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Lagrange and practical stability criteria for dynamical systems with nonlinear perturbations

ASSEN V. KRUMOV

In the paper two classes of nonlinear dynamical system with perturbations are considered. The sufficient conditions for robust Lagrange and practical stability are proven with theorems, applying the theory of nonlinear operators of the functional analysis. The presented criteria give also the bounds of the analyzed dynamical processes. Three examples comparing the numerical computer solutions and the analytical investigation of the stability of the systems are given. The method can be applied to analytical and computer modeling of nonlinear dynamical systems, synthesis of computer control and optimization.

Key words: Lagrange, stability criteria, dynamical systems, nonlinear perturbation

1. Introduction

In the known scientific literature on the perturbation methods [3], [9], [4], [13] usually the problem of stability, especially of systems with nonlinear perturbations is rarely considered. In [8] stability bounds are defined in the case of linear perturbation; in [11] conditions, which ensure closed loop asymptotic stability are given; in [2] exponential stability of nonlinear system with delayed perturbation is investigated; in [16] exponential stability of singularly perturbed nonlinear system is considered. The sufficient conditions for robust application of the perturbation method are investigated in [7], but the bounds of the nonlinear dynamic process are not explicitly given.

In none of the above mentioned paper the problems of Lagrange and practical stability are solved. Yet another shortcoming of these papers is that their results are not applicable in the case of perturbations containing derivatives of the variables, describing the dynamical system.

In the present paper, method and criteria for determining the sufficient conditions for robust Lagrange and practical stability of a class of dynamical systems with nonlinear perturbations are suggested. They are proven with theorems, applying the theory of nonlinear operators of the functional analysis. The presented criteria give also the bounds of the analyzed dynamical processes. The robustness of the suggested criteria can be achieved by taking into account the worst case values of the parameters and functions,

The author is with the Higher School of Transport, Sofia, Bulgaria. E-mail: assenkrumov@hotmail.com
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when estimating the norms of the operators. The perturbations in this method can contain derivatives of the variables.

The mathematical methods of the functional analysis and especially the methods of nonlinear functional analysis are applied in this paper. The are outlined in [14], [15], [1], [6], [10].

2. Lagrange stability criterion for dynamical systems with nonlinear perturbations

General case of dynamical systems with nonlinear perturbations can be described by the following equation

$$\frac{d\boldsymbol{X}}{dt} + \boldsymbol{A}(P)\boldsymbol{X} + \boldsymbol{F}(t,P) - \varphi[\boldsymbol{X}, \frac{d\boldsymbol{X}}{dt}, P] = 0, \tag{1}$$

where $\boldsymbol{A}$, $\boldsymbol{X}$, $\boldsymbol{F}(t)$ are matrices, P is a set of parameters, and φ is a nonlinear function of its arguments. In the sequel P will be taken into account only when it is necessary and usually to estimate of the norms. The linear part of (1), namely

$$L = \frac{d\boldsymbol{X}}{dt} + \boldsymbol{A}\boldsymbol{X} + \boldsymbol{F}(t) \tag{1a}$$

can be considered as first approximation of (1) and φ as residual or perturbation.

In the general case $\boldsymbol{F}(t)$ contains the input signals, including control signals $\boldsymbol{U}(t)$:

$$\boldsymbol{F}(t) = \boldsymbol{B}(t)\boldsymbol{U}(t) \tag{1b}$$

For the purpose of the proposed method a "big' parameter μ is introduced [7]:

$$L - \mu\varphi = 0. \tag{2}$$

Obviously if $\mu = 1$ then the system (2) is identical to (1) and this is the case when the suggested method can be applied.

