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A stabilization method of inhomogeneous ladder networks with nonlinear elements

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In the paper, different structures of electric ladder networks are considered: RC, RL, and RLC. Such systems are composed of resistors, inductors and capacitors connected in series. The elements of the network are not identical and have nonlinear characteristics. The network's dynamic behavior can be mathematically described by nonlinear differential equations. A class of robust feedback controls is designed to stabilize the system. The asymptotic stability of the closed-loop system is analyzed and proved by the use of Lyapunov functionals and LaSalle's invariance principle. The results of computer simulations are included to verify theoretical analysis and mathematical formulation.

Key words: electric ladder network, nonlinear circuit, stabilization, feedback control

1. Introduction

1.1. Motivation

In recent years there has been a growing attention to studies on ladder networks because they are strictly correlated to integrated interconnection problems, coupled mechanical systems, analog neural nets, distributed amplifiers, and so on. Ladder networks may be described as networks formed by numerous repetitions of an elementary cell. In case of an electric ladder network, the elementary cell may consist of resistors, inductance coils, and capacitors connected in series or in parallel. If all the elementary cells are identical, the ladder network is said to be homogeneous; if the elementary cells are not identical, the ladder network is called inhomogeneous.

Electric ladder networks may be employed to model both electrical and nonelectrical systems with distributed parameters. They may be used to calculate the voltage distribution in insulator string and in the windings of electric machines and transformers. They may also be employed to compute the pressure distribution in mechanical and thermal systems with distributed parameters. Ladder networks composed of reactive elements,

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such as inductance coils and capacitors, are used as artificial delay lines, in which the output signal lags behind the input signal; in such delay lines, the delay time is determined by the network parameters. Ladder networks are also deployed as electric filters.

Nonlinear circuit elements have a wide range of use in many areas of electrical engineering. They are incorporated into a circuit to design electronic devices with specific features that could not be achieved with linear elements. For example, nonlinear circuit elements are essential building blocks in many electronic circuits, such as parametric amplifiers, up-converters, mixers, low-power microwave oscillators, electronic tuning devices, etc. Typical nonlinear elements include nonlinear capacitors (varactor diode, junction diode), nonlinear inductors (saturable core inductor, Josephson junctions, ferroresonant power systems), and nonlinear resistors (tunnel diode, thyristor, dead-zone conductor, serially connected Zener diodes, neon bulb, etc.).

In this paper, inhomogeneous structures of electric ladder networks are considered and mathematically analyzed. It is assumed that the elements in these structures are not identical and have in general nonlinear characteristics. The primary goal of the paper is to construct stabilizing feedback control laws that asymptotically stabilize the system.

1.2. Related work

The properties of electrical ladder networks have been already studied in the past. Control problems for linear RL, RC, LC, and RLC electrical circuits are widely discussed in [8, 9, 10]. The dynamics and detailed characteristics of nonlinear electrical circuits are considered in [1, 7]. The papers [16, 17] cope with linear and nonlinear stabilization techniques for a nonlinear RLC circuit. To stabilize the system, authors have constructed various forms of the feedback. The asymptotic stability (in the Lyapunov sense [6]) of the closed-loop system has been proved by LaSalle's invariance principle [5] using special Lyapunov functions. Control problems for nonlinear RLC circuits are discussed in [2, 3, 15, 22]. The main motivation and the source of inspiration during preparation of this material were results obtained in [4, 11, 12, 13, 14, 18, 19, 20, 21, 23, 24]. They have played the crucial role and cleared the way to the main results.

1.3. Organization of the paper

The paper is organized as follows. In Section 2, nonlinear circuit elements are mathematically characterized. Mathematical models of selected electrical ladder networks are described in Section 3. Section 4 is dedicated to synthesis of stabilizing feedback controls. A computational example is presented in Section 5. Conclusions are in Section 6.

2. Nonlinear circuit elements

2.1. Nonlinear resistors

The most common nonlinear circuit element is a nonlinear resistor. The element is fully characterized by the relationship between voltage, current, and resistance. This relationship is usually described by an algebraic equation. The paper focuses on so called current-controlled resistors what means that the voltage drop v_r across the resistor can be written as

$$v_r(t) = r(i)i(t), \tag{1}$$

where r stands for the resistance, i denotes the current.

In this paper, we assume that all resistors are dissipative. A resistor is called dissipative if for all real numbers v_r and i it holds that $v_r i \geqslant 0$. The product $v_r i$ represents the power supplied to the component, therefore a dissipative resistor is characterized by the property that no voltage-current pair can produce negative power.

2.2. Nonlinear inductors

A nonlinear inductor is an example of a dynamic nonlinear circuit element. The relationship between the voltage drop v_l across the inductor, the flux ϕ, and the current i is described by the nonlinear differential equation of the form

$$v_l(t) = \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = \frac{\mathrm{d}(l(i)i(t))}{\mathrm{d}t} = g(i)\frac{\mathrm{d}i(t)}{\mathrm{d}t}, \tag{2}$$

where l denotes the inductance, $g(i) = l(i) + i(t)\frac{\mathrm{d}l(i)}{\mathrm{d}i}$.

2.3. Nonlinear capacitors

A capacitor is defined as an electronic component whose charge is a function of voltage. In this paper, rather than defining the capacitance, it is used the function $v_c(q)$ that gives the voltage drop across the capacitor.

3. Mathematical description of electric ladder networks

3.1. RC ladder network

Consider a nonlinear analogue circuit shown in Fig. 1. The circuit consists of a set

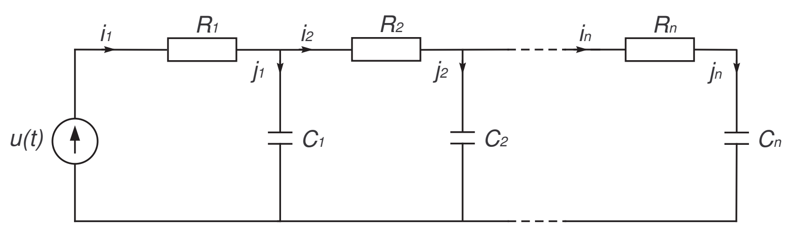

Figure 1. Schematic diagram of an RC ladder network.

of resistors R_k and capacitors C_k that are connected together to form a network. The resistors and capacitors are different and have nonlinear characteristics: $r_k(i_k)$ and $v_{c_k}(q_k)$, respectively, $k = 1,2,\ldots,n$, $i_k(t)$, $j_k(t)$ denote the currents in the circuit, $p_k(t)$, $q_k(t)$ stand for the corresponding electric charges, that is $\dot{p}_k(t) = i_k(t)$, $\dot{q}_k(t) = j_k(t)$. The circuit is powered by a voltage source $u(t)$.

