Observer-based fault estimation
for linear systems with distributed time delay

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The paper is engaged with the framework of designing adaptive fault estimation for linear continuous-time systems with distributed time delay. The Lyapunov-Krasovskii functional principle is enforced by imposing the integral partitioning method and a new equivalent delay-dependent design condition for observer-based assessment of faults are established in terms of linear matrix inequalities. Asymptotic stability conditions are derived and regarded with respect to the incidence of structured matrix variables in the linear matrix inequality formulation. Simulation results illustrate the design approach, and demonstrates power and performance of the actuator fault assessment.

Key words: adaptive fault estimation, distributed time delay systems, Lyapunov-Krasovskii functional, integral partitioning technique, time delay segmentation, linear matrix inequalities

1. Introduction

The use of Lyapunov method for stability analysis of the time delay systems has been ever growing subject of interest, starting with the pioneering works of Krasovskii [14], [15]. Usually nowadays, for the stability issue, different kind of modified Lyapunov-Krasovskii functionals are used to obtain delay-dependent stabilization. The results based on these functionals are applied to controllers synthesis, as well as to state observers design. Much research was done, and different stability criteria were derived for systems with time-delays in state variables (e.g. [7], [13], [17], [20]), especially formulated with respect to linear matrix inequality (LMI) principles. Some progress review in this research arrea can be found, e.g., in [19], [24], [26].

Systems with distributed time delays are applied e.g. in the modeling of combustion chambers rocket motor with pressure feeding [4], [25]. Because of the importance

of such systems, growing attention is devoted to studying distributed delay systems in recent years. Reflecting the fact that standard time-delay control representation schemes are not complete related to systems with distributed time delays, some alternative stability conditions are derived [10], [12], [21]. The readers are referred to [3], and the reference therein, for recent reports about the stability analysis of these systems.

The presented approach to observer-based adaptive actuator fault estimation exploits an extended family of Lyapunov-Krasovskii functionals, established by introducing triple integral terms [22], and the stability conditions are formulated with respect to the use of structured matrix variables and integral partitioning method [8]. Exploiting the duality principle to control law parameters design conditions [2], [5], and considering the standard approaches to observer-based fault estimation for the linear systems with discrete time-delay [1], [11], [24], the author's results, presented in [6], are improved and generalized with respect to the linear systems with distributed time delay. Because of using Lyapunov-Krasovskii functional, only sufficient conditions in terms of LMIs for estimator stability are obtained.

The outline of this paper is as follows. Section 2 introduces the model of continuous-time linear multi-input, multi-output (MIMO) systems with distributed time delays, as well as the proposed principle for the actuator or component faults assessment, and in Section 3 the preliminary results are presented. According to the system model, Section 4 discusses and proves the performance of the separation principle for systems with distributed time delays and preserves the desired properties for fault estimation. In accordance to the observer-based fault estimation asymptotic stability, the LMI form of design conditions is introduced in Section 5, specifically formulated to respect the structural LMI variables implementation in LMI solvers. A numerical example is given in Section 6, to illustrate basic properties of the proposed method, and Section 7 presents some concluding remarks.

Throughout the paper, the following notations are used: $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $\mathrm{diag}[\cdot]$ denotes a block diagonal matrix, for a square matrix $\boldsymbol{X} > 0$ (respectively $\boldsymbol{X} < 0$) means that $\boldsymbol{X}$ is a symmetric positive definite matrix (respectively, negative definite matrix), the symbol $\boldsymbol{I}_n$ represents the n-th order unit matrix, $\mathbb{N}$ denotes the set of integers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}^{n \times r}$ the set of all $n \times r$ real matrices.

2. Problem formulation

The systems under consideration are MIMO dynamic systems with distributed time delay. Without lose of generalization, in the state-space family, this class of systems is represented by the set of equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{A}_h \int_{t-h}^{t} \boldsymbol{q}(s)\mathrm{d}s + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{f}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{2}$$

while initial conditions are

$$\boldsymbol{q}(\theta) = \varphi(\theta), \quad \forall \theta \in \left\langle -\left(h + \frac{h}{m}\right), 0\right\rangle \tag{3}$$

where $0 < h \leqslant h_m$, $h \in \mathbb{R}$ is a value describing the magnitude of time-delay, $m \leqslant m_m$, $m \in \mathbb{N}$ is a partitioning factor, $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^r$, and $\boldsymbol{y}(t) \in \mathbb{R}^p$ are vectors of the system, input and output variables, respectively, $\boldsymbol{f}(t) \in \mathbb{R}^s$ is the unknown fault vector and matrices $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\boldsymbol{A}_h \in \mathbb{R}^{n \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n \times r}$, $\boldsymbol{C} \in \mathbb{R}^{p \times n}$, and $\boldsymbol{E} \in \mathbb{R}^{n \times s}$ are real matrices.

For the system (1)-(3), the following assumptions are considered:

  i The value h of the distributed delay is known and constant.

 ii The couple $(\boldsymbol{A}, \boldsymbol{B})$ is controllable and (1) is stabilized by the linear memoryless state feedback control

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}(t), \quad \boldsymbol{K} \in \mathbb{R}^{r \times n} \tag{4}$$

in such way that the faulty-free closed-loop system

$$\dot{\boldsymbol{q}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{q}(t) + \boldsymbol{A}_h \int_{t-h}^{t} \boldsymbol{q}(s)\mathrm{d}s \tag{5}$$

is asymptotically stable for given h and any initial state.

iii The unknown fault vector, changing unexpectedly when a fault occurs, is differentiable and bounded, i.e., $|\boldsymbol{f}(t)| < \boldsymbol{f}$, $\boldsymbol{f}$ is known, and the value of $\boldsymbol{f}(t)$ is set to zero until a fault occurs.

Note, design of the control algorithm (4) is not a subject of the paper and can be found, e.g., in [5].