The operator equations, corresponding to (2) are

$$Q(\boldsymbol{X}, \mu) = 0, \tag{3}$$

$$Q(\boldsymbol{X}_0, \mu_0) = 0. \tag{4}$$

Further will be considered that :

$$\mu_0 = 0, \quad \boldsymbol{X}(\mu_0) = \boldsymbol{X}_0. \tag{5}$$

The element $\boldsymbol{X}_0$ can be found for $\mu = \mu_0 = 0$ using the Green matrix $\boldsymbol{G}(t-\tau)$ [5] for the linear part L:

$$\boldsymbol{X}_0(t) = \boldsymbol{G}(t)\boldsymbol{X}(0) - \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{F}(\tau)\tau, \tag{6}$$

where the initial conditions are:

$$\boldsymbol{X}(0) = \boldsymbol{X}_0(0). \tag{7}$$

For the nonlinear operator $Q(\boldsymbol{X},\mu)$ the homogeneous forms C_{pq} of the polylinear operator $\widetilde{C}_{pq}$ can be found as Gateaux or Frechet derivatives [1], [6]:

$$C_{pq} = \frac{1}{p!q!} Q^{(p+q)}_{\mu^p x^q}(\boldsymbol{X}_0,\mu_0). \tag{8}$$

The inverse operator C_{01}^{-1} is:

$$\Gamma_{01} = C_{01}^{-1} = \left[\frac{\partial}{\partial \boldsymbol{X}} Q(\boldsymbol{X}_0,\mu_0)\right]^{-1} = \left[\frac{\partial}{\partial c} Q(\boldsymbol{X}_0 + c\boldsymbol{H},\mu_0)_{c=0}\right]^{-1}. \tag{9}$$

The operators $\widetilde{\prod}_{pq}$

$$\widetilde{\prod}_{pq} = -\Gamma_{01}[\widetilde{C}_{pq}] \tag{10}$$

and $\widetilde{C}_{pq}$ are polylinear symmetrical operators [1], [6], whose homogeneous forms are $\prod_{pq}$ and C_{pq} . The norms of $\widetilde{C}_{pq}$ and C_{pq} are:

$$||\widetilde{C}_{pq}|| = \sup_{||H_1||\ldots||H_q||\leqslant 1} ||\widetilde{C}_{pq}(H_1,H_2,\ldots H_q)|| \tag{11}$$

$$||C_{pq}|| = \sup_{||H||=1} ||\widetilde{C}_{pq}(H)|| \tag{12}$$

These two norms are connected with the inequality:

$$||C_{pq}|| \leqslant ||\widetilde{C}_{pq}|| \leqslant \frac{q^q}{q!}||C_{pq}|| \tag{13}$$

For the further consideration the majorant estimations v_{pq} of the norms of operators will be introduced:

$$v_{pq} = \Pi_{01} = 0 \tag{14}$$

$$||\widetilde{\Pi}_{pq}|| \leqslant ||\Gamma_{01}|| \cdot ||\widetilde{C}_{pq}|| \leqslant v_{pq}. \tag{15}$$

They can always be used for a construction of equation, called majorant:

$$\eta = \sum_{p+q\geqslant 1}^{\infty} v_{pq}\xi^p\eta^q. \tag{16}$$

If the right part of (16) is convergent with a sum $\rho(\eta,\xi)$ then:

$$\eta = \rho(\eta,\xi) \tag{17}$$

There can exist solution $\eta^*(\xi)$ of (17) satisfying the equation:

$$\eta^*(0) = 0. \tag{18}$$

The above definitions and formulas will be used in the following theorem, giving the sufficient conditions for Lagrange stability of the nonlinear system (1). The most useful norms of the Banach spaces can be found in [12].

Theorem 1 *The sufficient conditions for Lagrange stability of the nonlinear system* (1) *are:*

a) The solution $\boldsymbol{X}_0$ of the linear system (1a)*: $L = 0$ is bounded for $t \in [0,T]$, $p \in P$.*

b) The operator $Q(\boldsymbol{X},\mu)$ is analytical, there exist bounded inverse operator C_{01}^{-1} and the Gateaux derivative of $Q(\boldsymbol{X}_0,\mu_0)$ is continuous for:

$$||\boldsymbol{X}-\boldsymbol{X}_0|| \leqslant a, |\mu-\mu_0| \leqslant 1. \tag{19}$$

c) There is solution $\eta^*(\xi)$ of the majorant equation (16) *for the operator $Q(\boldsymbol{X},\mu)$ which satisfies the conditions:*

$$0 < \eta^*(1) \leqslant a, \quad \eta^*(0) = 0 \tag{20}$$

and there is no singular point of $\eta^*(\xi)$ for $\xi \leqslant 1$.