According to Kirchhoff's voltage law, the sum of the voltage drops in a closed circuit is equal to zero, therefore

$$r_1(i_1)i_1(t) + v_{c_1}(q_1) = u(t), \tag{3}$$

$$r_2(i_2)i_2(t) + v_{c_2}(q_2) = v_{c_1}(q_1), \tag{4}$$

and so on

$$r_n(i_n)i_n(t) + v_{c_n}(q_n) = v_{c_{n-1}}(q_{n-1}). \tag{5}$$

Let introduce the notation: $\boldsymbol{i}(t) = [i_1(t)\, i_2(t) \ldots i_n(t)]^{\mathrm{T}}$, $\boldsymbol{j}(t) = [j_1(t)\, j_2(t) \ldots j_n(t)]^{\mathrm{T}}$, $\boldsymbol{p}(t) = [p_1(t)\, p_2(t) \ldots p_n(t)]^{\mathrm{T}}$, $\boldsymbol{q}(t) = [q_1(t)\, q_2(t) \ldots q_n(t)]^{\mathrm{T}}$, $\dot{\boldsymbol{p}}(t) = \boldsymbol{i}(t)$, $\dot{\boldsymbol{q}}(t) = \boldsymbol{j}(t)$. Without loss of generality it can be considered that

$$r_k(i_k) = r_k(\dot{p}_k) = r_k(\dot{\boldsymbol{p}}), \quad k = 1,2,\ldots,n, \tag{6}$$

$$v_{c_k}(q_k) = v_{c_k}(p_k, p_{k+1}) = v_{c_k}(\boldsymbol{p}), \quad k = 1,2,\ldots,n-1, \tag{7}$$

$$v_{c_n}(q_n) = v_{c_n}(p_n) = v_{c_n}(\boldsymbol{p}). \tag{8}$$

Then, the circuit's dynamic behavior can be governed by the following equation

$$\boldsymbol{R}(\dot{\boldsymbol{p}})\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} + \boldsymbol{V}_C(\boldsymbol{p}) = \boldsymbol{B}u(t), \quad \boldsymbol{p}(0) = \boldsymbol{p}_0, \tag{9}$$

where $\boldsymbol{p}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $\boldsymbol{p}_0 \in \mathbb{R}^n$ is a given initial condition, $t > 0$,

$$\boldsymbol{R}(\dot{\boldsymbol{p}}) = \operatorname{diag}\left(r_1(\dot{\boldsymbol{p}}), r_2(\dot{\boldsymbol{p}}), \ldots, r_n(\dot{\boldsymbol{p}})\right), \tag{10}$$

$$\boldsymbol{V}_C(\boldsymbol{p}) = \begin{bmatrix} v_{c_1}(\boldsymbol{p}) \\ v_{c_2}(\boldsymbol{p}) - v_{c_1}(\boldsymbol{p}) \\ \vdots \\ v_{c_n}(\boldsymbol{p}) - v_{c_{n-1}}(\boldsymbol{p}) \end{bmatrix}, \tag{11}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^{\mathrm{T}}. \tag{12}$$

The fundamental assumptions on which further results are based can be characterized in the form:

Assumption 1 *The resistances r_k, $k = 1,2,\ldots,n$ are continuous functions with continuous derivatives and $r_k(\dot{p}_k) > 0$ for $\dot{p}_k \in \Omega_{r_k} \subset \mathbb{R}$, where Ω_{r_k} is a neighborhood of zero.*

Assumption 2 *The characteristics v_{c_k}, $k = 1,2,\ldots,n$ of the capacitors are continuous functions with continuous derivatives and $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{V}_C(\boldsymbol{p}) > 0$ for $\boldsymbol{p} \neq \boldsymbol{0}$ in some neighborhood $\Omega_c \subset \mathbb{R}^n$ of zero.*

Lemma 1 ([22]) *The matrix $\boldsymbol{R}(\dot{\boldsymbol{p}})$ is positive definite for each $\dot{\boldsymbol{p}} \in \Omega_r \subset \mathbb{R}^n$, where Ω_r is a neighborhood of zero.*

3.2. RL ladder network

Consider an RL ladder network which schematic diagram is shown in Fig. 2. The

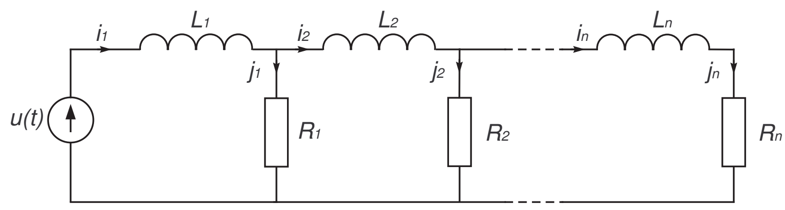

Figure 2. Schematic diagram of an RL ladder network.

network consists of inductors L_k, resistors R_k, and a voltage source $u(t)$. The inductors and resistors are different and have nonlinear characteristics: $l_k(i_k)$ and $r_k(j_k)$, respectively, where $k = 1,2,\dots,n$, $i_k(t)$, $j_k(t)$ denote the currents in the circuit.

