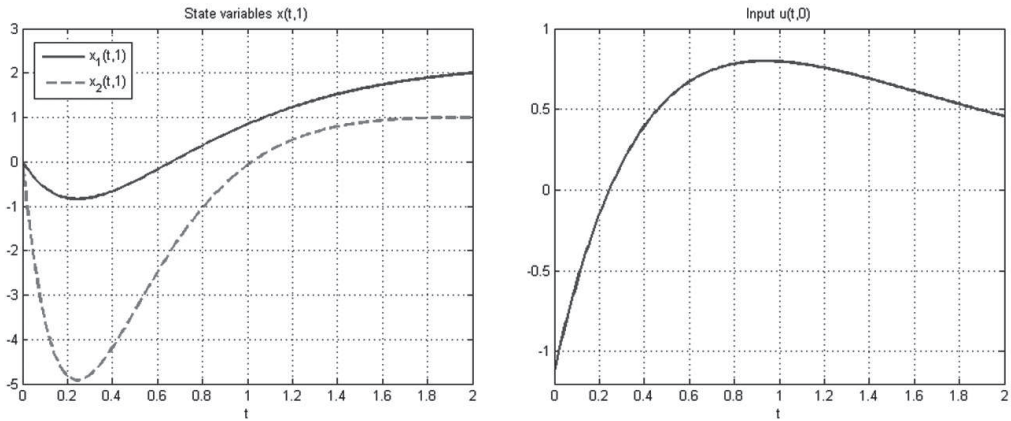


Figure 1. State variables and input of the system.

Taking into account (14) and (16) for (9) we obtain

$$\begin{bmatrix} v_0(t_f) \\ v_1(t_f) \end{bmatrix} = [B_0, A_2 B_0]^{-1} x_f = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (18)$$

From (10) for (17) we have

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \int_0^{t_1} c_0(t_f - \tau) d\tau & \int_0^{t_f} c_0(t_f - \tau) d\tau \\ \int_0^{t_1} c_1(t_f - \tau) d\tau & \int_0^{t_f} c_1(t_f - \tau) d\tau \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (19)$$

Using (8) it is easy to show that

$$c_0(t) = 2e^t - e^{2t}, \quad c_1(t) = e^{2t} - e^t. \quad (20)$$

Using formula (19) we may compute values of the system input for arbitrary t_1 and t_f ($0 < t_1 < t_f$).

For $t_1 = 1$ and $t_f = 2$, we obtain

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -0.0481 \\ 1.2948 \end{bmatrix}.$$

The plots of state variables and input for $q = 1$ and $t_f \in [0, 2]$ are shown in Fig. 2.

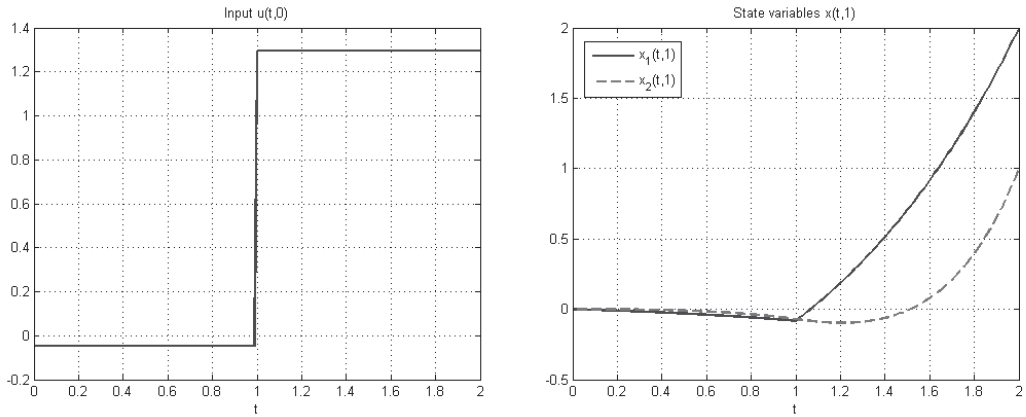


Figure 2. State variables and piecewise constant input of the system.

Definition 3 The positive system (1) is called reachable at the point (t_f, q) if for any given final state $x_f \in \mathfrak{R}_+^n$ there exists a nonnegative input $u(t, i) \in \mathfrak{R}_+^m$, $0 \leq t \leq t_f$, $0 \leq i \leq q$ which steers the system from zero boundary conditions to the state x_f , i.e. $x(t_f, q) = x_f$.

Theorem 5 The positive model (1) is reachable at the point $(t_f, 1)$ if the matrix

$$R_f = \int_0^{t_f} e^{A_2 \tau} B_0 B_0^T e^{A_2^T \tau} d\tau, \quad t_f > 0 \quad (21)$$

is a monomial matrix. The input that steers the system in time t_f from zero boundary conditions to the state x_f is given by the formula (12).

Proof If R_f is a monomial matrix, then there exists the inverse matrix $R_f^{-1} \in \mathfrak{R}_+^{n \times n}$ and the input (12) is well defined and nonnegative for $0 \leq t \leq t_f$. Similarly as in proof of theorem 4, it can be shown that the input (12) steers the system from zero boundary conditions to nonnegative final state $x_f \in \mathfrak{R}_+^n$. \square

The considerations for the controllability to zero of the general model (1) are similar.

4. Concluding remarks

The reachability of standard and positive hybrid linear systems described by the general model (1) have been considered. Necessary and sufficient conditions for the reachability of the general model (1) have been established (Theorem 3). The sufficient

condition for the reachability of positive general model have been established in Theorem 5. The piecewise constant and standard input control are considered and illustrated by numerical examples. An open problems are the observability of standard and positive general model of hybrid systems and an extension of these considerations for fractional hybrid linear systems.

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