To estimate faults (actuator faults or system component faults), the adaptive state estimator is proposed

$$\dot{\boldsymbol{q}}_e(t) = \boldsymbol{A}\boldsymbol{q}_e(t) + \boldsymbol{J}(\boldsymbol{y}(t) - \boldsymbol{y}_e(t)) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{A}_h \int_{t-h}^{t} \boldsymbol{q}_e(s)\mathrm{d}s + \boldsymbol{E}\boldsymbol{f}_e(t) \tag{6}$$

$$\boldsymbol{y}_e(t) = \boldsymbol{C}\boldsymbol{q}_e(t) \tag{7}$$

where $\boldsymbol{q}_e(t) \in \mathbb{R}^n$ is the estimator state vector, $\boldsymbol{y}_e(t) \in \mathbb{R}^p$ is the observed system output vector, $\boldsymbol{f}_e(t)$ is an estimate of $\boldsymbol{f}(t)$ and $\boldsymbol{J} \in \mathbb{R}^{n \times p}$ is the estimator gain matrix. The state estimator (6), (7), is combined with the law for the fault estimate updating of the form

$$\dot{\boldsymbol{f}}_e(t) = \boldsymbol{G}\boldsymbol{H}^T \boldsymbol{e}_y(t) \tag{8}$$

where $\boldsymbol{H} \in \mathbb{R}^{p \times s}$ is the law gain matrix and $\boldsymbol{G} = \boldsymbol{G}^T > 0$, $\boldsymbol{G} \in \mathbb{R}^{s \times s}$ is a learning weight matrix, being setting interactively.

The matrix parameters of the above given estimators have to be designed in such way that ensures asymptotic convergence to zero of the estimation errors

$$\boldsymbol{e}_f(t) = \boldsymbol{f}(t) - \boldsymbol{f}_e(t), \qquad \boldsymbol{e}_y(t) = \boldsymbol{y}(t) - \boldsymbol{y}_e(t). \tag{9}$$

Moreover, assumption iii) implies that the derivative $\boldsymbol{e}_f(t)$ with respect to time can be considered as

$$\dot{\boldsymbol{e}}_f(t) = -\dot{\boldsymbol{f}}_e(t) \tag{10}$$

while the initial conditions can be considered as $\boldsymbol{q}_e(\theta) = \boldsymbol{q}(\theta)$ for θ from (3), since the selection of $\boldsymbol{J}$ should ensure that the state estimator is stable, and so the influence of the initial error decreases to zero asymptotically. Such adaptation law (8) ensures $|\boldsymbol{f}_e(t)| \leqslant \boldsymbol{f}$ for all t if $|\boldsymbol{f}_e(\vartheta)| \leqslant \boldsymbol{f}$.

Based on (8), the main goal is to design the delay-dependent stability criterion of observer-based fault estimation for distributed time-delay systems, using an integral delay partitioning in a Lyapunov-Krasovskii functional.

3. Preliminary results

Assumption 1 *The couple $(\boldsymbol{A}, \boldsymbol{C})$ is observable.*

Proposition 1 *If $\boldsymbol{N}$ is a positive definite symmetric matrix, and $\boldsymbol{M}$ is a square matrix of the same dimension then*

$$\boldsymbol{M}^{-T} \boldsymbol{N} \boldsymbol{M}^{-1} \geqslant \boldsymbol{M}^{-1} + \boldsymbol{M}^{-T} - \boldsymbol{N}^{-1}. \tag{11}$$

Proof See, e.g., [6]

Proposition 2 *(Schur complement lemma) Let $\boldsymbol{S}$, $\boldsymbol{Q} = \boldsymbol{Q}^T$, $\boldsymbol{R} = \boldsymbol{R}^T$, $\det \boldsymbol{R} \neq 0$ are real matrices of appropriate dimensions, then the next inequalities are equivalent*

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} > 0 \Leftrightarrow \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T > 0, \ \boldsymbol{R} > 0. \tag{12}$$

Proof See, e.g., [16].

The Schur complement lemma is needed in the observer design to deduce an LMI feasible problem.

Proposition 3 *(Jensen's inequalities) Let $f(\boldsymbol{x}(p))$, $\boldsymbol{x}(p) \in \mathbb{R}^n$, $\boldsymbol{X} = \boldsymbol{X}^T > 0$, $\boldsymbol{X} \in \mathbb{R}^{n \times n}$ is a real positive and integrable vector function of the form*

$$f(\boldsymbol{x}(p)) = \boldsymbol{x}^T(p) \boldsymbol{X} \boldsymbol{x}(p) \tag{13}$$

such that there exist well defined integrations

$$\int_{-ct+r}^{0}\int^{t} f(\boldsymbol{x}(p))dp\,dr > 0 \tag{14}$$

$$\int_{t-c}^{t} f(\boldsymbol{x}(p))dp > 0 \tag{15}$$

for $c > 0$, $c \in \mathbb{R}$, $t \in \langle 0, \infty)$, then

$$\int_{-ct+r}^{0}\int^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)dp\,dr \geqslant \frac{2}{c^2}\int_{-ct+r}^{0}\int^{t} \boldsymbol{x}^T(p)dp\,dr\,\boldsymbol{X}\int_{-ct+r}^{0}\int^{t} \boldsymbol{x}(p)dp\,dr \tag{16}$$

$$\int_{t-c}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)dp \geqslant \frac{1}{c}\int_{t-c}^{t} \boldsymbol{x}^T(p)dp\,\boldsymbol{X}\int_{t-c}^{t} \boldsymbol{x}(p)dp \tag{17}$$

respectively.

Proof See, e.g., [5], [9].

The integral inequalities, given in Proposition 3, are of crucial significance for the observer-based adaptive actuator fault estimation stability analysis, which exploits a Lyapunov-Krasovskii functional with double and triple integral terms, and will be used in the proof of the main results in the paper. Note that Lyapunov-Krasovskii functional with at least double integral terms has to be used to obtain the delay-dependent design condition.

4. Separation principle

The state estimator should estimate system states based on the available input and output information and preserve desired properties specified at the fault estimation principle design stage.

Theorem 1 *If no fault is occurred, the interconnection of the controlled system* (1), (2) *and the observer* (6), (7) *is asymptotically stable only if each of these connected parts is asymptotically stable.*

Proof The plant (1), (2) and the observer (6), (7) can be written compactly as

$$\begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \dot{\boldsymbol{q}}_e(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{JC} & \boldsymbol{A}-\boldsymbol{JC} \end{bmatrix}\begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{q}_e(t) \end{bmatrix} + $$
$$+ \int_{t-h}^{t}\begin{bmatrix} \boldsymbol{A}_h & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_h \end{bmatrix}\begin{bmatrix} \boldsymbol{q}(r) \\ \boldsymbol{q}_e(r) \end{bmatrix}\mathrm{d}r + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B} \end{bmatrix}\boldsymbol{u}(t) + \begin{bmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix}\begin{bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{f}_e(t) \end{bmatrix} \tag{18}$$

$$\dot{q}^\circ(t) = A^\circ q^\circ(t) + A_h^\circ \int_{t-h}^{t} q^\circ(s)\mathrm{d}s + B^\circ u(t) + E^\circ f^\circ(t) \tag{19}$$

where

$$q^{\circ T}(t) = \begin{bmatrix} q^T(t) & q_e^T(t) \end{bmatrix}, \qquad f^{\circ T}(t) = \begin{bmatrix} f^T(t) & f_e^T(t) \end{bmatrix} \tag{20}$$

$$A^\circ = \begin{bmatrix} A & 0 \\ JC & A - JC \end{bmatrix}, \qquad A_h^\circ = \begin{bmatrix} A_h & 0 \\ 0 & A_h \end{bmatrix} \tag{21}$$

$$B^\circ = \begin{bmatrix} B \\ B \end{bmatrix}, \qquad E^\circ = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}. \tag{22}$$