By applying Kirchhoff's voltage and current laws to the circuit, we can obtain the differential equation that describes the dynamics of electric current flow

$$\boldsymbol{G}(\boldsymbol{i})\frac{\mathrm{d}\boldsymbol{i}(t)}{\mathrm{d}t} + \boldsymbol{R}(\boldsymbol{i})\boldsymbol{i}(t) = \boldsymbol{B}u(t), \quad \boldsymbol{i}(0) = \boldsymbol{i}_0, \tag{13}$$

where $\boldsymbol{i}(t) = [i_1(t)\, i_2(t)\, \dots\, i_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$, $\boldsymbol{i}_0 \in \mathbb{R}^n$ is a given initial condition, $u(t) \in \mathbb{R}$, $t > 0$,

$$\boldsymbol{G}(\boldsymbol{i}) = \operatorname{diag}\left(g_1(\boldsymbol{i}), g_2(\boldsymbol{i}), \dots, g_n(\boldsymbol{i})\right), \tag{14}$$

$$g_k(\boldsymbol{i}) = l_k(i_k) + i_k(t)\frac{\mathrm{d}l_k(i_k)}{\mathrm{d}i_k}, \quad k = 1,2,\dots,n, \tag{15}$$

$$\boldsymbol{R}(\boldsymbol{i}) = \begin{bmatrix} a_1 & b_2 & 0 & \dots & 0 & 0 \\ c_2 & a_2 & b_3 & \dots & 0 & 0 \\ 0 & c_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_n \\ 0 & 0 & 0 & \dots & c_n & a_n \end{bmatrix}, \tag{16}$$

$$a_1 = r_1(\boldsymbol{i}),\ a_i = r_{k-1}(\boldsymbol{i}) + r_k(\boldsymbol{i}),\ b_i = c_i = -r_{k-1}(\boldsymbol{i}), \quad k = 2,3,\dots,n, \tag{17}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^{\mathrm{T}}. \tag{18}$$

The objective of the paper is to study the RL ladder network system (13) under the following conditions:

Assumption 3 *The resistances r_k, $k = 1,2,\dots,n$ are continuous functions with continuous derivatives and $r_k(i_k) > 0$ for $i_k \in \Omega_{r_k} \subset \mathbb{R}$, where Ω_{r_k} is a neighborhood of zero.*

Assumption 4 *The inductances l_k, $k = 1,2,\dots,n$ are continuous functions with continuous derivatives, $l_k(i_k) > 0$ for $i_k \in \Omega_{l_k} \subset \mathbb{R}$, $\boldsymbol{i}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{i})\boldsymbol{i} > 0$ for $\boldsymbol{i} \in \Omega_l \subset \mathbb{R}^n$ and $\boldsymbol{i} \neq \boldsymbol{0}$, Ω_{l_k}, Ω_l are some neighborhoods of zero.*

3.3. RLC ladder network

Consider an RLC ladder network which schematic diagram is presented in Fig. 3. The circuit consists of a set of resistors R_k, inductors L_k, and capacitors C_k that are

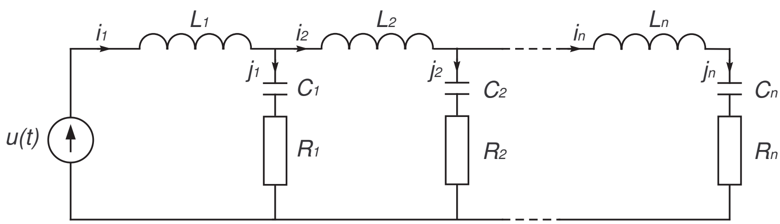

Figure 3. Schematic diagram of an RLC ladder network.

connected in the network form. The circuit is powered by a voltage source $u(t)$. The elements in the circuit have nonlinear characteristics described by the functions $r_k(j_k)$, $l_k(i_k)$, $v_{c_k}(q_k)$, $k = 1,2,\dots,n$, where $i_k(t)$, $j_k(t)$ denote the currents in the circuit, $p_k(t)$, $q_k(t)$ stand for the corresponding electric charges, that is $\dot{p}_k(t) = i_k(t)$, $\dot{q}_k(t) = j_k(t)$.

Modeling of the RLC ladder network can be done using Kirchhoff's voltage and current laws. This approach leads to the following differential equation

$$\boldsymbol{G}(\dot{\boldsymbol{p}})\frac{\mathrm{d}^2\boldsymbol{p}(t)}{\mathrm{d}t^2} + \boldsymbol{R}(\dot{\boldsymbol{p}})\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} + \boldsymbol{V}_C(\boldsymbol{p}) = \boldsymbol{B}u(t), \quad \boldsymbol{p}(0) = \boldsymbol{p}_0,\ \dot{\boldsymbol{p}}(0) = \dot{\boldsymbol{p}}_0, \tag{19}$$

where $\boldsymbol{p}(t) = [p_1(t)\ p_2(t)\ \dots\ p_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$, $\boldsymbol{i}(t) = \dot{\boldsymbol{p}}(t)$, $\boldsymbol{p}_0 \in \mathbb{R}^n$, $\dot{\boldsymbol{p}}_0 \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $t > 0$,

$$\boldsymbol{G}(\dot{\boldsymbol{p}}) = \operatorname{diag}\left(g_1(\dot{\boldsymbol{p}}), g_2(\dot{\boldsymbol{p}}), \dots, g_n(\dot{\boldsymbol{p}})\right), \tag{20}$$

$$g_k(\dot{\boldsymbol{p}}) = g_k(\boldsymbol{i}) = l_k(i_k) + i_k(t)\frac{\mathrm{d}l_k(t)}{\mathrm{d}i_k}, \quad k = 1,2,\dots,n, \tag{21}$$

$$\boldsymbol{R}(\dot{\boldsymbol{p}}) = \begin{bmatrix} a_1 & b_2 & 0 & \dots & 0 & 0 \\ c_2 & a_2 & b_3 & \dots & 0 & 0 \\ 0 & c_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_n \\ 0 & 0 & 0 & \dots & c_n & a_n \end{bmatrix}, \tag{22}$$

$$a_1 = r_1(\dot{\boldsymbol{p}}),\ a_i = r_{k-1}(\dot{\boldsymbol{p}}) + r_k(\dot{\boldsymbol{p}}),\ b_i = c_i = -r_{k-1}(\dot{\boldsymbol{p}}), \quad k = 2,3,\ldots,n, \tag{23}$$

$$\boldsymbol{V}_C(\boldsymbol{p}) = \begin{bmatrix} v_{c_1}(\boldsymbol{p}) \\ v_{c_2}(\boldsymbol{p}) - v_{c_1}(\boldsymbol{p}) \\ \vdots \\ v_{c_n}(\boldsymbol{p}) - v_{c_{n-1}}(\boldsymbol{p}) \end{bmatrix}, \tag{24}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^{\mathrm{T}}. \tag{25}$$