Then the solution $\boldsymbol{X}$ of the nonlinear system (1) *is bounded and stable for $t \in [0,T]$, $p \in P$:*

$$||\boldsymbol{X}|| \leqslant ||\boldsymbol{X}_0|| + \eta^*(\xi = 1). \tag{21}$$

Proof The analytical operator $Q(\boldsymbol{X},\mu)$ can be developed in convergent Taylor series [1], [6]:

$$Q(\boldsymbol{X},\mu) = Q(\boldsymbol{X}_0,\mu_0) + \sum_{p+q\geqslant 1}^{\infty} (\mu-\mu_0)^p C_{pq}(\boldsymbol{X}-\boldsymbol{X}_0). \tag{22}$$

If the operator C_{01}^{-1} is bounded and the Gateaux derivative is continuous in the intervals (19), then there exist only one analytical solution $\boldsymbol{X}(\mu)$ of $Q(\boldsymbol{X},\mu)=0$:

$$\boldsymbol{X}(\mu)-\boldsymbol{X}_0=(\mu-\mu_0)\boldsymbol{X}_1+(\mu-\mu_0)^2\boldsymbol{X}_2+\ldots \tag{23}$$

Taking into account (10) and (14) the equation (22) can be transformed:

$$\boldsymbol{X}-\boldsymbol{X}_0=\sum_{p+q\geqslant 1}^{\infty}(\mu-\mu_0)^p\Pi_{pq}(\boldsymbol{X}-\boldsymbol{X}_0). \tag{24}$$

After replacement of $\boldsymbol{X}-\boldsymbol{X}_0$ in (24) with (23) the result is:

$$||\boldsymbol{X}_{m+1}||=\sum\frac{(\alpha_1+\alpha_2+\ldots\alpha_{\mathrm{m}})!}{\alpha_1\,!\alpha_2\,!\ldots\alpha_{\mathrm{m}}\,!}\widetilde{\Pi}_{pq}(\boldsymbol{X}_1)^{\alpha_1}\ldots(\boldsymbol{X}_m)^{\alpha_m} \tag{25}$$

where $p+\alpha_1+\cdots+m\alpha_m=m+1$, $\alpha_1+\cdots+\alpha_m=q$ and for the norms the following inequality is valid:

$$||\boldsymbol{X}_{m+1}||\leqslant\sum\frac{(\alpha_1+\alpha_2+\ldots\alpha_{\mathrm{m}})!}{\alpha_1\,!\alpha_2\,!\ldots\alpha_{\mathrm{m}}\,!}||\widetilde{\Pi}_{pq}||\cdot||\boldsymbol{X}_1||^{\alpha_1}\ldots||\boldsymbol{X}_m||^{\alpha_m}. \tag{26}$$

The majorant equation (16) has only one solution if the operator:

$$g(\xi,\eta)=\eta-\sum_{p+q\geqslant 1}^{\infty}v_{pq}\xi^p\eta^q \tag{27}$$

has a continuous reverse operator $g(\xi,\eta)$ found as Gateaux derivative

$$g'_{\eta}(0,0)=\boldsymbol{H}-v_{01}\boldsymbol{H}=\boldsymbol{H}. \tag{28}$$

$g'_{\eta}(0,0)$ satisfies this condition and hence there is only one solution:

$$\eta=\sum_{p=1}^{\infty}\beta_p\xi^p,\quad \eta(0)=0. \tag{29}$$

For small enough ξ the solution 29) is convergent and r_0 is its radius of convergence.

To determine β_k, the series (29) is substituted in (16). The quantities containing ξ with the same power form the following equations for different m:

$$\beta_{m+1}=\sum\frac{(\alpha_1+\alpha_2+\ldots\alpha_{\mathrm{m}})!}{\alpha_1\,!\alpha_2\,!\ldots\alpha_{\mathrm{m}}\,!}v_{pq}\beta_1^{\alpha_1}\ldots\beta_m^{\alpha_m}\xi^p \tag{30}$$

where $p+\alpha_1+\cdots+m\alpha_m=m+1$, $\alpha_1+\cdots+\alpha_m=q$.

