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An analytical method for solving the two-phase inverse Stefan problem

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Abstract. In the paper we present an application of the homotopy analysis method for solving the two-phase inverse Stefan problem. In the proposed approach a series is created, having elements which satisfy some differential equation following from the investigated problem. We reveal, in the paper, that if this series is convergent then its sum determines the solution of the original equation. A sufficient condition for this convergence is formulated. Moreover, the estimation of the error of the approximate solution, obtained by taking the partial sum of the considered series, is given. Additionally, we present an example illustrating an application of the described method.

Key words: inverse Stefan problem, homotopy analysis method, heat transfer, solidification.

1. Introduction

Solidification of pure metals is customarily modeled by means of the Stefan problem [1, 2]. This problem consists in simultaneous determination of the temperature distribution in the investigated region and of the location of the interface dividing the region into two subregions taken by the liquid phase and solid phase. In the inverse Stefan problem one usually assumes that the additional information, compensating the lack of input data, is given by the knowledge of the location of the interface, its velocity in the normal direction or temperature at selected points of the region. Solving the inverse problem, discussed in this paper, lies in determination of the temperature distribution in considered region, as well as in determination of the temperature and the heat flux on one of boundaries of the region. Supplementary information of this inverse problem is given by the location of interface, therefore the considered problem is called the design problem.

It is possible to find an exact analytical solution of the inverse Stefan problem only in few simple cases. Therefore for solving problems of that type the numerical methods of different kinds are the most often used [3–15]. Application of analytical or analytic-numerical methods is rather insignificant. In particular, in papers [16, 17] an application of the homotopy perturbation method for solving the one- and two-phase inverse Stefan problem is presented. Whereas, the use of the Adomian decomposition method and variational iteration method for solving the same problem is discussed in papers [18, 19].

The homotopy analysis method has been developed in the 90's, its author is Shijun Liao [20–23] and it belongs to the group of analytical methods. This method has found an application for solving many problems formulated with the aid of ordinary and partial differential equations [24–27], including the heat conduction problems [28–31], fractional differential equations [32,33] (for some other applications of the fractional calculus see for example [34–36]), integral equations [37–39], integro-differential equations [40, 41] and others. A particular case of the homotopy analysis method is the homotopy perturbation method [16, 17, 42].

By using the considered method we create a series, elements of which fulfill some differential equation resulting from the investigated problem. Obtained equation is easier to solve in comparison with the original one. We reveal in the paper that if this series is convergent then its sum determines the solution of discussed equation. Sufficient condition of this convergence is formulated in this paper as well as the estimation of error of approximate solution received by taking only the partial sum of considered series. We also give some example illustrating the application of described approach.

2. Statement of the problem

Let us consider two regions: D_1 denoting the region taken by the liquid phase and D_2 describing the region taken by the solid phase (see Fig. 1):

$$D_1 = \{(x,t);\ x \in [0, \xi(t)],\ t \in [0, t^*]\},$$

$$D_2 = \{(x,t);\ x \in [\xi(t), d],\ t \in [0, t^*]\}$$

and their boundaries:

$$\Gamma_1 = \{(x,0);\ x \in (0, s),\ s = \xi(0)\},$$

$$\Gamma_2 = \{(x,0);\ x \in (s, d),\ s = \xi(0)\},$$

$$\Gamma_3 = \{(0,t);\ t \in [0, t^*]\},$$

$$\Gamma_4 = \{(d,t);\ t \in [0, t^*]\},$$

$$\Gamma_5 = \{(x,t);\ t \in [0, t^*],\ x = \xi(t)\},$$

where function $x = \xi(t)$ defines the location of interface.

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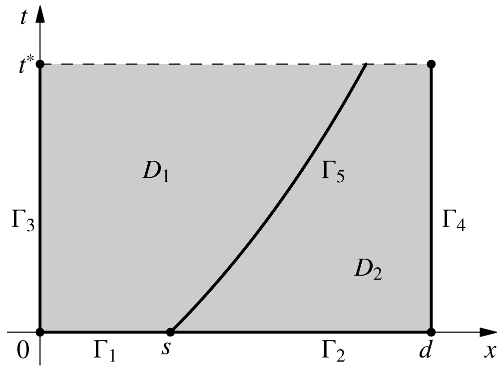

Fig. 1. Domain of the problem

The two-phase Stefan problem [1,2] consists in determining the location of a moving interface described by means of function $x = \xi(t)$ and in determining functions u_1 and u_2, defined in domains D_1 and D_2, respectively, which fulfill the heat conduction equations ($k = 1, 2$):

$$\frac{\partial u_k(x,t)}{\partial t} = a_k \frac{\partial^2 u_k(x,t)}{\partial x^2} \qquad \text{in} \qquad D_k, \tag{1}$$

where a_k denotes the thermal diffusivity in liquid phase ($k = 1$) and solid phase ($k = 2$), whereas t and x refer to the time and spatial location, respectively. On boundaries Γ_1 and Γ_2 the sought functions fulfill the initial conditions