The following assumptions play an important role in further results:

Assumption 5 *The resistances r_k, $k = 1,2,\ldots,n$ are continuous functions with continuous derivatives and $r_k(\dot{p}_k) \geqslant 0$ for $\dot{p}_k \in \Omega_{r_k} \subset \mathbb{R}$, where Ω_{r_k} is a neighborhood of zero.*

Assumption 6 *The inductances l_k, $k = 1,2,\ldots,n$ are continuous functions with continuous derivatives, $l_k(\dot{p}_k) > 0$ for $\dot{p}_k \in \Omega_{l_k} \subset \mathbb{R}$, $\dot{\boldsymbol{p}}^{\mathrm{T}}\boldsymbol{G}(\dot{\boldsymbol{p}})\dot{\boldsymbol{p}} > 0$ for $\dot{\boldsymbol{p}} \in \Omega_l \subset \mathbb{R}^n$ and $\dot{\boldsymbol{p}} \neq \boldsymbol{0}$, Ω_{l_k}, Ω_l are some neighborhoods of zero.*

Assumption 7 *The characteristics v_{c_k}, $k = 1,2,\ldots,n$ of the capacitors are continuous functions with continuous derivatives and $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{V}_C(\boldsymbol{p}) > 0$ for $\boldsymbol{p} \neq \boldsymbol{0}$ in some neighborhood $\Omega_c \subset \mathbb{R}^n$ of zero.*

Lemma 2 ([22]) *The matrix $\boldsymbol{R}(\dot{\boldsymbol{p}})$ is semi-positive definite for each $\dot{\boldsymbol{p}} \in \Omega_r \subset \mathbb{R}^n$, where Ω_r is a neighborhood of zero.*

4. Synthesis of a stabilizing feedback controller

Consider a dynamic compensator that is connected in parallel to the ladder network system as shown in Fig. 4

$$\dot{w}(t) = -\alpha w(t) + \beta u(t), \quad w(0) = w_0, \tag{26}$$

and the following feedback

$$u(t) = -\frac{1}{K_0 + \gamma(w(t) + y(t))^2}\,(w(t) + y(t)), \tag{27}$$

where $w(t) \in \mathbb{R}$, $t > 0$, $w_0 \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, $\gamma \geqslant 0$, $K_0 > 0$, $y(t) = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{p}(t)$ in case of charge-feedback control or $y(t) = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{i}(t)$ in case of current-feedback control.

Theorem 8 *Suppose assumptions 1 and 2 hold. If there exists a neighborhood $\Omega_{pw} \subset \mathbb{R}^n \times \mathbb{R}$ of zero in which the closed-loop system* (9), (26), (27) *has only one equilibrium point $(\boldsymbol{p}_{\mathrm{e}}, w_{\mathrm{e}}) = (\boldsymbol{0}, 0)$, then this point is locally asymptotically stable in the Lyapunov sense.*

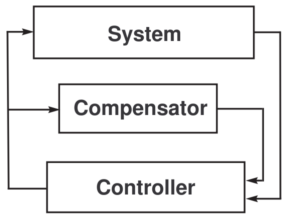

Figure 4. Block diagram of the closed-loop system with parallel compensation.

Proof Consider the following Lyapunov function

$$V(\boldsymbol{z}) = \int_{\boldsymbol{0}}^{\boldsymbol{p}(t)} \boldsymbol{V}_C(\boldsymbol{\xi})^{\mathrm{T}} \mathrm{d}\boldsymbol{\xi} + \frac{\alpha}{2\beta} w(t)^2 + V_\gamma(\boldsymbol{z}), \tag{28}$$

where $\boldsymbol{z}(t) = \mathrm{col}(\boldsymbol{p}(t), w(t))$, $\int_{\boldsymbol{0}}^{\boldsymbol{p}(t)} (\ldots)\, \mathrm{d}\boldsymbol{\xi}$ denotes a line integral along the straight line in the space $\mathbb{R}^n$ from the beginning point $\boldsymbol{0}$ to the ending point $\boldsymbol{p}(t)$,

$$V_\gamma(\boldsymbol{z}) = \begin{cases} \frac{1}{2K_0} \left(w(t) + y(t)\right)^2, & \text{for } \gamma = 0, \\ \frac{1}{2\gamma} \ln\left(1 + \frac{\gamma}{K_0} \left(w(t) + y(t)\right)^2\right), & \text{for } \gamma > 0. \end{cases} \tag{29}$$

The time derivative of the function (28) becomes

$$\begin{aligned} \dot{V}(\boldsymbol{z}) &= \nabla_{\boldsymbol{p}} \left(\int_{\boldsymbol{0}}^{\boldsymbol{p}} \boldsymbol{V}_C(\boldsymbol{\xi})^{\mathrm{T}} \mathrm{d}\boldsymbol{\xi} \right) \dot{\boldsymbol{p}}(t) + \frac{\alpha}{\beta} w(t) \dot{w}(t) + \dot{V}_\gamma(\boldsymbol{z}) \\ &= \boldsymbol{V}_C(\boldsymbol{p})^{\mathrm{T}} \dot{\boldsymbol{p}}(t) + \frac{\alpha}{\beta} w(t) \dot{w}(t) - u(t) \left(\dot{w}(t) + \dot{y}(t)\right). \end{aligned} \tag{30}$$