Applying the method of mathematical induction it can be proven that:

$$||\boldsymbol{X}_k||\leqslant\beta_k \tag{31}$$

for each k. Indeed,

$$\beta_1 = v_{10}, \quad ||\boldsymbol{X}_1|| \leqslant v_{10}. \tag{32}$$

Hence

$$||\boldsymbol{X}_1|| \leqslant \beta_1. \tag{33}$$

Equations (30) and (26) can be compared to find out that if inequality (31) is satisfied for $k = m$, then it will be satisfied also for $k = m+1$, which means that (31) is true for arbitrary k.

Inequality (31) shows that the series (29) is majorant in relation to (23) and, (23) is convergent for $|\mu - \mu_0| \leqslant r_0$:

$$||\boldsymbol{X} - \boldsymbol{X}_0|| \leqslant ||\boldsymbol{X}_1|| \cdot |\mu - \mu_0| + ||\boldsymbol{X}_2|| \cdot |\mu - \mu_0|^2 + \cdots \leqslant \sum_{p=1}^{\infty} \beta_k |\mu - \mu_0|^p. \tag{34}$$

If the right partof the majorant equation (16) is convergent with a sum $\rho(\eta, \xi)$, then the solution of (16) will not be the series (29) but a function $\eta^*(\xi)$, satisfying the condition (18). If this function has no singular point for $\xi \leqslant 1$, then it can be developed in series (29) with a sum equal to $\eta^*(\xi)$ and hence for $\mu = 1$, $\mu_0 = 0$:

$$||\boldsymbol{X} - \boldsymbol{X}_0|| \leqslant \eta^*\left(|\mu - \mu_0|\right) = \eta^*(1) \leqslant a \tag{35}$$

or

$$||\boldsymbol{X}|| - ||\boldsymbol{X}_0|| \leqslant ||\boldsymbol{X} - \boldsymbol{X}_0|| \leqslant \eta^*(1) \leqslant a$$

$$||\boldsymbol{X}|| \leqslant ||\boldsymbol{X}_0|| + \eta^*(\xi = 1) \tag{36a}$$

$$||\boldsymbol{X}|| \leqslant ||\boldsymbol{X}_0|| + a. \tag{36b}$$

So the solution $\boldsymbol{X}$ is bounded and has Lagrange stability. □

3. Definition and analysis of the practical stability

The method described in the previous section can be used to analyze practical stability of the dynamical system with nonlinear perturbation (1). The following definition of practical stability can be accepted. For a given sets Q and Q^0:

$$Q\{\boldsymbol{X} : ||\boldsymbol{X}|| \leqslant R_Q\}, \quad Q^0 \subseteq Q, \tag{37}$$

where Q is the set containing $\boldsymbol{X}$ and Q^0 is a subset containing the initial values of $\boldsymbol{X}$: $\boldsymbol{X}(0) \in Q^0$, the dynamical system is practically stable if $\boldsymbol{X} \in Q$ for $t \in [0, T]$, where T can be finite or $T \to \infty$. The definition of practical stability is illustrated graphically in Fig. 1.

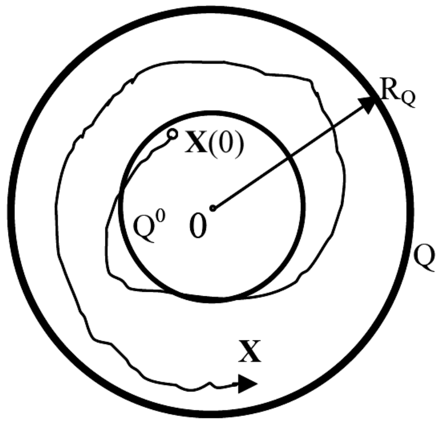

Figure 1. Graphical illustration of the practical stability.

The investigation of Lagrange and practical stability is much easier in the case of zero initial conditions; especially the calculation of the norm of the inverse operator Γ_{01}. For this purpose the following substitution can be done:

$$\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{X}(0). \tag{38}$$

After replacement in (1) one obtains:

$$\frac{d\boldsymbol{Y}}{dt} + \boldsymbol{A}(P)\boldsymbol{Y} + \boldsymbol{F}(t,P) + \boldsymbol{A}(P)\boldsymbol{X}(0) - \varphi\left[\boldsymbol{Y}, \frac{\boldsymbol{Y}}{dt}, \boldsymbol{X}(0), P\right] = 0, \tag{39}$$

where the initial conditions for $\boldsymbol{Y}$ are zero and $\boldsymbol{A}(P)\boldsymbol{X}(0)$ is constant and can be considered as an input signal. The element of the solution $\boldsymbol{X}_0$ is defined in this case by the input signals $\boldsymbol{F}(t,P)$ and $\boldsymbol{A}(P)X(0)$.