Instead of writing a differential equation governing $q(t)$ and $q_e(t)$, the extended system behavior can be described using $q(t)$ and the equations for the error vectors

$$e_q(t) = q(t) - q_e(t), \qquad e_f(t) = f(t) - f_e(t). \tag{23}$$

Thus, to perform the coordinate change, the congruence transform matrix T_c can be defined, with respect to (23), as

$$T_c = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}, \qquad T_c^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \tag{24}$$

where, evidently, it yields

$$T_c q^\circ(t) = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} q(t) \\ q_e(t) \end{bmatrix} = \begin{bmatrix} q(t) \\ e_q(t) \end{bmatrix} = q^\bullet(t) \tag{25}$$

$$T_c q^\circ(t) = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} f(t) \\ f_e(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ e_f(t) \end{bmatrix} = f^\bullet(t) \tag{26}$$

and

$$q^{\bullet T}(t) = \begin{bmatrix} q^T(t) & e_q^T(t) \end{bmatrix}, \qquad f^{\bullet T}(t) = \begin{bmatrix} f^T(t) & e_f^T(t) \end{bmatrix}. \tag{27}$$

Thus, multiplying the left-hand site of (19) by T_c, it is obtained

$$T_c \dot{q}^\circ(t) = T_c A^\circ T_c^{-1} T_c q^\circ(t) + T_c B^\circ u(t) + T_c A_h^\circ \int_{t-h}^{t} q^\circ(s)\mathrm{d}s + T_c E^\circ f^\circ(t) \tag{28}$$

$$\dot{q}^\bullet(t) = A^\bullet q^\bullet(t) + A_h^\circ \int_{t-h}^{t} q^\bullet(s)\mathrm{d}s + B^\bullet u(t) + E^\circ f^\bullet(t) \tag{29}$$

respectively, where a very elementary calculation gives

$$
\boldsymbol{T}_c \boldsymbol{A}_h^{\circ} \boldsymbol{q}^{\circ}(t) = \begin{bmatrix} \boldsymbol{A}_h & \boldsymbol{0} \\ \boldsymbol{A}_h & -\boldsymbol{A}_h \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{q}_e(t) \end{bmatrix} = \boldsymbol{A}_h^{\circ} \boldsymbol{e}^{\bullet}(t) \tag{30}
$$

$$
\boldsymbol{T}_c \boldsymbol{E}_h^{\circ} \boldsymbol{f}^{\circ}(t) = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{E} & -\boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{f}_e(t) \end{bmatrix} = \boldsymbol{E}^{\circ} \boldsymbol{f}^{\bullet}(t) \tag{31}
$$

and

$$
\boldsymbol{A}^{\bullet} = \boldsymbol{T}_c \boldsymbol{A}^{\circ} \boldsymbol{T}_c^{-1} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{J}\boldsymbol{C} \end{bmatrix}, \quad \boldsymbol{B}^{\bullet} = \boldsymbol{T}_c \boldsymbol{B}^{\circ} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix}. \tag{32}
$$

Note, (29) is an alternative description of the same connected system and contain the same information as (18), since $\boldsymbol{q}(t)$, $\boldsymbol{f}(t)$ and $\boldsymbol{e}_q(t)$, $\boldsymbol{e}_f(t)$ uniquely determine $\boldsymbol{q}_e(t)$, $\boldsymbol{f}_e(t)$, respectively.

Inserting the control law (4), the open form of (29) is

$$
\begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \dot{\boldsymbol{e}}_q(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{J}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{e}_q(t) \end{bmatrix} +
$$
$$
+ \int_{t-h}^{t} \begin{bmatrix} \boldsymbol{A}_h & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_h \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(r) \\ \boldsymbol{e}_q(r) \end{bmatrix} \mathrm{d}r + \begin{bmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{e}_f(t) \end{bmatrix}. \tag{33}
$$

Because of the block-diagonal structure of the connected system (33), the separation property holds, i.e., the gain $\boldsymbol{K}$ of the state-feedback control and the state estimator gains $\boldsymbol{J}$ for the state observer, can be designed independently for the fault-free system. This concludes the proof.
□

This result deserves an important remark. Evidently, combined with the law to update the fault estimation, the matrix $\boldsymbol{H}$ may be designed only if any additional restriction is established in the stability conditions.

5. Estimator parameter design

The results of the previous sections can be exploited to investigate the stabilizability problem in the estimator design for systems with distributed time delays, subject to the fault estimate updating constraint. It is shown that the corresponding delay-dependent problem, defined in the sense of quadratic stability, can be formulated using structured variables and explained in the terms of LMIs, conditioned by an matrix equality constraint.

Theorem 2 *The estimator system matrix is stable if for given $h > 0$, $m > 1$ exist symmetric positive definite matrices $\boldsymbol{P}, \boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{n \times n}$, $\boldsymbol{W} \in \mathbb{R}^{mn \times mn}$, matrices $\boldsymbol{H} \in \mathbb{R}^{p \times s}$, $\boldsymbol{Y} \in \mathbb{R}^{n \times p}$ and a positive scalar $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \quad \varepsilon > 0, \quad \boldsymbol{U} = \boldsymbol{U}^T > 0, \quad \boldsymbol{V} = \boldsymbol{V}^T > 0, \quad \boldsymbol{W} = \boldsymbol{W}^T > 0 \quad (34)$$

$$\begin{bmatrix} \boldsymbol{\Delta}_{11} & * \\ \boldsymbol{Y}^{\circ}\boldsymbol{T}_A & -2b^{-2}\boldsymbol{P} + b^{-2}\boldsymbol{V} \end{bmatrix} < 0 \quad (35)$$

$$\boldsymbol{E}^T \boldsymbol{P} \boldsymbol{E} - \boldsymbol{E}^T \boldsymbol{C}^T \boldsymbol{H} = 0 \quad (36)$$