$$u_1(x,0) = \varphi_1(x) \qquad \text{on} \qquad \Gamma_1, \tag{2}$$

$$u_2(x,0) = \varphi_2(x) \qquad \text{on} \qquad \Gamma_2, \tag{3}$$

$$\xi(0) = s. \tag{4}$$

On boundary Γ_3 function u_1 satisfies the Dirichlet boundary condition

$$u_1(0,t) = \theta_1(t). \tag{5}$$

Next, on boundary Γ_4 function u_2 fulfills the Dirichlet or Neumann boundary conditions

$$u_2(d,t) = \theta(t), \tag{6}$$

$$-\lambda_2 \frac{\partial u_2(d,t)}{\partial x} = q(t), \tag{7}$$

where λ_k denotes the thermal conductivity in liquid phase ($k = 1$) and solid phase ($k = 2$). Finally, on the moving interface (Γ_5) the sought functions fulfil the condition of temperature continuity and the Stefan condition

$$u_1(\xi(t),t) = u_2(\xi(t),t) = u^*, \tag{8}$$

$$\kappa \frac{d\xi(t)}{dt} = \lambda_2 \frac{\partial u_2(x,t)}{\partial x}\Big|_{x=\xi(t)} - \lambda_1 \frac{\partial u_1(x,t)}{\partial x}\Big|_{x=\xi(t)}, \tag{9}$$

where u^* is the phase change temperature and κ denotes the latent heat of fusion per unit volume.

Solving of the discussed inverse Stefan problem lies in finding the functions u_1 and u_2 describing the temperature distribution in domains D_1 and D_2 as well as the functions θ and q defining the temperature and the heat flux, respectively, on boundary Γ_4 such that they will satisfy Eqs. (1) with conditions (2)–(9). All the other functions (φ_k, θ_1, ξ) and parameters (a_k, λ_k, κ, u^*, s) are known.

3. Homotopy analysis method

For solving the investigated problem we intend to apply the homotopy analysis method. By using this method we are able to solve the operator equation

$$N(v(z)) = 0, \qquad z \in \Omega, \tag{10}$$

where N denotes the operator (in particular it can be the nonlinear operator), whereas v describes the unknown function. At the beginning we need to determine the operator $\mathcal{H}$ in the form

$$\mathcal{H}(\Phi, p) \equiv (1-p)L\left(\Phi(z;p) - v_0(z)\right) - phN\left(\Phi(z;p)\right), \tag{11}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ denotes the convergence control parameter, v_0 describes the initial approximation of solution of problem (10) and L is the auxiliary linear operator with property $L(0) = 0$.

By considering equation $\mathcal{H}(\Phi, p) = 0$ we obtain the so called zero-order deformation equation

$$(1-p)L\big(\Phi(z;p) - v_0(z)\big) = phN\big(\Phi(z;p)\big). \tag{12}$$

For $p = 0$ we have $L(\Phi(z;0) - u_0(z)) = 0$, from which we get $\Phi(z;0) = v_0(z)$. Next, since for $p = 1$ we have $N(\Phi(z;1)) = 0$, therefore $\Phi(z;1) = v(z)$, where v is the sought solution of Eq. (10). In this way the change of parameter p from zero to one corresponds with the change of problem from the trivial form to the originally given form (which means the change of solution from v_0 to v).

On the way of expanding the function Φ into the Maclaurin series with respect to the parameter p we receive

$$\Phi(z;p) = \Phi(z;0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(z;p)}{\partial p^m}\Big|_{p=0} p^m. \tag{13}$$

Introducing the notation

$$v_m(z) = \frac{1}{m!} \frac{\partial^m \Phi(z;p)}{\partial p^m}\Big|_{p=0}, \qquad m = 1, 2, 3, \ldots, \tag{14}$$

the previous relation can be described in the following way

$$\Phi(z;p) = v_0(z) + \sum_{m=1}^{\infty} v_m(z) p^m. \tag{15}$$

If the above series is convergent in the appropriate region, then for $p = 1$ we obtain the desired solution

$$v(z) = \sum_{m=0}^{\infty} v_m(z). \tag{16}$$

In order to determine the function v_m we differentiate m times the left and the right side of the relation (12) with respect to parameter p. Obtained result is divided by $m!$ and next, by substituting $p = 0$, we get the so called mth-order deformation equation ($m > 0$):

$$L\big(v_m(z) - \chi_m v_{m-1}(z)\big) = hR_m\big(\overline{v}_{m-1}, z\big), \tag{17}$$

where $\overline{v}_{m-1} = \{v_0(z), v_1(z), \ldots, v_{m-1}(z)\}$,

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1 \end{cases} \tag{18}$$

and

$$R_m(\overline{v}_{m-1}, z) = \frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial p^{m-1}} N\left(\sum_{i=0}^{\infty} v_i(z) p^i\right)\right)\Bigg|_{p=0}. \tag{19}$$

If we are not able to determine the sum of series in (16), then as the approximate solution of considered equation we can accept the partial sum of this series

$$\widehat{v}_n(z) = \sum_{m=0}^{n} v_m(z). \tag{20}$$

By selecting properly the convergence control parameter h we can influence the region of convergence of the series in (15) as well as the rapidness of this convergence [23, 43–45]. One of the methods enabling to choose the value of convergence control parameter is the so called "optimization method" [23, 46, 47]. In this method we define the squared residual of the governing equation

$$E_n(h) = \int\limits_{\Omega} \Big(N\big[\widehat{v}_n(z)\big]\Big)^2 dz. \tag{21}$$

The optimal value of the convergence control parameter is determined by minimizing the above squared residual. The effective region of the convergence control parameter is defined in the following way

$$\mathbf{R}_h = \big\{h : \lim_{n\to\infty} E_n(h) = 0\big\}. \tag{22}$$

If we take the value of convergence control parameter different than the optimal value but still belonging to the effective region, we obtain the convergent series as well, however rapidness of this convergence will be smaller.