Example 1

The equation:

$$\frac{1}{3}\frac{d^2x}{dt^2} + \frac{5}{3}\frac{dx}{dt} + 2.2\sin(25^0 + x) = M, \quad M \in [1.1,\ 2.4] \tag{40}$$

can be represented in the following way, which reduces the norms of the perturbation:

$$\frac{1}{3}\frac{d^2x}{dt^2} + \frac{5}{3}\frac{dx}{dt} + 2x + \mu\{2.2\sin(25^0 + x) - 2x - 0.931\} = M - 0.931. \tag{41}$$

The described method will be applied directly to (41) without transforming it into a system of two equations of first order.

The operators C_{pq} and their norms for $M = 1.1$ are:

$$\begin{aligned} C_{10} &= 2.2\sin(25^0 + x_0) - 2x_0 - 0.931, \\ C_{11} &= 2.2\cos(25^0) + x_0)H - 2H, \\ C_{1q} &= 2.2\sin\left(25^0 + x_0 + q\frac{\pi}{2}\right) H^q \end{aligned} \tag{42}$$

$$C_{01}(H) = \frac{1}{3}\frac{d^2H}{dt^2} + \frac{5}{3}\frac{dH}{dt} + 2H \tag{43}$$

$$g(t-\tau) = 3\{\exp[-2(t-\tau)] - \exp[-3(t-\tau)]\} \tag{44}$$

For $\mu = 0$ and $x = x_0$:

$$\begin{aligned} &x_0(t) = \int_0^t g(t-\tau)M d\tau = \frac{1}{2}(M - 0.931)[1 + 2\exp(-3t) - 3\exp(-2t)], \\ &||x_0(t)|| = \sup_{0\leqslant t\leqslant\infty, M=1.1} |x_0(t)| = 0.0845 \end{aligned} \tag{45}$$

$$||C_{10}|| = \sup_{0\leqslant t\leqslant\infty, M=1.1} \left|2.2\sin(25^0 + x_0) - 2x_0 - 0.931\right| = 0.079 \tag{46}$$

$$\begin{aligned} &||C_{11}|| = \sup_{0\leqslant t\leqslant\infty, M=1.1} ||C_{11}(H)|| \leqslant \sup_{0\leqslant t\leqslant\infty, M=1.1} \left|2.2\sin(25^0 + x_0) - 2\right| \, ||H||, \\ &||C_{11}|| = 0.0915 \end{aligned} \tag{47}$$

$$||C_{1q}|| = \sup_{0\leqslant t\leqslant\infty, M=1.1} \left|2.2\cos\left(25^0 + x_0 + q\frac{\pi}{2}\right)\right| = 1.906 \tag{48}$$

$$\begin{aligned} &||\Gamma^{-1}y|| = \sup_{0\leqslant t\leqslant\infty} \left|\int_0^t g(t-\tau)y(\tau)d\tau\right| \leqslant \sup_{0\leqslant t\leqslant\infty} |y(t)| \sup_{0\leqslant t\leqslant\infty} \int_0^t |g(t-\tau)|d\tau = \\ &\qquad ||y|| \sup_{0\leqslant t\leqslant\infty} \{0.5(1 + 2\exp(-3t) - 3\exp(-2t))\} \end{aligned} \tag{49}$$

In this case:

$$||\Gamma^{-1}|| = \sup_{||y||=1} ||\Gamma^{-1}y|| = 0.5 \tag{50}$$

According to (16) the following mojorant equation is constructed:

$$\eta = v_{10}\xi + v_{11}\xi\eta + \sum_{q=2}^{\infty} v_{1q}\xi\eta^q \tag{51}$$

and after applying the formula of geometric progression:

$$\eta = v_{10}\xi + v_{11}\xi\eta + \frac{v_{1q}\eta^2\xi}{1-\eta} \tag{52}$$

where

$$v_{10} = 0.0395, \quad v_{11} = 0.0458, \quad v_{1q} = 0.953 \tag{53}$$

are calculated according to (15).