Evaluating $\dot{V}(\boldsymbol{z})$ along the trajectory of the closed-loop system (9), (26), (27) gives

$$\begin{aligned} \dot{V}(\boldsymbol{z}) = {} & \boldsymbol{V}_C(\boldsymbol{p})^{\mathrm{T}} \left(\boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \boldsymbol{B} u(t) - \boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \boldsymbol{V}_C(\boldsymbol{p}) \right) + \frac{\alpha}{\beta} w(t) \left(-\alpha w(t) + \beta u(t)\right) \\ & - u(t) \left(-\alpha w(t) + \beta u(t)\right) - u(t) \boldsymbol{B}^{\mathrm{T}} \left(\boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \boldsymbol{B} u(t) - \boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \boldsymbol{V}_C(\boldsymbol{p}) \right). \end{aligned} \tag{31}$$

After some elementary calculations

$$\begin{aligned} \dot{V}(\boldsymbol{z}) = {} & \boldsymbol{V}_C(\boldsymbol{p})^{\mathrm{T}} \boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \boldsymbol{B} u(t) - \boldsymbol{V}_C(\boldsymbol{p})^{\mathrm{T}} \boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \boldsymbol{V}_C(\boldsymbol{p}) - \frac{\alpha^2}{\beta} w(t)^2 \\ & + 2\alpha w(t) u(t) - \beta u(t)^2 - u(t) \boldsymbol{B}^{\mathrm{T}} \boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \boldsymbol{B} u(t) + u(t) \boldsymbol{B}^{\mathrm{T}} \boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \boldsymbol{V}_C(\boldsymbol{p}), \end{aligned} \tag{32}$$

it can be seen that

$$\dot{V}(\boldsymbol{z}) = -\left(\boldsymbol{B}u(t) - \boldsymbol{V}_C(\boldsymbol{p})\right)^{\mathrm{T}} \boldsymbol{R}(\dot{\boldsymbol{p}})^{-1} \left(\boldsymbol{B}u(t) - \boldsymbol{V}_C(\boldsymbol{p})\right) - \beta\left(\frac{\alpha}{\beta}w(t) - u(t)\right)^2, \tag{33}$$

and finally

$$\dot{V}(\boldsymbol{z}) = -\dot{\boldsymbol{p}}(t)^{\mathrm{T}}\boldsymbol{R}(\dot{\boldsymbol{p}})\dot{\boldsymbol{p}}(t) - \frac{1}{\beta}\dot{w}(t)^2. \tag{34}$$

Let Ω_h be a compact set defined as

$$\Omega_h = \left\{\boldsymbol{z} \in \Omega_{pw} \subset \mathbb{R}^{n+1} : V(\boldsymbol{z}) < h\right\}, \tag{35}$$

where $h > 0$ is a real positive number. With the help of lemma 3 it can be noticed that $V(\boldsymbol{z}) > 0$ for $\boldsymbol{z} \in \Omega_h \backslash \{\boldsymbol{0}\}$, $V(\boldsymbol{0}) = 0$ and $\dot{V}(\boldsymbol{z}) \leqslant 0$ for $\boldsymbol{z} \in \Omega_h$. As a consequence of LaSalle's invariance principle [5], the trajectories of the closed-loop system (9), (26), (27) enter the largest invariant set in S, where

$$S = \{\boldsymbol{z} \in \Omega_h : \dot{V}(\boldsymbol{z}) = 0\}. \tag{36}$$

From $\dot{V}(\boldsymbol{z}) = 0$ it follows that

$$\dot{\boldsymbol{p}}(t) = 0 \quad \text{and} \quad \dot{w}(t) = 0. \tag{37}$$

This means that S contains only equilibrium points of the system (9), (26), (27). Since the system has only one equilibrium point in the considered neighborhood, thus $S = \{\boldsymbol{0}\}$ and according to LaSalle's principle, the origin $\boldsymbol{0} \in \mathbb{R}^{n+1}$ is asymptotically stable (in the Lyapunov sense).

Theorem 9 *Suppose assumptions 3 and 4 hold. If there exists a neighborhood* $\Omega_{iw} \subset \mathbb{R}^n \times \mathbb{R}$ *of zero in which the closed-loop system* (13), (26), (27) *has only one equilibrium point* $(\boldsymbol{i}_{\mathrm{e}}, w_{\mathrm{e}}) = (\boldsymbol{0}, 0)$, *then this point is locally asymptotically stable in the Lyapunov sense.*

Proof The proof can be carried out almost exactly like the proof of theorem 8.

Theorem 10 *Suppose assumptions 5, 6, and 7 hold. If there exists a neighborhood* $\Omega_{p\dot{p}w} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ *of zero in which the closed-loop system* (19), (26), (27) *has only one equilibrium point* $(\boldsymbol{p}_{\mathrm{e}}, \dot{\boldsymbol{p}}_{\mathrm{e}}, w_{\mathrm{e}}) = (\boldsymbol{0}, \boldsymbol{0}, 0)$, *then this point is locally asymptotically stable in the Lyapunov sense.*

Proof Consider the following Lyapunov function candidate

$$V(\boldsymbol{z}) = \int\limits_{\boldsymbol{0}}^{\dot{\boldsymbol{p}}(t)} \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} + \int\limits_{\boldsymbol{0}}^{\boldsymbol{p}(t)} \boldsymbol{V}_C(\boldsymbol{\xi})^{\mathrm{T}}\,\mathrm{d}\boldsymbol{\xi} + \frac{\alpha}{2\beta}w(t)^2 + V_\gamma(\boldsymbol{z}), \tag{38}$$

where $z(t) = \text{col}(\boldsymbol{p}(t), \dot{\boldsymbol{p}}(t), w(t))$, $\int_{\mathbf{0}}^{\boldsymbol{p}(t)} (\ldots)\, \mathrm{d}\boldsymbol{\xi}$, $\int_{\mathbf{0}}^{\dot{\boldsymbol{p}}(t)} (\ldots)\, \mathrm{d}\boldsymbol{\xi}$ denote line integrals along the straight line in the space $\mathbb{R}^n$ from the beginning point $\mathbf{0}$ to the ending points $\boldsymbol{p}(t)$ and $\dot{\boldsymbol{p}}(t)$, respectively,

$$V_\gamma(z) = \begin{cases} \frac{1}{2K_0}\left(w(t)+y(t)\right)^2, & \text{for } \gamma = 0, \\ \frac{1}{2\gamma}\ln\left(1+\frac{\gamma}{K_0}\left(w(t)+y(t)\right)^2\right), & \text{for } \gamma > 0. \end{cases} \tag{39}$$