with

$$\boldsymbol{\Delta}_{11} = \boldsymbol{T}_I^T \boldsymbol{Y}^{\circ} \boldsymbol{T}_A + \boldsymbol{T}_A^T \boldsymbol{Y}^{\circ T} \boldsymbol{T}_I - \boldsymbol{T}_V^T \boldsymbol{V} \boldsymbol{T}_V + \boldsymbol{T}_U^T \boldsymbol{U}^{\circ} \boldsymbol{T}_U + \boldsymbol{T}_W^T \boldsymbol{W}^{\circ} \boldsymbol{T}_W \quad (37)$$

$$\boldsymbol{T}_V = b^{-1} \begin{bmatrix} a^2 \boldsymbol{I}_n & -\boldsymbol{I}_n & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \quad (38)$$

$$\boldsymbol{T}_U = \begin{bmatrix} a\boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & a^{-1}\boldsymbol{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_n & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \quad (39)$$

$$\boldsymbol{T}_W = \begin{bmatrix} \boldsymbol{0}_w & \begin{bmatrix} \boldsymbol{I}_{mn} & \boldsymbol{0}_w \\ \boldsymbol{0}_w & \boldsymbol{I}_{mn} \end{bmatrix} \end{bmatrix} \quad (40)$$

$$a = \sqrt{\frac{h}{m}}, \qquad b = \frac{h}{\sqrt{2}m} \quad (41)$$

$$\boldsymbol{U}^{\circ} = \begin{bmatrix} \boldsymbol{U} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{U} \end{bmatrix}, \boldsymbol{W}^{\circ} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{W} \end{bmatrix}, \quad (42)$$

$$\boldsymbol{Y}^{\circ} = \begin{bmatrix} \begin{bmatrix} \boldsymbol{P} & \boldsymbol{Y} \end{bmatrix} & \boldsymbol{P} & \begin{bmatrix} \boldsymbol{P} & \cdots & \boldsymbol{P} \end{bmatrix} & \boldsymbol{P} \end{bmatrix} \quad (43)$$

$$\boldsymbol{T}_A = \mathrm{diag} \begin{bmatrix} \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{C} \end{bmatrix} & \boldsymbol{A}_h & \mathrm{diag} \begin{bmatrix} \boldsymbol{A}_h & \cdots & \boldsymbol{A}_h \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \quad (44)$$

$$\boldsymbol{T}_I = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \quad (45)$$

where matrices $\boldsymbol{Y}^{\circ} \in \mathbb{R}^{n \times ((m+2)n+p)}$, $\boldsymbol{W}^{\circ} \in \mathbb{R}^{2mn \times 2mn}$, and $\boldsymbol{U}^{\circ} \in \mathbb{R}^{2n \times 2n}$ are the structured matrix variables, and $\boldsymbol{T}_A \in \mathbb{R}^{((m+2)n+p) \times (m+2)n}$, $\boldsymbol{T}_U \in \mathbb{R}^{2n \times (m+2)n}$, $\boldsymbol{T}_I, \boldsymbol{T}_V \in \mathbb{R}^{n \times (m+2)n}$, $\boldsymbol{T}_W \in \mathbb{R}^{2mn \times (m+2)n}$.

When the above conditions hold, the estimator gain matrix is given by

$$\boldsymbol{J} = \boldsymbol{P}^{-1} \boldsymbol{Y} \quad (46)$$

and the adaptive fault estimation algorithm is

$$
\boldsymbol{f}_e(t) = \boldsymbol{G}\boldsymbol{H}^T\boldsymbol{C}\int_{t_f}^{t} \boldsymbol{e}_q(r)dr \tag{47}
$$

where $\boldsymbol{e}_q(t)$ is given in (23), and t_f denotes the time instant when the fault occurs.

Proof By (33), with $\boldsymbol{e}_f(t)$ defined in (23), it yields

$$
\dot{\boldsymbol{e}}_q(t) = (\boldsymbol{A}-\boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) + \boldsymbol{A}_h\int_{t-h}^{t} \boldsymbol{e}_q(s)ds + \boldsymbol{E}\boldsymbol{e}_f(t). \tag{48}
$$

Considering an additive constraint for $\boldsymbol{e}_f(t)$, the Lyapunov-Krasovskii functional is defined as follows

$$
v(\boldsymbol{e}_q(t)) = v_0(\boldsymbol{e}_q(t)) + v_1(\boldsymbol{e}_q(t)) + v_2(\boldsymbol{e}_q(t)) + v_3(\boldsymbol{e}_q(t)) \tag{49}
$$

where, with $\boldsymbol{P}=\boldsymbol{P}^T>0$, $\boldsymbol{W}=\boldsymbol{W}^T>0$, $\boldsymbol{G}=\boldsymbol{G}^T>0$, $\boldsymbol{U}=\boldsymbol{U}^T>0$, $\boldsymbol{V}=\boldsymbol{V}^T>0$, it is

$$
v_0(\boldsymbol{e}_q(t)) = \boldsymbol{e}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) \tag{50}
$$

$$
v_1(\boldsymbol{e}_q(t)) = \int_{t-\frac{h}{m}}^{t} \boldsymbol{e}_p^T(s)\boldsymbol{W}\boldsymbol{e}_p(s)ds \tag{51}
$$

$$
v_2(\boldsymbol{e}_q(t)) = \int_{-\frac{h}{m}}^{0}\int_{t+\vartheta}^{t} \boldsymbol{e}_q^T(s)\boldsymbol{U}\boldsymbol{e}_q(s)dsd\vartheta \tag{52}
$$

$$
v_3(\boldsymbol{e}_q(t)) = \int_{-\frac{h}{m}}^{0}\int_{\vartheta}^{0}\int_{t+\lambda}^{t} \dot{\boldsymbol{e}}_q^T(s)\boldsymbol{V}\dot{\boldsymbol{e}}_q(s)dsd\lambda d\vartheta \tag{53}
$$

and the integral partition is considered as follows

$$
\boldsymbol{e}_p^T(t) = \int_{t-h}^{t} \boldsymbol{e}_q^T(s)ds \sim \left[\ \int_{t-\frac{h}{m}}^{t} \boldsymbol{e}_q^T(s)ds \quad \int_{t-\frac{2h}{m}}^{t-\frac{h}{m}} \boldsymbol{e}_q^T(s)ds \quad \cdots \quad \int_{t-h}^{t-(m-1)\frac{h}{m}} \boldsymbol{e}_q^T(s)ds\ \right] \tag{54}
$$

$$
\boldsymbol{e}_p^T(s) \sim \left[\ \boldsymbol{e}_{p1}^T(s) \quad \boldsymbol{e}_{p2}^T(s)\ \right] \tag{55}
$$

$$
\boldsymbol{e}_{p1}^T(s) = \int_{t-\frac{h}{m}}^{t} \boldsymbol{e}_q^T(s)ds, \quad \boldsymbol{e}_{p2}^T(s) = \left[\ \int_{t-\frac{2h}{m}}^{t-\frac{h}{m}} \boldsymbol{e}_q^T(s)ds \quad \cdots \quad \int_{t-h}^{t-(m-1)\frac{h}{m}} \boldsymbol{e}_q^T(s)ds\ \right]. \tag{56}
$$