Equation (52) is quadratic with two roots for $\xi = 1$:

$$\eta_1(\xi = 1) = 0.478 \quad \text{and} \quad \eta_2(\xi = 1) = 0.043. \tag{54}$$

However, only η_2 satisfies the condition (20):

$$\eta_2(\xi = 0) = 0$$

Hence according to (21), (45), (54):

$$||x|| \leqslant ||x_0|| + \eta^*(\xi = 1) = 0.0845 + 0.043 = 0.1275 \text{ rad.} = 7.3^0.$$

Obviously the stationary solution of (40) is 5^0 $(2.2\sin 30^0 = 1.1)$. The present criterion, however, gives the sufficient conditions applying majorant estimation of the norms and as the result, obtained bound of the solution is higher than the real one.

The criterion can be applied also in the case when $M = 2.4$. Then $v_{10} = 0.1806$, $v_{11} = 0.572$, $v_{1q} = 1$ and in this case the majorant equation has no real solution for $\xi = 1$, which means that $||x||$ is not bounded and the equation is not stable.

The above cases are calculated numerically and the results are presented in Table 1. It can be seen, that the process is not stable for $M = 2.4$ and that for $M = 1.1$ the maximal difference between x and x_0 is 0.027 for finite t and for $t = \infty$ (calculated for the stationary regimes), which is less than $\eta^*(\xi = 1) = 0.043$.

Example 2

Let consider one dimensional nonlinear system of second order:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x - \mu\varphi\left(x, \frac{dx}{dt}\right) = 0.76, \quad \mu = 1, \mu_0 = 0. \tag{55}$$

For the linear part of (55) one can write:

Table 1. Numerical solution of (40) for $M = 1.1$ and $M = 2.4$

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t, s	0.5	1	2	5	8	10	$t = \infty$
x_0, $M = 1.1$	0.0137	0.0268	0.0463	0.0733	0.0812	0.0830	0.0845
x, $M = 1.1$	0.0138	0.0271	0.0469	0.0750	0.0836	0.0857	0.0872
x, $M = 2.4$	0.119	0.235	0.414	0.733	0.917	1.01	∞

$$g(t-\tau) = e^{-(t-\tau)} - e^{-2(t-\tau)}, \tag{56}$$

$$x_0(t) = 0.38\left(e^{-2t} - 2e^{-t} + 1\right), \tag{57}$$

$$\frac{dx_0(t)}{dt} = 0.38\left(-2e^{-2t} + 2e^{-t}\right), \tag{58}$$

$$C_{01}(H) = \frac{d^2H}{dt^2} + 3\frac{dH}{dt} + 2H. \tag{59}$$

For the nonlinear part of (55) the following variant will be considered:

$$\varphi = P\left\{\frac{dx}{dt}\right\}^2. \tag{60}$$

The Lagrange stability of this variant will be analyzed by construction of majorant equation and numerically verified. The corresponding operators are as follows:

$$C_{10} = P\left\{\frac{dx}{dt}\right\}^2, \tag{61}$$

$$C_{11}(H) = -2P\frac{dx_0}{dt}\frac{dH}{dt}, \quad x = x_0 + cH, \tag{62}$$

$$C_{12}(H) = -\frac{1}{2}P\left\{\frac{\partial^2 x_0}{\partial c^2}\right\}_{c=0} = -P\left\{\frac{dH}{dt}\right\}^2, \tag{63}$$

$$C_{pq} = 0 \quad \text{for} \quad p > 1 \ \text{ or } \ q > 2, \quad C_{02} = 0. \tag{64}$$

To calculate norms, Banach spaces are used with norms [12]:

$$||y|| = \sup_{0\leqslant t\leqslant\infty} |y(t)|, \quad ||y|| = \sum_{k=0}^{N} \sup_{0\leqslant t\leqslant\infty} |y^k(t)|. \tag{65}$$

Table 2. Numerical solution of (55).

t, s	1	2	3	10
x_0	0.1510	0.2840	0.3430	0.3799
x (Runge-Kutta)	0.1569	0.2936	0.3493	0.3799