Another way for determining the value of convergence control parameter is the so-called h-curve which is obtained by investigating the behavior of a certain quantity of the solution as a function of parameter h [21, 45]. This method enables to determine the effective region of the convergence control parameter, whereas it makes no possibility to calculate the value ensuring the fastest convergence [23].

4. Solution of the problem

Let us proceed now to the application of the homotopy analysis method for solving the above formulated two-phase inverse Stefan problem. In this case the operators N_k and L_k, for $k = 1, 2$, can be defined as follows

$$N_k(v) = \frac{\partial v}{\partial t} - a_k \frac{\partial^2 v}{\partial x^2} \tag{23}$$

and

$$L_k(v) = \frac{\partial^2 v}{\partial x^2}. \tag{24}$$

For the investigated problem we have (under assumption that the series is convergent) for $k = 1, 2$:

$$\begin{aligned} & R_{k,m}\big(\overline{u}_{k,m-1}, x, t\big) \\ &= \frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial p^{m-1}} N_k\left(\sum_{i=0}^{+\infty} u_{k,i}(x,t)\, p^i\right)\right)\Bigg|_{p=0} \\ &= \frac{1}{(m-1)!} N_k \\ &\cdot \left((m-1)! u_{k,m-1}(x,t) + \sum_{i=m}^{+\infty} u_{k,i}(x,t) w_k(i) p^{i-m+1}\right)\Bigg|_{p=0} \\ &= N_k(u_{k,m-1}(x,t)), \end{aligned} \tag{25}$$

for $m = 1, 2, \ldots$, where $w_k(i) \in \mathbb{N}$ for $i = m, m+1, \ldots$.

In this way for $m = 1$ we get the system of two partial differential equations ($k = 1, 2$):

$$\frac{\partial^2 u_{k,1}(x,t)}{\partial x^2} = h\left(\frac{\partial u_{k,0}(x,t)}{\partial t} - a_k \frac{\partial^2 u_{k,0}(x,t)}{\partial x^2}\right), \tag{26}$$

whereas for $m \geq 2$ we receive the following systems ($k = 1, 2$):

$$\begin{aligned} \frac{\partial^2 u_{k,m}(x,t)}{\partial x^2} &= \frac{\partial^2 u_{k,m-1}(x,t)}{\partial x^2} \\ &+ h\left(\frac{\partial u_{k,m-1}(x,t)}{\partial t} - a_k \frac{\partial^2 u_{k,m-1}(x,t)}{\partial x^2}\right). \end{aligned} \tag{27}$$

For uniqueness of solution the above systems of partial differential equations suppose to be completed by additional conditions. For this purpose we use conditions (5), (8) and (9). Thus, the first of above systems is completed by conditions of the form

$$u_{1,0}(0,t) + u_{1,1}(0,t) = \theta_1(t), \tag{28}$$

$$u_{1,0}(\xi(t),t) + u_{1,1}(\xi(t),t) = u^*, \tag{29}$$

$$u_{2,0}(\xi(t),t) + u_{2,1}(\xi(t),t) = u^*, \tag{30}$$

$$\begin{aligned} & -\lambda_1 \frac{\partial (u_{1,0} + u_{1,1})(\xi(t),t)}{\partial x} \\ & + \lambda_2 \frac{\partial (u_{2,0} + u_{2,1})(\xi(t),t)}{\partial x} = \kappa \frac{d\xi(t)}{dt}, \end{aligned} \tag{31}$$

whereas for the remaining systems ($m \geq 2$) we define the following conditions

$$u_{1,m}(0,t) = 0, \tag{32}$$

$$u_{1,m}(\xi(t),t) = 0, \tag{33}$$

$$u_{2,m}(\xi(t),t) = 0, \tag{34}$$

$$-\lambda_1 \frac{\partial u_{1,m}(\xi(t),t)}{\partial x} + \lambda_2 \frac{\partial u_{2,m}(\xi(t),t)}{\partial x} = 0. \tag{35}$$

The above conditions ensure that any approximate solution (20) satisfies the assumed boundary conditions. As the initial approximation we can take the functions defining the initial conditions ($k = 1, 2$):

$$u_{k,0}(x,t) = \varphi_k(x). \tag{36}$$

Thus the problem has been reduced to the sequence of systems of differential Eqs. (26) and (27) with conditions (28)–(31) and (32)–(35), respectively. The obtained systems of equations are easier to solve in comparison with the original system of partial differential equations.

Now we proceed to present the theorem ensuring that the sums of determined series represent the solutions of considered equations.