Hence, the derivative of $v(\boldsymbol{e}_q(t))$ with respect to t is given by

$$\dot{v}(\boldsymbol{e}_q(t)) = \dot{v}_0(\boldsymbol{e}_q(t)) + \dot{v}_1(\boldsymbol{e}_q(t)) + \dot{v}_2(\boldsymbol{e}_q(t)) + \dot{v}_3(\boldsymbol{e}_q(t)). \tag{57}$$

Taking the first element time-derivative result and assuming that

$$\dot{\boldsymbol{f}}_e(t) = \boldsymbol{GH}^T \boldsymbol{e}_y(t) = \boldsymbol{GH}^T \boldsymbol{C} \boldsymbol{e}_q(t) \tag{58}$$

then, with (10), (48), (54), it follows

$$\dot{v}_0(\boldsymbol{e}_q(t)) =$$
$$= \boldsymbol{e}_q^T(t)\boldsymbol{P}\dot{\boldsymbol{e}}_q(t) + \dot{\boldsymbol{e}}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\dot{\boldsymbol{e}}_f(t) + \dot{\boldsymbol{e}}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) =$$
$$= [(\boldsymbol{A} - \boldsymbol{JC})\boldsymbol{e}_q(t) + \boldsymbol{A}_h\boldsymbol{e}_p(t) + \boldsymbol{E}\boldsymbol{e}_f(t)]^T \boldsymbol{P}\boldsymbol{e}_q(t) + \tag{59}$$
$$+ \boldsymbol{e}_q^T(t)\boldsymbol{P}[(\boldsymbol{A} - \boldsymbol{JC})\boldsymbol{e}_q(t) + \boldsymbol{A}_h\boldsymbol{e}_p(t) + \boldsymbol{E}\boldsymbol{e}_f(t)] -$$
$$- \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\dot{\boldsymbol{f}}_e(t) - \dot{\boldsymbol{f}}_e^T(t)\boldsymbol{G}^{-1}\boldsymbol{e}_f(t)$$

and, using (58), it is

$$\boldsymbol{e}_q^T(t)\left[(\boldsymbol{A} - \boldsymbol{JC})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{JC})\right]\boldsymbol{e}_q(t) +$$
$$+ \boldsymbol{e}_q^T(t)\boldsymbol{P}\boldsymbol{A}_h\boldsymbol{e}_p(t) + \boldsymbol{e}_p^T(t)\boldsymbol{A}_h^T\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_f(t) + \boldsymbol{e}_f^T(t)\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{e}_q(t) - \tag{60}$$
$$- \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{GH}^T\boldsymbol{C}\boldsymbol{e}_q(t) - \boldsymbol{e}_q^T(t)\boldsymbol{C}^T\boldsymbol{HGG}^{-1}\boldsymbol{e}_f(t).$$

Setting

$$\boldsymbol{PE} - \boldsymbol{C}^T\boldsymbol{H} = 0 \tag{61}$$

the effect of the fault estimation error $\boldsymbol{e}_f(t)$ to the state estimation error $\boldsymbol{e}_q(t)$ is removed and (60) is simplified as

$$\dot{v}_0(\boldsymbol{e}_q(t)) = \tag{62}$$
$$= \boldsymbol{e}_q^T(t)\boldsymbol{P}\boldsymbol{A}_h\boldsymbol{e}_p(t) + \boldsymbol{e}_p^T(t)\boldsymbol{A}_h^T\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\left[(\boldsymbol{A} - \boldsymbol{JC})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{JC})\right]\boldsymbol{e}_q(t).$$

Analogously, taking into consideration (17), (54), it yields

$$\dot{v}_1(\boldsymbol{e}_q(t)) = \frac{d}{dt}\left\{ \int_{t-\frac{h}{m}}^{t} \boldsymbol{e}_p^T(s)\boldsymbol{W}\boldsymbol{e}_p(s)\mathrm{d}s \right\} = \boldsymbol{e}_p^T(t)\boldsymbol{W}\boldsymbol{e}_p(t) - \boldsymbol{e}_p^T\left(t - \frac{h}{m}\right)\boldsymbol{W}\boldsymbol{e}_p\left(t - \frac{h}{m}\right)$$
$$\tag{63}$$

and

$$\dot{v}_2(\boldsymbol{e}_q(t)) = \frac{d}{dt}\left\{ \int_{-\frac{h}{m}}^{0} \left\{ \int_{t+\vartheta}^{t} \boldsymbol{e}_q^T(s)\boldsymbol{U}\boldsymbol{e}_q(s)\mathrm{d}s \right\}\mathrm{d}\vartheta \right\} =$$
$$= \int_{-\frac{h}{m}}^{0} \boldsymbol{e}_q^T(t)\boldsymbol{U}\boldsymbol{e}_q(t)\mathrm{d}\vartheta - \int_{t-\frac{h}{m}}^{0} \boldsymbol{e}_q^T(t+\vartheta)\boldsymbol{U}\boldsymbol{e}_q(t+\vartheta)\mathrm{d}\vartheta = \tag{64}$$
$$= \frac{h}{m}\boldsymbol{e}_q^T(t)\boldsymbol{U}\boldsymbol{e}_q(t) - \int_{t-\frac{h}{m}}^{t} \boldsymbol{e}_q^T(s)\boldsymbol{U}\boldsymbol{e}_q(s)\mathrm{d}s$$

$$\dot{v}_2(\boldsymbol{e}_q(t)) \leqslant \tfrac{h}{m}\boldsymbol{e}_q^T(t)\boldsymbol{U}\boldsymbol{e}_q(t) - \tfrac{m}{h}\int\limits_{t-\frac{h}{m}}^{0}\boldsymbol{e}_q^T(s)\mathrm{d}s\,\boldsymbol{U}\int\limits_{t-\frac{h}{m}}^{0}\boldsymbol{e}_q(s)\mathrm{d}s =$$

$$= \tfrac{h}{m}\boldsymbol{e}_q^T(t)\boldsymbol{U}\boldsymbol{e}_q(t) - \tfrac{m}{h}\boldsymbol{e}_{p1}^T(t)\boldsymbol{U}\boldsymbol{e}_{p1}(t) \tag{65}$$

respectively.