The majorant assessment of the operators' norms can be made in the following way:

$$||C_{10}|| = \sup_{0\leqslant t\leqslant\infty} \left|P\left\{0.38\left(-2e^{-2t}+2e^{-t}\right)\right\}^2\right| = P0.19^2, \tag{66}$$

$$||C_{11}(H)|| = \sup_{0\leqslant t\leqslant\infty} \left|-2P\frac{dx_0}{dt}\frac{dH}{dt}\right| = 0.38P||H||, \quad ||C_{11}(H)|| = 0.38P, \tag{67}$$

$$||C_{12}(H)|| = \sup_{0\leqslant t\leqslant\infty,\ ||H||=1} \left|P\left\{\frac{dH}{dt}\right\}^2\right| = P, \tag{68}$$

$$||C_{01}^{-1}|| \leqslant \left|\left|\int_0^t |g(t-\tau)|d\tau\right|\right| < 1, \tag{69}$$

$$||x_0(t)|| = 0.38, \tag{70}$$

$$v_{10} = 0.036P, \quad v_{11} = 0.38P, \quad v_{12} = P. \tag{71}$$

The majorant equation is:

$$\eta = 0.036P\xi + 0.38P\xi\eta + P\xi\eta^2. \tag{72}$$

This quadratic equation has for $\xi = 1$ and $P = 1$ two real solutions. The second solution for $\xi = 1$ is: $\eta_2(\xi = 1) = 0.0655$ and according to (21), (70):

$$||x|| \leqslant ||x_0|| + \eta^*(\xi = 1) = 0.38 + 0.0655 = 0.4455.$$

In Table 2 the numerical solution of (55) for $P = 1$ is given. It confirms the result achieved from the majorant equation (72), that the norm of the solution of x is bounded by the value 0.4455, i.e. it is Lagrange stable. Of course, this value is greater than the real values of the process, because of the majorant estimation of the norms.

4. Determination and verification of the stability bounds for a class of nonlinear perturbations

For some classes of nonlinear perturbations the stability bounds, calculated during the stability analysis or obtained on the basis of physical laws can be verified. Let us consider the following class of dynamical systems:

$$Lx = X(x,t) + P(x,t) \tag{73}$$

where L is linear differential operator of n-th order:

$$L = a_n(t)\frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_0(t). \tag{74}$$

$X(x,t)$ and $P(x,t)$ are nonlinear continuous functions and $P(x,t)$ should be considered as perturbation. If $\boldsymbol{x}$, $\boldsymbol{X}(\boldsymbol{x},t)$ and $\boldsymbol{P}(\boldsymbol{x},t)$ are vectors, the same problem can be described with a set of differential equation of the first order with linear left part:

$$\dot{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x},t) + \boldsymbol{P}(\boldsymbol{x},t) \tag{75}$$

where $\boldsymbol{A}$ is the matrix:

$$\boldsymbol{A} = [a_{ik}(t)], \quad i,k = 1,2,\ldots n.$$

Further the scalar variant will be considered. If the initial conditions of the derivatives of x are zero, then equation (3) can be used with substitution (38), if necessary. If the initial conditions of the derivatives of x are not zero, then equation (75) and substitution (38) have to be applied.

To determine and verify the stability bounds, the following theorem for practical stability can be used.

Theorem 2 : *Let us consider L, $X(x,t)$, $P(x,t)$ defined in the Banach space E_1, of which x is element and that the values of L, $X(x,t)$, $P(x,t)$ are elements of the Banach space E_2. The sufficient condition for practical stability of* (3) *for $t \in [0,T]$ when $x(0) \in Q^0$* (Fig.1) *and $||P(x,t)|| \leqslant \delta$ is:*

$$||L^{-1}||\,\{||X(x,t)||_{x\in Q} + \delta\} + ||x(0)|| \leqslant R_Q \tag{76}$$

where L^{-1} is operator inverse to L for zero initial conditions.