Theorem 1. Let functions $u_{k,m}$, $k = 1, 2$, $m \geq 1$, be determined from the systems of Eqs. (26) and (27) with conditions (28)–(31) and (32)–(35), respectively. If the series $\sum\limits_{m=0}^{\infty} u_{k,m}$, for $k = 1, 2$, are convergent then their sums designate the solutions of considered equations.

Proof. Let the series $\sum_{m=0}^{\infty} u_{k,m}$, for $k = 1, 2$, be convergent and let

$$s_k(x,t) = \sum_{m=0}^{\infty} u_{k,m}(x,t).$$

From the necessary condition for the series convergence we get that for any $x \in [a, b]$ and $t \in [0, t^*]$:

$$\lim_{m\to\infty} u_{k,m}(x,t) = 0, \qquad k = 1, 2.$$

According to the method we have (for $k = 1, 2$):

$$L_k\big(u_{k,m}(x,t) - \chi_m\, u_{k,m-1}(x,t)\big) = hR_{k,m}\big(\overline{u}_{k,m-1}, x, t\big),$$

which implies

$$\begin{aligned} h &\sum_{m=1}^{\infty} R_{k,m}\big(\overline{u}_{k,m-1}, x, t\big) \\ &= \lim_{n\to\infty} \sum_{m=1}^{n} L_k\big(u_{k,m}(x,t) - \chi_m u_{k,m-1}(x,t)\big) \\ &= \lim_{n\to\infty} L_k\left(\sum_{m=1}^{n} \big(u_{k,m}(x,t) - \chi_m u_{k,m-1}(x,t)\big)\right) \\ &= \lim_{n\to\infty} L_k\big(u_{k,n}(x,t)\big) \\ &= L_k\big(\lim_{n\to\infty} u_{k,n}(x,t)\big) = L_k(0) = 0, \end{aligned}$$

where we have used the continuity of operator L_k. Since $h \neq 0$, therefore for $k = 1, 2$, we have

$$\sum_{m=1}^{\infty} R_{k,m}\big(\overline{u}_{k,m-1}, x, t\big) = 0.$$

Next, by using relation (25) we get

$$\begin{aligned} 0 = \sum_{m=1}^{\infty} R_{k,m}\left(\overline{u}_{k,m-1}, x, t\right) &= \sum_{m=1}^{\infty} N_k\big(u_{k,m-1}(x,t)\big) \\ = N_k\left(\sum_{m=1}^{\infty} u_{k,m-1}(x,t)\right) &= N_k\big(s_k(x,t)\big). \end{aligned}$$

The following theorem describes the sufficient condition for convergence of series $\sum_{m=0}^{\infty} u_{k,m}$ for $k = 1, 2$.

Theorem 2. Let functions $u_{k,m}$, $k = 1, 2$, $m \geq 1$, be determined from the systems of Eqs. (26) and (27) with conditions (28)–(31) and (32)–(35), respectively. If parameter h is selected in such a way that there exist the constants $\gamma_h \in (0, 1)$ and $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$ and $k = 1, 2$ the following inequality

$$\|u_{k,m+1}\| \leq \gamma_h \|u_{k,m}\| \tag{37}$$

is satisfied, then the series $\sum_{m=0}^{\infty} u_{k,m}$ are uniformly convergent.

Proof. Let $S_{k,n}$, for $k = 1, 2$, denotes the partial sums of considered series

$$S_{k,n}(x,t) = \sum_{m=0}^{n} u_{k,m}(x,t).$$

We intend to show that for any $x \in [a, b]$ and $t \in [0, t^*]$ the sequences $\{S_{k,n}(x,t)\}$ are the Cauchy sequences. For this purpose let us begin by estimating the following norm (for $k = 1, 2$):

$$\begin{aligned} &\|S_{k,n}(x,t) - S_{k,n-1}(x,t)\| \\ &= \|u_{k,n}(x,t)\| \leq \gamma_h \|u_{k,n-1}\| \leq \ldots \leq \gamma_h^{n-m_0} \|u_{k,m_0}\| \end{aligned}$$

if only $n \geq m_0$. Now for any $n, m \in \mathbb{N}$, $n \geq m \geq m_0$, and $k = 1, 2$, we have

$$\begin{aligned} \|S_{k,n}(x,t) - S_{k,m}(x,t)\| &\leq \|S_{k,n}(x,t) - S_{k,n-1}(x,t)\| + \ldots \\ &+ \|S_{k,m+1}(x,t) - S_{k,m}(x,t)\| \\ &\leq \gamma_h^{n-m_0} \|u_{k,m_0}\| + \ldots + \gamma_h^{m+1-m_0} \|u_{k,m_0}\| \\ &= \gamma_h^{m+1-m_0}\big(\gamma_h^{n-m-1} + \ldots + \gamma_h + 1\big)\|u_{k,m_0}\| \\ &= \gamma_h^{m+1-m_0} \frac{1-\gamma_h^{n-m}}{1-\gamma_h} \|u_{k,m_0}\|. \end{aligned}$$

Since $\gamma_h \in (0, 1)$, therefore we deduce that sequences $\{S_{k,n}(x,t)\}$ are the Cauchy sequences. Using the completeness of space $\mathbb{R}$ we conclude that these sequences are convergent, which implies the convergence of discussed series.