Differentiate $V(z)$ with respect to time t

$$\begin{aligned} \dot{V}(z) &= \nabla_{\dot{\boldsymbol{p}}}\left(\int_{\mathbf{0}}^{\dot{\boldsymbol{p}}} \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\right)\ddot{\boldsymbol{p}}(t) + \nabla_{\boldsymbol{p}}\left(\int_{\mathbf{0}}^{\boldsymbol{p}} \boldsymbol{V}_C(\boldsymbol{\xi})^{\mathrm{T}}\,\mathrm{d}\boldsymbol{\xi}\right)\dot{\boldsymbol{p}}(t) + \frac{\alpha}{\beta}w(t)\dot{w}(t) + \dot{V}_\gamma(z) \\ &= \dot{\boldsymbol{p}}(t)^{\mathrm{T}}\boldsymbol{G}(\dot{\boldsymbol{p}})\ddot{\boldsymbol{p}}(t) + \boldsymbol{V}_C(\boldsymbol{p})^{\mathrm{T}}\dot{\boldsymbol{p}}(t) + \frac{\alpha}{\beta}w(t)\dot{w}(t) - u(t)\left(\dot{w}(t)+\dot{y}(t)\right), \end{aligned} \tag{40}$$

and next substitute (19) and (26) into (40)

$$\begin{aligned} \dot{V}(z) &= \dot{\boldsymbol{p}}(t)^{\mathrm{T}}\boldsymbol{G}(\dot{\boldsymbol{p}})\left(\boldsymbol{G}(\dot{\boldsymbol{p}})^{-1}\boldsymbol{B}u(t) - \boldsymbol{G}(\dot{\boldsymbol{p}})^{-1}\boldsymbol{R}(\dot{\boldsymbol{p}})\dot{\boldsymbol{p}}(t) - \boldsymbol{G}(\dot{\boldsymbol{p}})^{-1}\boldsymbol{V}_C(\boldsymbol{p})\right) + \boldsymbol{V}_C(\boldsymbol{p})^{\mathrm{T}}\dot{\boldsymbol{p}}(t) \\ &\quad + \frac{\alpha}{\beta}w(t)\left(-\alpha w(t)+\beta u(t)\right) - u(t)\left(-\alpha w(t)+\beta u(t)\right) - u(t)\boldsymbol{B}^{\mathrm{T}}\dot{\boldsymbol{p}}(t). \end{aligned} \tag{41}$$

After some elementary calculations

$$\dot{V}(z) = -\dot{\boldsymbol{p}}(t)^{\mathrm{T}}\boldsymbol{R}(\dot{\boldsymbol{p}})\dot{\boldsymbol{p}}(t) - \beta\left(\frac{\alpha}{\beta}w(t) - u(t)\right)^2, \tag{42}$$

what can be also expressed in the shorter form

$$\dot{V}(z) = -\dot{\boldsymbol{p}}(t)^{\mathrm{T}}\boldsymbol{R}(\dot{\boldsymbol{p}})\dot{\boldsymbol{p}}(t) - \frac{1}{\beta}\dot{w}(t)^2. \tag{43}$$

With the help of lemmas 3 and 4 it can be noticed that $V(z) > 0$ for $z \in \Omega_h \backslash \{\mathbf{0}\}$, $V(\mathbf{0}) = 0$, and $\dot{V}(z) \leqslant 0$ for $z \in \Omega_h$, where Ω_h is a compact set defined as follows

$$\Omega_h = \left\{z \in \Omega_{p\dot{p}w} \subset \mathbb{R}^{2n+1} : V(z) < h\right\}, \tag{44}$$

and h is a real positive number. According to LaSalle's theorem [5], the trajectories enter the largest invariant set in S, where

$$S = \{z \in \Omega_h : \dot{V}(z) = 0\}. \tag{45}$$

To prove that all solutions starting from Ω_h tend to zero, it is sufficient to show that S contains only the zero solution. The condition $\dot{V}(z) = 0$ holds if and only if $\dot{\boldsymbol{p}}(t) = 0$ and $\dot{w}(t) = 0$, therefore S contains all equilibrium points of the system (19), (26), (27). Since the system has only one equilibrium point, thus $S = \{\mathbf{0}\}$.

Lemma 3 ([24]) *If* $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{V}_C(\boldsymbol{p}) > 0$ *for* $\boldsymbol{p} \neq \boldsymbol{0}$*, then the line integral* $\int_{\boldsymbol{0}}^{\boldsymbol{p}} \boldsymbol{V}_C(\boldsymbol{\xi})^{\mathrm{T}}\,\mathrm{d}\boldsymbol{\xi}$ *along the straight line in the space* $\mathbb{R}^n$ *from the beginning point* $\boldsymbol{0}$ *to the ending point* $\boldsymbol{p}$ *(* $\boldsymbol{p} \neq \boldsymbol{0}$*) is positive.*

Lemma 4 ([24]) *If* $\dot{\boldsymbol{p}}^{\mathrm{T}}\boldsymbol{G}(\dot{\boldsymbol{p}})\dot{\boldsymbol{p}} > 0$ *for* $\dot{\boldsymbol{p}} \neq \boldsymbol{0}$*, then the line integral* $\int_{\boldsymbol{0}}^{\dot{\boldsymbol{p}}} \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$ *along the straight line in the space* $\mathbb{R}^n$ *from the beginning point* $\boldsymbol{0}$ *to the ending point* $\dot{\boldsymbol{p}}$ *(* $\dot{\boldsymbol{p}} \neq \boldsymbol{0}$*) is positive.*

5. Numerical example

Consider an RLC circuit presented in Fig. 5. The characteristics of the resistor R_2,