Proof To make the initial conditions zero let us use the substitution:

$$\begin{aligned} x &= z + x(0), \\ Lx &= Lz + Lx(0) = Lz. \end{aligned} \tag{77}$$

The initial conditions for z are zero. Then equation (3) takes the form:

$$Lz = X(x,t) + P(x,t) \tag{78}$$

and

$$z = L^{-1}\{X(x,t) + P(x,t)\} \tag{79}$$

$$||z|| \leqslant ||L^{-1}||\{||X(x,t)|| + ||P(x,t)||\}. \tag{80}$$

The norm of z can be obtained from the equation (77):

$$x - x_0 = z, \quad ||x|| - ||x(0)|| \leqslant ||z||. \tag{81}$$

It follows from (80) and (81):

$$||x|| \leqslant ||L^{-1}||\{||X(x,t)||_{x\in Q} + \delta\} + ||x(0)||. \tag{82}$$

According to the definition of practical stability, $x \in Q$ if the process is stable, i.e.

$$||x|| \leqslant R_Q. \tag{83}$$

Inequality (83) can be used to calculate the norm || X(x,t) || for a given time interval $t \in [0,T]$, because if it is not true, then the calculated from the left part of the inequality (76) R_Q will be greater than the value used for calculating the norms. Hence from (82) and (83) the criterion (76) is obtained. Thus the theorem giving sufficient conditions for practical stability is proven.

The criterion (76) can be used in two ways:

1. Considering (76) as equation R_Q can be calculated as a function of δ, $||x(0)||$, time interval $t \in [0,T]$ and other parameters.

2. The left part of (76) can be calculated using a required or expected value of R_Q, and if the inequality (76) in this case is true, this means that the suggested value of R_Q is the bound of $||x||$. Of course it is not the lowest bound of $||x||$, because the majorant estimation of the norms.

Example 3

Illustration of the last criterion can be shown with equation (41):

$$\frac{1}{3}\frac{d^2x}{dt^2} + \frac{5}{3}\frac{dx}{dt} + 2x + \{2.2\sin(20^0 + x) - 2x\} = 0.931 + M_k(t). \tag{84}$$

In this case:

$$X(x,t) = -2.2\sin(20^0 + x) + 2x + 0.931,$$

$$||P(x,t)|| = ||M_k(t)|| = 0.169, \quad x(0) = 5^0 = 0.087 \text{ rad}.$$

The norms of the operators can be estimated as follows:

$$||X(x,t)|| = \sup_{x \in Q} |-2.2\sin(20^0 + x) + 2x + 0.931| =$$

$$= |-2.2\sin(20^0 + R_Q) + 2R_Q + 0.931| = -2.2\sin(20^0 + R_Q) + 2R_Q + 0.931.$$

It follows from (50)

$$||\Gamma^{-1}|| = ||L^{-1}|| = 0.5.$$

Then the criterion (76) will be:

$$0.5\{-2.2\sin(20^0 + R_Q) + 2R_Q + 0.931 + \delta\} + 5^0 \leqslant R_Q. \tag{85}$$

This inequality gives the sufficient conditions for practical stability which depend on the values of R_Q, δ, and $x(0)$. One of these three values can be found if the other two are given or the condition (76) for stability can be checked if all three of them are given. When $\delta = 0.169$, $x(0) = 5^0$ one can be calculated that $R_Q = 14^0$ 50min:

$$1.1 - 2.2\sin(20^0 + R_Q) + 2x(0) \leqslant 0,$$
$$1.27 \leqslant 2.2\sin(20^0 + R_Q),$$
$$0.57 \leqslant \sin(20^0 + R_Q),$$
$$R_Q \geqslant 14^0 \text{ 50min}.$$

For given $x(0)$ and δ the value of R_Q determined from (76) is greater than the real value of R_Q because it includes majorant norms. It is easy to find that the stationary value of R_Q is 10^0. On the other hand we can verify a suggestion that $R_Q = 5^0$, when $x(0) = 0$. For this purpose the left part of inequality (76) is calculated applying $x(0) = 0$:

$$0.5\{-2.2\sin(20^0 + 5^0) + 2.5^0 + 0.931 + 0.169\} = 0.5\{-2.2(0.4226) + 2\cdot 0.087 + 1.1\} =$$
$$= 0.17215 \text{ rad.} = 9.86^0 > 5^0.$$

Obviously the suggestion $R_Q = 5^0$ is not true.

5. Conclusions

1. Criteria suggested in this paper give robust Lagrange and practical stability bounds for dynamical systems with nonlinear perturbations.
2. Nonlinear perturbations can include derivatives of the variables.
3. Analytical results obtained from the criteria are compared with numerical calculations, applying computer modeling.

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