Under the same conditions

$$\dot{v}_3(\boldsymbol{e}_q(t)) = \tfrac{d}{dt}\left\{\int\limits_{-\frac{h}{m}}^{0}\int\limits_{\vartheta}^{0}\left\{\int\limits_{t+\lambda}^{t}\dot{\boldsymbol{e}}_q^T(s)\boldsymbol{V}\dot{\boldsymbol{e}}_q(s)\mathrm{d}s\right\}\mathrm{d}\lambda\mathrm{d}\vartheta\right\} =$$

$$= \int\limits_{-\frac{h}{m}}^{0}\int\limits_{\vartheta}^{0}\left\{\dot{\boldsymbol{e}}_q^T(t)\boldsymbol{V}\dot{\boldsymbol{e}}_q(t) - \dot{\boldsymbol{e}}_q^T(t+\lambda)\boldsymbol{V}\dot{\boldsymbol{e}}_q(t+\lambda)\right\}\mathrm{d}\lambda\mathrm{d}\vartheta =$$

$$= \int\limits_{-\frac{h}{m}}^{0} -\vartheta\dot{\boldsymbol{e}}_q^T(t)\boldsymbol{V}\dot{\boldsymbol{e}}_q(t)\mathrm{d}\vartheta - \int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{e}}_q^T(s)\boldsymbol{V}\dot{\boldsymbol{e}}_q(s)\mathrm{d}s\mathrm{d}\vartheta = \tag{66}$$

$$= \tfrac{1}{2}(\tfrac{h}{m})^2\dot{\boldsymbol{e}}_q^T(t)\boldsymbol{V}\dot{\boldsymbol{e}}_q(t) - \int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{e}}_q^T(s)\boldsymbol{V}\dot{\boldsymbol{e}}_q(s)\mathrm{d}s\mathrm{d}\vartheta$$

and, since (16), (54) implies,

$$\int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{e}}_q^T(s)\boldsymbol{V}\dot{\boldsymbol{e}}_q(s)\mathrm{d}s\mathrm{d}\vartheta \geqslant$$

$$\geqslant \tfrac{2}{(\frac{h}{m})^2}\int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{e}}_q^T(s)\mathrm{d}s\mathrm{d}\vartheta\,\boldsymbol{V}\int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{e}}_q(s)\mathrm{d}s\mathrm{d}\vartheta =$$

$$= \tfrac{2}{(\frac{h}{m})^2}\int\limits_{-\frac{h}{m}}^{0}\left(\boldsymbol{e}_q(t)-\boldsymbol{e}_q(t+\vartheta)\right)^T\mathrm{d}\vartheta\,\boldsymbol{V}\int\limits_{-\frac{h}{m}}^{0}\left(\boldsymbol{e}_q(t)-\boldsymbol{e}_q(t+\vartheta)\right)\mathrm{d}\vartheta = \tag{67}$$

$$= \tfrac{2}{(\frac{h}{m})^2}\left(\tfrac{h}{m}\boldsymbol{e}_q^T(t) - \int\limits_{t-\frac{h}{m}}^{t}\boldsymbol{e}_q^T(s)\mathrm{d}s\right)\boldsymbol{V}\left(\tfrac{h}{m}\boldsymbol{e}_q(t) - \int\limits_{t-\frac{h}{m}}^{t}\boldsymbol{e}_q(s)\mathrm{d}s\right) =$$

$$= \tfrac{2}{(\frac{h}{m})^2}\left(\tfrac{h}{m}\boldsymbol{e}_q^T(t) - \boldsymbol{e}_{p1}^T(t)\right)\boldsymbol{V}\left(\tfrac{h}{m}\boldsymbol{e}_q(t) - \boldsymbol{e}_{p1}(t)\right)$$

then

$$\dot{v}_3(\boldsymbol{e}_q(t)) \leqslant \tfrac{1}{2}(\tfrac{h}{m})^2\dot{\boldsymbol{e}}_q^T(t)\boldsymbol{V}\dot{\boldsymbol{e}}_q(t) - \tfrac{2}{(\frac{h}{m})^2}\left(\tfrac{h}{m}\boldsymbol{e}_q^T(t) - \boldsymbol{e}_{p1}^T(t)\right)\boldsymbol{V}\left(\tfrac{h}{m}\boldsymbol{e}_q(t) - \boldsymbol{e}_{p1}(t)\right). \tag{68}$$

Hence, expressing with respect to (56), (55) that

$$\boldsymbol{e}_p(t-\tfrac{h}{m}) = \boldsymbol{e}_{p2}(t) + \int\limits_{t-h-\frac{h}{m}}^{t-h}\boldsymbol{e}_p(s)\mathrm{d}s = \boldsymbol{e}_{p2}^T(t) + \boldsymbol{e}_{p3}^T(t) \tag{69}$$

constructing the composite vector

$$\boldsymbol{e}^{\circ T}(t) = \begin{bmatrix} \boldsymbol{e}_q^T(t) & \boldsymbol{e}_{p1}^T(t) & \boldsymbol{e}_{p2}^T(t) & \boldsymbol{e}_{p3}^T(t) \end{bmatrix} \tag{70}$$

and introducing the notation

$$\boldsymbol{T}_A^{\circ} = \begin{bmatrix} (\boldsymbol{A} - \boldsymbol{JC}) & \boldsymbol{A}_h & \begin{bmatrix} \boldsymbol{A}_h & \cdots & \boldsymbol{A}_h \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \tag{71}$$

then, using (45), $\dot{v}_0(\boldsymbol{e}_q(t))$ can be written as

$$\dot{v}_0(\boldsymbol{e}_q(t)) = \boldsymbol{e}^{\circ T}(t)\left(\boldsymbol{T}_I^T \boldsymbol{P}\boldsymbol{T}_A^{\circ} + \boldsymbol{T}_A^{\circ T}\boldsymbol{P}\boldsymbol{T}_I\right)\boldsymbol{e}^{\circ}(t). \tag{72}$$

Same as above, using (40), it yields

$$\dot{v}_1(\boldsymbol{e}_q(t)) \leqslant \boldsymbol{e}^{\circ T}(t)\boldsymbol{T}_W^T \begin{bmatrix} \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{W} \end{bmatrix} \boldsymbol{T}_W \boldsymbol{e}^{\circ}(t) = \boldsymbol{e}^{\circ T}(t)\boldsymbol{T}_W^T \boldsymbol{W}^{\circ}\boldsymbol{T}_W \boldsymbol{e}^{\circ}(t) \tag{73}$$

and with (39)

$$\dot{v}_2(\boldsymbol{e}_q(t)) \leqslant \boldsymbol{e}^{\circ T}(t)\boldsymbol{T}_U^T \begin{bmatrix} \boldsymbol{U} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{U} \end{bmatrix} \boldsymbol{T}_U \boldsymbol{e}^{\circ}(t) = \boldsymbol{e}^{\circ T}(t)\boldsymbol{T}_U^T \boldsymbol{U}^{\circ}\boldsymbol{T}_U \boldsymbol{e}^{\circ}(t). \tag{74}$$