In the last theorem we give the estimation of error of approximate solution obtained by taking the partial sums of the series.

Theorem 3. If assumptions of Theorem 2 are satisfied and additionally $n \in \mathbb{N}$ and $n \geq m_0$, then for $k = 1, 2$ we get the following estimation of error of approximate solution

$$\|u_k - \widehat{u}_{k,n}\| \leq \frac{\gamma_h^{n+1-m_0}}{1-\gamma_h} \|u_{k,m_0}\|. \tag{38}$$

Proof. Let $n \in \mathbb{N}$ and $n \geq m_0$. Then for every $x \in [a, b]$ and $t \in [0, t^*]$ we get

$$\begin{aligned} \|u_k(x,t) - \widehat{u}_{k,n}(x,t)\| &= \Big\|\sum_{m=n+1}^{\infty} u_{k,m}(x,t)\Big\| \\ \leq \sum_{m=n+1}^{\infty} \|u_{k,m}(x,t)\| &\leq \sum_{m=n+1}^{\infty} \gamma_h^{m-m_0} \|u_{k,m_0}\| \\ &= \frac{\gamma_h^{n+1-m_0}}{1-\gamma_h} \|u_{k,m_0}\|. \end{aligned}$$

5. Example

The theoretical considerations introduced in the previous sections will be illustrated now with example. In the discussed example we take the following values of parameters: $s = 3/2$, $d = 3$, $a_1 = 5/2$, $a_2 = 5/4$, $\lambda_1 = 6$, $\lambda_2 = 2$, $\kappa = 8/10$, $t^* = 1$. Location of the interface is described with the aid of function $\xi(t) = (t+3)/2$. Initial and boundary conditions are defined by means of functions

$$\varphi_1(x) = \exp\left(\frac{3-2\,x}{10}\right),$$

$$\varphi_2(x) = \exp\left(\frac{3-2\,x}{5}\right),$$

$$\theta_1(t) = \exp\left(\frac{t+3}{10}\right).$$

For such initial data the exact solution of the problem is given by functions

$$u_{1,e}(x,t) = \exp\left(\frac{t-2\,x+3}{10}\right),$$

$$u_{2,e}(x,t) = \exp\left(\frac{t-2\,x+3}{5}\right).$$

As the initial approximations we take

$$u_{1,0}(x,t) = \exp\left(\frac{3-2\,x}{10}\right),$$

$$u_{2,0}(x,t) = \exp\left(\frac{3-2\,x}{5}\right).$$

In our case we can calculate the squared residual (21) for each of equations (1). In this way we obtain the vector $\mathbf{E}_n(h) = [E_{1,n}(h), E_{2,n}(h)]$. Therefore it is natural to determine the optimal value of the convergence control parameter h as the element for which we get the minimal value of the norm of squared residual $\mathbf{E}_n$:

$$h_{opt} := \arg\min \|\mathbf{E}_n(h)\|.$$

In the considered case the numerically computed optimal value of the convergence control parameter is equal to 0.4053849 ($h = 0.4053849$). Logarithm of the norm of squared residual $\mathbf{E}_3$ is displayed in Fig. 2. Whereas, Fig. 3 presents the h-curves of $u_x(\xi(0),0)$ and $v_x(\xi(0),0)$. On the basis of this plot we conclude that as the effective region of the convergence control parameter we can take the interval $(0, 0.6)$. The same interval we receive for the other h-curves.