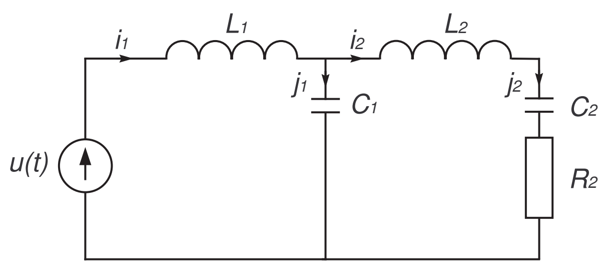

Figure 5. RLC ladder network with $n = 2$.

the inductors L_1, L_2, and the capacitors C_1, C_2 are nonlinear and can be written in the following analytic form:

$$r_2(\dot{\boldsymbol{p}}) = \frac{0.2}{1+0.5\dot{p}_2^2}, \tag{46}$$

$$l_1(\dot{\boldsymbol{p}}) = 0.1\mathrm{e}^{\dot{p}_1^2}, \quad l_2(\dot{\boldsymbol{p}}) = 0.2\mathrm{e}^{\dot{p}_2^2}, \tag{47}$$

$$v_{c_1}(\boldsymbol{p}) = (p_1 - p_2)\mathrm{e}^{0.3(p_1-p_2)^2}, \quad v_{c_2}(\boldsymbol{p}) = p_2\mathrm{e}^{0.5p_2^2}. \tag{48}$$

The dynamics of electric charge flow in the circuit can be described in the form

$$\boldsymbol{G}(\dot{\boldsymbol{p}})\frac{\mathrm{d}^2\boldsymbol{p}(t)}{\mathrm{d}t^2} + \boldsymbol{R}(\dot{\boldsymbol{p}})\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} + \boldsymbol{V}_C(\boldsymbol{p}) = \boldsymbol{B}u(t), \tag{49}$$

where $\boldsymbol{p}(t) = \mathrm{col}(p_1(t), p_2(t)) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, $t > 0$,

$$\boldsymbol{G}(\dot{\boldsymbol{p}}) = \begin{bmatrix} 0.1\mathrm{e}^{\dot{p}_1^2}\left(1+2\dot{p}_1^2\right) & 0 \\ 0 & 0.2\mathrm{e}^{\dot{p}_2^2}\left(1+2\dot{p}_2^2\right) \end{bmatrix}, \tag{50}$$

$$\boldsymbol{R}(\dot{\boldsymbol{p}}) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{0.2}{1+0.5\dot{p}_2^2} \end{bmatrix}, \tag{51}$$

$$\boldsymbol{V}_C(\boldsymbol{p}) = \begin{bmatrix} (p_1 - p_2)\,\mathrm{e}^{0.3(p_1-p_2)^2} \\ p_2 \mathrm{e}^{0.5p_2^2} \end{bmatrix}, \tag{52}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{53}$$

The following initial conditions are used for the differential equation (49):

$$p_1(0) = 0.3, \quad p_2(0) = 0.2, \tag{54}$$

$$\dot{p}_1(0) = 0.0, \quad \dot{p}_2(0) = 0.0. \tag{55}$$

Introduce one-dimensional parallel compensator

$$\dot{w}(t) + 0.2w(t) = 0.5u(t), \quad w(0) = 0, \tag{56}$$

and design the controller

$$u(t) = -50.0\,(w(t) + p_1(t))\,. \tag{57}$$

It is easy to check that assumptions 5, 6, and 7 hold. In some neighborhood of zero the closed-loop system (49), (56), (57) has only zero equilibrium point. According to theorem 10 the closed-loop system is then asymptotically stable. The trajectories of the open-loop system (dot line) and closed-loop system (solid line) are shown in Figs. 6–8. The control voltage $u(t)$ and the state variable $w(t)$ of the compensator are presented in Figs. 9 and 10.

6. Conclusions

The paper has addressed the stabilization problem for inhomogeneous electric ladder networks with nonlinear elements. A dynamic feedback control law has been proposed to make the state asymptotically stable. The asymptotic stability (in the Lyapunov sense) of the closed-loop systems have been proved by the use of Lyapunov functionals and concluded by LaSalle's invariance principle. The designed dynamic controller is one-dimensional and system size independent. Stabilization in a wide range of the controller parameters improves the system's robustness. The controller provides also excellent damping and dynamic performance improvement in comparison with open-loop systems. Numerical calculations and computer simulations have been performed in the MathWorks™ MATLAB®/Simulink® environment to show the effectiveness of the proposed method.

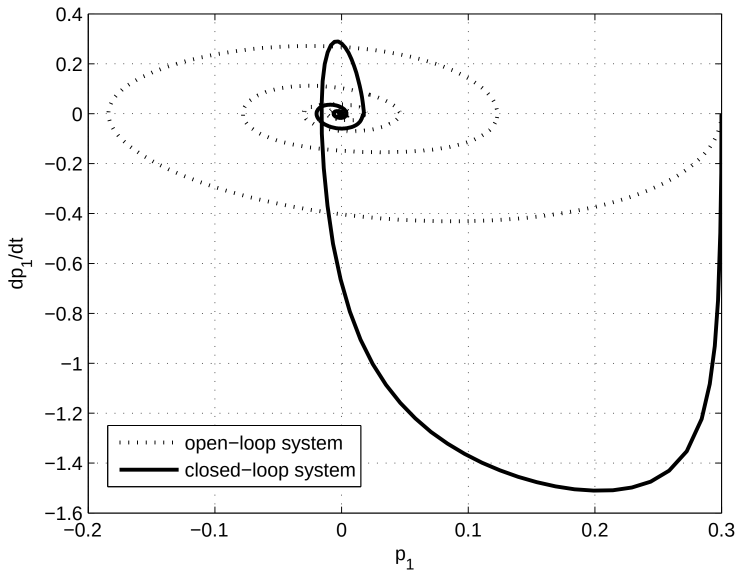

Figure 6. The electric charge $p_1(t)$ and the current $\dot{p}_1(t)$ in the open-loop circuit (dot line) and closed-loop circuit (solid line).