Inserting (48) and denoting

$$\boldsymbol{e}_c(t) = (\boldsymbol{A} - \boldsymbol{JC})\boldsymbol{e}_q(t) + \boldsymbol{A}_h \boldsymbol{e}_p(t) \tag{75}$$

$$\boldsymbol{V}^{\circ} = \boldsymbol{P}^{-1}\boldsymbol{V}\boldsymbol{P}^{-1} \tag{76}$$

then, with respect to (61), it yields

$$
\begin{aligned}
\dot{\boldsymbol{e}}_q^T(t)\boldsymbol{V}\dot{\boldsymbol{e}}_q(t) &= (\boldsymbol{e}_c(t) + \boldsymbol{E}\boldsymbol{e}_f(t))^T \boldsymbol{V}(\boldsymbol{e}_c(t) + \boldsymbol{E}\boldsymbol{e}_f(t)) = \\
&= (\boldsymbol{P}\boldsymbol{e}_c(t) + \boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_f(t))^T \boldsymbol{V}^{\circ}(\boldsymbol{P}\boldsymbol{e}_c(t) + \boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_f(t)) \geqslant \\
&\geqslant (\boldsymbol{P}\boldsymbol{e}_c(t) + (\boldsymbol{P}\boldsymbol{E} - \boldsymbol{C}^T \boldsymbol{H})\boldsymbol{e}_f(t))^T \boldsymbol{V}^{\circ}(\boldsymbol{P}\boldsymbol{e}_c(t) + (\boldsymbol{P}\boldsymbol{E} - \boldsymbol{C}^T \boldsymbol{H})\boldsymbol{e}_f(t)) = \\
&= (\boldsymbol{P}\boldsymbol{A}_h \boldsymbol{e}_p(t) + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{JC})\boldsymbol{e}_q(t))^T \boldsymbol{V}^{\circ}(\boldsymbol{P}\boldsymbol{A}_h \boldsymbol{e}_p(t) + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{JC})\boldsymbol{e}_q(t))
\end{aligned} \tag{77}
$$

or, equivalently, with the notation

$$\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{J} \tag{78}$$

$$\dot{\boldsymbol{e}}_q^T(t)\boldsymbol{V}\dot{\boldsymbol{e}}_q(t) \leqslant (\boldsymbol{P}\boldsymbol{A}_h \boldsymbol{e}_p(t) + (\boldsymbol{P}\boldsymbol{A} - \boldsymbol{Y}\boldsymbol{C})\boldsymbol{e}_q(t))^T \boldsymbol{V}^{\circ}(\boldsymbol{P}\boldsymbol{A}_h \boldsymbol{e}_p(t) + (\boldsymbol{P}\boldsymbol{A} - \boldsymbol{Y}\boldsymbol{C})\boldsymbol{e}_q(t)). \tag{79}$$

Thus, using (43), (44), and (38), (41), respectively, it yields

$$(\boldsymbol{P}\boldsymbol{A} - \boldsymbol{Y}\boldsymbol{C})\boldsymbol{e}_q(t) + \boldsymbol{P}\boldsymbol{A}_h \boldsymbol{e}_p(t) = \boldsymbol{Y}^{\circ}\boldsymbol{T}_A \boldsymbol{e}^{\circ}(t) \tag{80}$$

$$\sqrt{2}\frac{m}{h}\left(\frac{h}{m}\boldsymbol{e}_q(t) - \boldsymbol{e}_{p1}(t)\right) = \boldsymbol{T}_V \boldsymbol{e}^\circ(t) \tag{81}$$

and the last term $\dot{v}_3(\boldsymbol{e}_q(t))$ in (68) can be written as

$$\dot{v}_3(\boldsymbol{e}_q(t)) \leqslant \boldsymbol{e}^{\circ T}(t)\left(\boldsymbol{T}_A^T \boldsymbol{Y}^{\circ T} b^2 \boldsymbol{V}^\circ \boldsymbol{Y}^\circ \boldsymbol{T}_A - \boldsymbol{T}_V^T \boldsymbol{V} \boldsymbol{T}_V\right) \boldsymbol{e}^\circ(t). \tag{82}$$

To incorporate (78) into (71), the matrix $\boldsymbol{T}_A^\circ$ is written as

$$\boldsymbol{T}_A^\circ = \boldsymbol{P}^{-1}\left[\begin{array}{ccc} \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}) & \boldsymbol{P}\boldsymbol{A}_h & \left[\begin{array}{ccc} \boldsymbol{P}\boldsymbol{A}_h & \cdots & \boldsymbol{P}\boldsymbol{A}_h \end{array}\right] & \boldsymbol{0}\end{array}\right] = \boldsymbol{P}^{-1}\boldsymbol{Y}^\circ \boldsymbol{T}_A \tag{83}$$

and with (83) so (72) takes the form

$$\dot{v}_0(\boldsymbol{e}_q(t)) = \boldsymbol{e}^{\circ T}(t)\left(\boldsymbol{T}_I^T \boldsymbol{Y}^\circ \boldsymbol{T}_A + \boldsymbol{T}_A^T \boldsymbol{Y}^{\circ T} \boldsymbol{T}_I\right) \boldsymbol{e}^\circ(t). \tag{84}$$

Therefore, the derivative of $v(\boldsymbol{e}_q(t))$ can be written as

$$\dot{v}(\boldsymbol{e}_q(t)) \leqslant \boldsymbol{e}^{\circ T}(t)\boldsymbol{P}^\circ \boldsymbol{e}^\circ(t) < 0 \tag{85}$$

$$\boldsymbol{P}^\circ = \boldsymbol{T}_I^T \boldsymbol{Y}^\circ \boldsymbol{T}_A + \boldsymbol{T}_A^T \boldsymbol{Y}^\circ \boldsymbol{T}_I - \boldsymbol{T}_V^T \boldsymbol{V} \boldsymbol{T}_V +$$
$$+ \boldsymbol{T}_U^T \boldsymbol{U}^\circ \boldsymbol{T}_U + \boldsymbol{T}_W^T \boldsymbol{W}^\circ \boldsymbol{T}_W + \boldsymbol{T}_A^T \boldsymbol{Y}^{\circ T} b^2 \boldsymbol{V}^\circ \boldsymbol{Y}^\circ \boldsymbol{T}_A < 0. \tag{86}$$