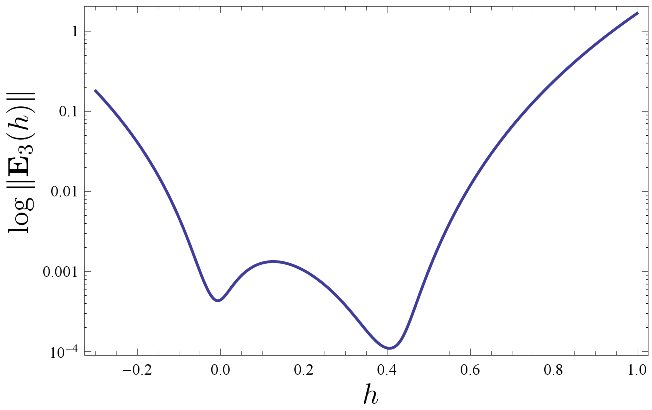

Fig. 2. Logarithm of the norm of squared residual

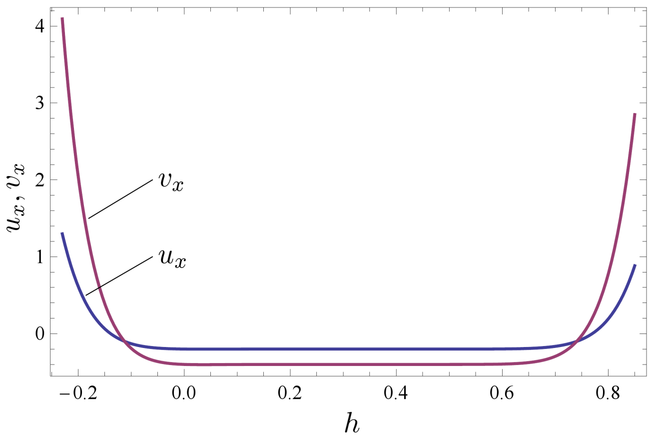

Fig. 3. The h-curves

In the example we were not able to find the sums of obtained series, therefore we present in Table 1 the absolute errors (Δ) and the relative percentage errors (δ) of approximating the exact solutions $u_{k,e}$ by functions $\widehat{u}_{k,n}$ for $k = 1, 2$. The errors of reconstruction of the temperature distribution θ and the heat flux q are collected in Table 2. As indicated by the example, if we have properly selected value of the convergence control parameter h then, if it is impossible to predict a general form of function $u_{k,m}$ or calculate the sum of series in (16), it is sufficient to make use of the sum of several first functions $u_{k,m}$ to provide a very good approximation of the sought solution. As revealed by the obtained results, together with an increase of the number of components in sum (20) the errors quickly decrease. In the presented example calculation of ten components ensures the absolute error at the level of 10^{-6} and the relative error at the level of $10^{-4}\%$ in case of the temperature and, respectively, 10^{-5} and $10^{-3}\%$ in case of the heat flux.

A plot of the reconstruction of initial conditions is displayed in Fig. 4. Next, Fig. 5 shows the plots of reconstruction errors for the missing boundary conditions (temperature θ and heat flux q). With regard to taken conditions (28)–(31) and (32)–(35) for systems of equations (26) and (27), the remaining conditions of the original problem are fulfilled exactly by each approximate solution.

Table 1
Errors of reconstruction of the temperature distributions (Δ – absolute error, δ – percentage relative error)

n	$\Delta\widehat{u}_{1,n}$	$\delta\widehat{u}_{1,n}$ [%]	$\Delta\widehat{u}_{2,n}$	$\delta\widehat{u}_{2,n}$ [%]
1	$1.40741 \cdot 10^{-2}$	1.16758	0.11313	14.25655
2	$1.60406 \cdot 10^{-3}$	0.13307	$7.36977 \cdot 10^{-3}$	0.92875
3	$2.32495 \cdot 10^{-4}$	$1.92876 \cdot 10^{-2}$	$4.52834 \cdot 10^{-3}$	0.57067
4	$4.26256 \cdot 10^{-5}$	$3.53619 \cdot 10^{-3}$	$8.66183 \cdot 10^{-4}$	0.10916
5	$9.62538 \cdot 10^{-6}$	$7.98515 \cdot 10^{-4}$	$5.22107 \cdot 10^{-4}$	$6.57969 \cdot 10^{-2}$
6	$2.59488 \cdot 10^{-6}$	$2.15269 \cdot 10^{-4}$	$1.78633 \cdot 10^{-4}$	$2.25116 \cdot 10^{-2}$
7	$8.13814 \cdot 10^{-7}$	$6.75134 \cdot 10^{-5}$	$8.12032 \cdot 10^{-5}$	$1.02334 \cdot 10^{-2}$
8	$2.91079 \cdot 10^{-7}$	$2.41477 \cdot 10^{-5}$	$3.01164 \cdot 10^{-5}$	$3.79533 \cdot 10^{-3}$
9	$1.16923 \cdot 10^{-7}$	$9.69984 \cdot 10^{-6}$	$1.29549 \cdot 10^{-5}$	$1.63260 \cdot 10^{-3}$
10	$5.21075 \cdot 10^{-8}$	$4.32280 \cdot 10^{-6}$	$4.82479 \cdot 10^{-6}$	$6.08029 \cdot 10^{-4}$

Table 2
Errors of reconstruction of the boundary conditions (Δ – absolute error, δ – percentage relative error)

n	$\Delta\widehat{\theta}_n$	$\delta\widehat{\theta}_n$ [%]	$\Delta\widehat{q}_n$	$\delta\widehat{q}_n$ [%]
1	0.20774	34.13608	0.41319	84.87152
2	$1.10863 \cdot 10^{-2}$	1.82175	$1.18744 \cdot 10^{-2}$	2.43906
3	$8.54057 \cdot 10^{-3}$	1.40342	$1.67003 \cdot 10^{-2}$	3.43034
4	$1.64159 \cdot 10^{-3}$	0.26975	$2.95667 \cdot 10^{-3}$	0.60731
5	$9.15306 \cdot 10^{-4}$	0.15041	$1.14107 \cdot 10^{-3}$	0.23438
6	$2.87311 \cdot 10^{-4}$	$4.72122 \cdot 10^{-2}$	$1.73445 \cdot 10^{-4}$	$3.56265 \cdot 10^{-2}$
7	$1.19841 \cdot 10^{-4}$	$1.96929 \cdot 10^{-2}$	$9.30739 \cdot 10^{-5}$	$1.91179 \cdot 10^{-2}$
8	$3.76351 \cdot 10^{-5}$	$6.18435 \cdot 10^{-3}$	$8.17854 \cdot 10^{-5}$	$1.67992 \cdot 10^{-2}$
9	$1.39789 \cdot 10^{-5}$	$2.29707 \cdot 10^{-3}$	$5.44008 \cdot 10^{-5}$	$1.11742 \cdot 10^{-2}$
10	$3.62801 \cdot 10^{-6}$	$5.96170 \cdot 10^{-4}$	$3.23537 \cdot 10^{-5}$	$6.64562 \cdot 10^{-3}$