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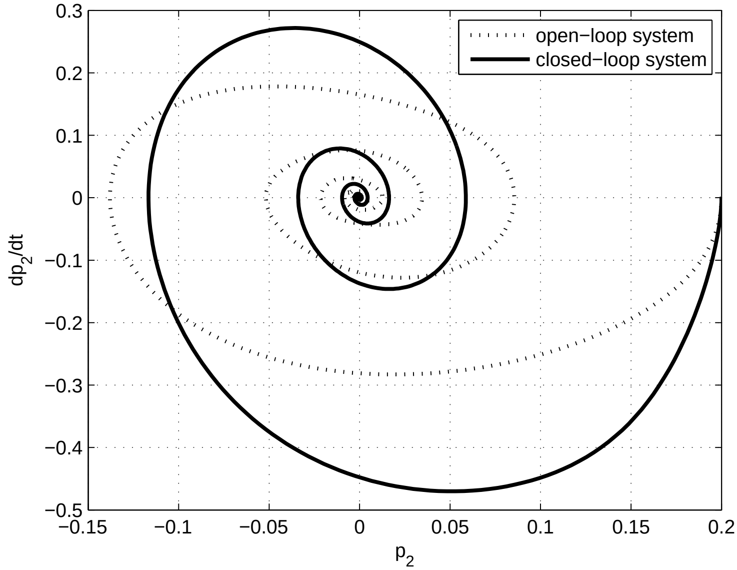

Figure 7. The electric charge $p_2(t)$ and the current $\dot{p}_2(t)$ in the open-loop circuit (dot line) and closed-loop circuit (solid line).

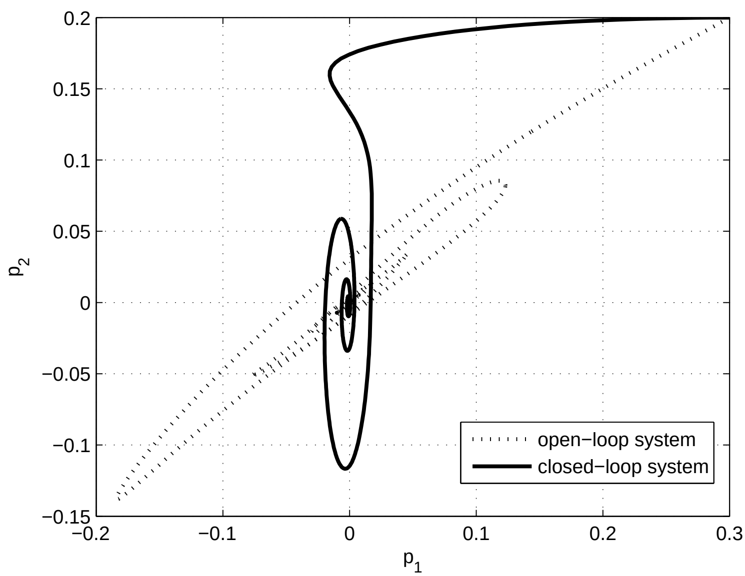

Figure 8. The trajectories of the open-loop system (dot line) and closed-loop system (solid line).

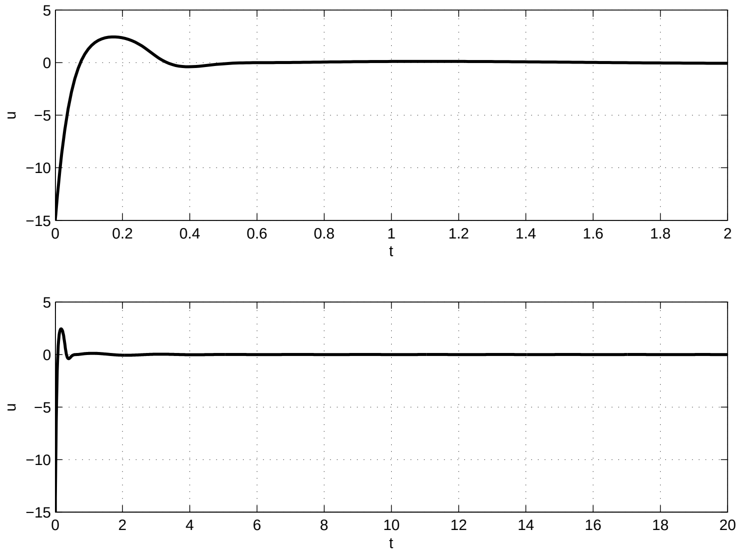

Figure 9. The control voltage $u(t)$. The same simulation results are shown in different time windows.

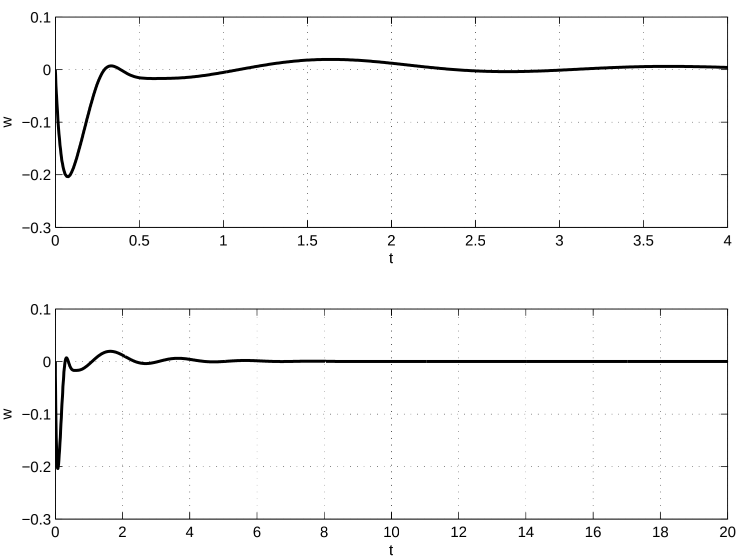

Figure 10. The compensator state variable $w(t)$. The same simulation results are shown in different time windows.

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