It is noticed, as (76) implies that $\boldsymbol{V}^\circ$ depends on $\boldsymbol{P}^{-1}$ that (86) is a nonlinear matrix inequality, and it is necessary to transform (86) into the LMI form. Therefore, using Schur complement, (86) is

$$\left[\begin{array}{cc} \boldsymbol{\Delta}_{11} & \boldsymbol{T}_A^T \boldsymbol{Y}^{\circ T} \\ \boldsymbol{Y}^\circ \boldsymbol{T}_A & -b^{-2}\boldsymbol{P}\boldsymbol{V}^{-1}\boldsymbol{P} \end{array}\right] \leqslant 0 \tag{87}$$

where $\boldsymbol{\Delta}_{11}$ is given in (37). Using (11), the item in the bottom right-hand corner of (87) is approximated as

$$-b^{-2}\boldsymbol{P}\boldsymbol{V}^{-1}\boldsymbol{P} \leqslant -b^{-2}2\boldsymbol{P} + b^{-2}\boldsymbol{V}. \tag{88}$$

Evidently, with (88), now (87) implies (35).

Since (61) is generally singular, to obtain a more regular expression the left-hand side of (61) can be pre-multiplied by $\boldsymbol{E}^T$, i.e.,

$$\boldsymbol{E}^T \boldsymbol{P}\boldsymbol{E} - \boldsymbol{E}^T \boldsymbol{C}^T \boldsymbol{H} = 0 \tag{89}$$

which implies (36). This concludes the proof. $\square$

Remark 1 To overcome equality problems in LMI solvers, instead of the equlity (36) the inequality

$$2\boldsymbol{E}^T \boldsymbol{P}\boldsymbol{E} - \boldsymbol{H}^T \boldsymbol{C}\boldsymbol{E} - \boldsymbol{E}^T \boldsymbol{C}^T \boldsymbol{H} + \varepsilon \boldsymbol{I}_s < 0 \tag{90}$$

can be placed into LMIs design conditions. It is obvious that for enough small $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$, (89) implies (90).

6. Illustrative example

To illustrate the proposed method, the system described by (1), (2) is considered, where

$$A = \begin{bmatrix} 2.6 & 0.0 & -0.8 \\ 1.2 & 0.2 & 0.0 \\ 0.0 & -0.5 & 3.0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$A_h = \begin{bmatrix} 0.00 & 0.02 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ -0.02 & 0.00 & 0.00 \end{bmatrix}, \quad B = E = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Setting $m = 2$, and solving (34)-(36) with respect the LMI matrix variables P, U, V, W, H, and Y using Self-Dual-Minimization (SeDuMi) package [18] for Matlab, the estimator parameter design problem was solved as feasible and regular up to $h \leqslant 3.4\,s$. Illustrating for $h = 1.5\,s$, conditioned by resulting positive definite matrices U, V, W, the design parameters were

$$P = \begin{bmatrix} 0.2979 & -0.0517 & -0.2042 \\ -0.0517 & 0.4942 & 0.2356 \\ -0.2042 & 0.2356 & 0.2742 \end{bmatrix}, \quad Y = \begin{bmatrix} -1.0355 & 2.2128 \\ 1.3281 & -0.8093 \\ 1.9446 & -2.5751 \end{bmatrix}$$

$$J = \begin{bmatrix} 5.9707 & -3.2030 \\ -3.7022 & 6.1622 \\ 14.7167 & -17.0679 \end{bmatrix}, \quad \rho(A_e) = \left\{ -1.7745 \quad -4.3339 \pm 1.7892\mathrm{i} \right\}$$

$$H = \begin{bmatrix} 1.0752 & -0.0697 \\ -1.1919 & 0.7007 \end{bmatrix}, \quad G = \begin{bmatrix} 1.50 & 4.75 \\ 4.75 & 10.00 \end{bmatrix}$$

ensuring the stable eigenvalue spectrum of the estimator.

For simulation purposes only, the equilibrium of the system was stabilized by the feedback controller

$$u(t) = -Kq(t)$$

where, using the method proposed in [5] which offers the possibility to design the linear state controller for the linear systems with distributed time-delay, the gain matrix was computed as

$$K = \begin{bmatrix} -6.7996 & -5.4356 & 25.7509 \\ 5.3858 & 3.4324 & -14.7765 \end{bmatrix}.$$

In simulations was considered the fault which doesn't cause closed-loop system instability, modeled by a fault starting at any time instant in the system equilibrium state.
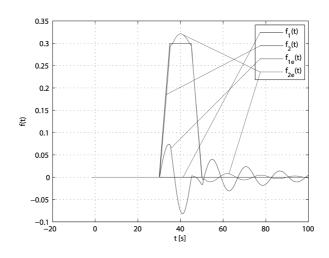
Figure 1. The second actuator fault and its estimation

Applying the above designed observer-based actuator fault estimation, the observer fault response is given in Fig. 1. This figure presents the fault signal, as well as its estimation, reflecting a single actuator fault in the the second actuator, starting at the time instant $t = 30$ s and continuing during the time 20 s. Note that equivalent results are obtained for the system working in a forced regime.

From the simulation results of Fig. 1 it can be found that the errors between the signals reflecting a single actuator fault and the observer approximate ones tends to zero. Moreover, the states of the system converge to the equilibrium when the actuator fault disappeared, via the used controller.

7.   Concluding remarks

Design conditions for observer-based fault estimation, introduced and explained with respect to formal limitations triggered by the existence of structured matrix variables in LMIs, are derived in the paper. The integral partitioning technique, as well an extended version of Lyapunov-Krasovskii functional, is used to reduce the conservatism, and to regularize the distributed delay dependent stability condition. Sufficient conditions are established in terms of LMI as a convex LMI problem, and the manipulation is accomplished in the manner giving the design conditions with guaranty of asymptotic stability of the observer-based adaptive actuator fault estimation.

The algorithm is enough robust to the system time-delay value h in that sense that, for given m, there exists such upper bound of h that the design task is feasible. The convergence of fault estimation dynamics can be modified by suitable choice of the learning weight matrix. Presented illustrative example confirms the effectiveness of the proposed

design method. In particular, with the use of such modification of Lyapunov-Krasovskii functional, it shows how to adapt the standard approaches to design optimal matrix parameters of Luenberger-like state estimators for linear systems with distributed time delays. Combined with the fault compensation, a fault tolerant control (FTC) structure for this fault can be developed.

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