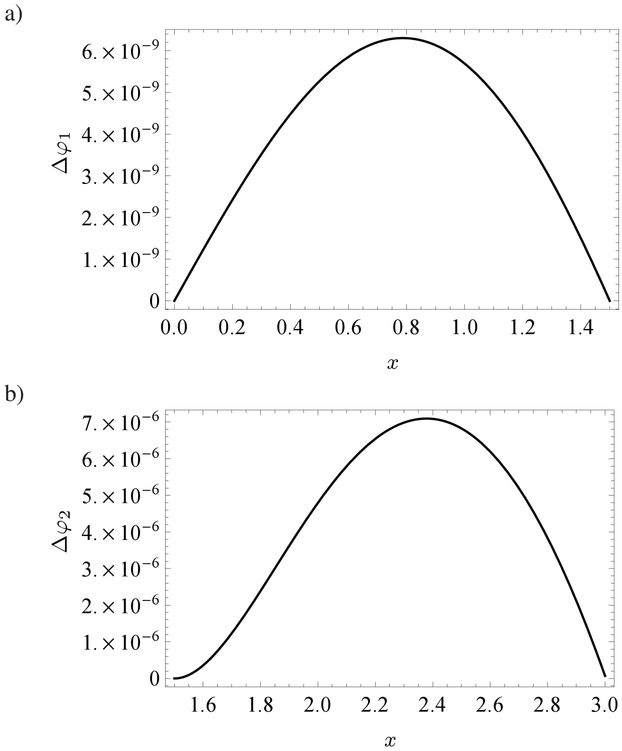

Fig. 4. Error of the 10-th order approximation of the initial conditions

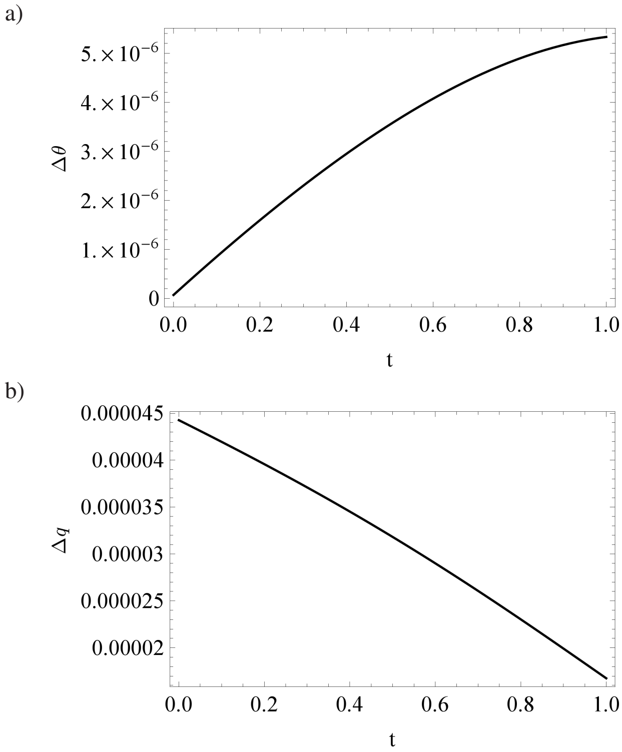

Fig. 5. Error of the 10-th order approximation of the boundary conditions

The calculations have been also executed for the perturbed input data. We have burdened the input data by the 0.5, 1.0 and 2.0% random error. In Table 3 we present the absolute and relative errors of reconstructing the temperature and the heat flux for various perturbations of input data and for the fifth-order approximate solution. Next, in Figs. 6 and 7 there are displayed the absolute errors of reconstruction of the boundary conditions in case of input data burdened by 1 and 2% error, obtained for the fifth-order approximate solution. On the basis of received results one can conclude that the investigated method is stable with regard to the errors of input data. Each time when the input data were burdened with errors, the error of the boundary conditions reconstruction did not exceed the error of the input data.

Table 3
Errors of the reconstructed boundary conditions for the burdened input data ($n = 5$, Δ – absolute error, δ – percentage relative error)

Error	$\Delta\widehat{\theta}_5$	$\delta\widehat{\theta}_5$ [%]	$\Delta\widehat{q}_5$	$\delta\widehat{q}_5$ [%]
0.5%	$1.26959 \cdot 10^{-3}$	0.20862	$1.15961 \cdot 10^{-3}$	0.23819
1.0%	$1.85554 \cdot 10^{-3}$	0.30491	$2.63330 \cdot 10^{-3}$	0.54089
2.0%	$4.57154 \cdot 10^{-3}$	0.79170	$7.90250 \cdot 10^{-3}$	1.62322

a)
b)

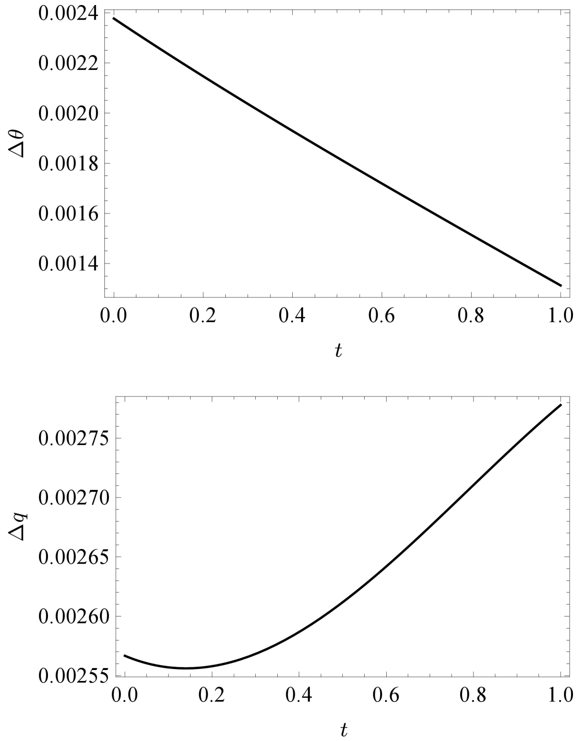

Fig. 6. Errors of reconstruction of the temperature (a) and the heat flux (b) on boundary Γ_4 for the input data burdened by 1% error (result obtained for the fifth-order approximate solution, $n = 5$)

a)
b)

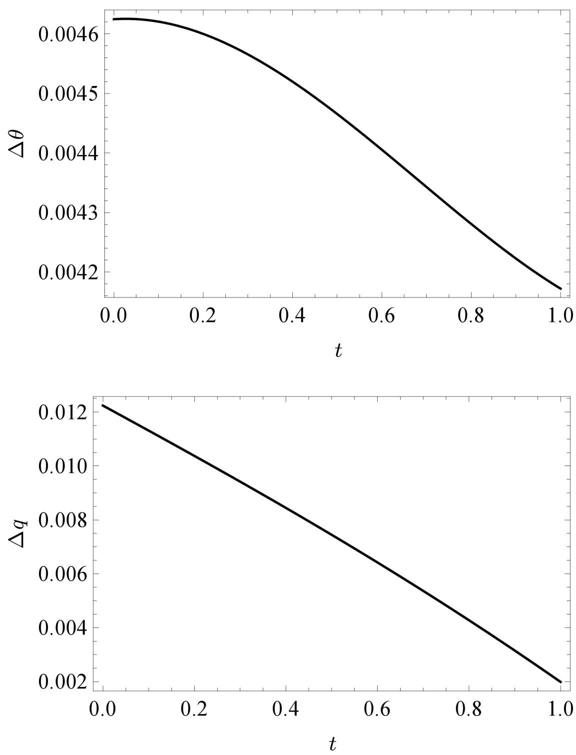

Fig. 7. Errors of reconstruction of the temperature (a) and the heat flux (b) on boundary Γ_4 for the input data burdened by 2% error (result obtained for the fifth-order approximate solution, $n = 5$)

6. Conclusions

In the paper we have presented the application of the homotopy analysis method for solving the two-phase inverse Stefan problem. A concept of the investigated method consists in creation of a series, terms of which satisfy the differential equation resulting from the considered task and easier to solve in comparison with the original equation. We have proven in the paper that if this series is convergent then its sum determines the solution of the considered equation. We have formulated the sufficient condition for this convergence and we have estimated the error of approximate solution obtained by taking only the partial sum of the discussed series. The received series are usually fast convergent, therefore the use of only several terms ensures very good approximation of the exact solution. Presented exemplary calculations show that this method is effective for solving the problems under consideration. Additionally, the method appears to be stable with regard to the input data errors. In each considered case the errors of reconstruction of the missing boundary conditions were smaller than the perturbations of input data. An advantage of the approach is the fact that discretization of the region is not required like it is happening in classical methods, for example in the finite difference method or finite element method. Another advantage is that the solution is obtained in the form of the continuous function which can be used for the further analysis.

REFERENCES

[1] M.N. Özişik, *Heat Conduction*, Wiley & Sons, New York, 1980.

[2] S.C. Gupta, *The Classical Stefan Problem. Basic Concepts, Modelling and Analysis*, Elsevier, Amsterdam, 2003.

[3] J. Hristov, "An inverse Stefan problem relevant to boilover: heat balance integral solutions and analysis", *Thermal Science* 11, 141–160 (2007).

[4] K. Okamoto and B.Q. Li, "A regularization method for the inverse design of solidification processes with natural convection", *Int. J. Heat Mass Transfer* 50, 4409–4423 (2007).

[5] H.-S. Ren, "Application of the heat-balance integral to an inverse Stefan problem", *Int. J. Therm. Sci.* 46, 118–127 (2007).

[6] D. Słota, "Solving the inverse Stefan design problem using genetic algorithms", *Inverse Probl. Sci. Eng.* 16, 829–846 (2008).

[7] D. Słota, "Identification of the cooling condition in 2-D and 3-D continuous casting processes", *Numer. Heat Transfer* B 55, 155–176 (2009).

[8] B. Johansson, D. Lesnic, and T. Reeve, "A method of fundamental solutions for the one-dimensional inverse Stefan problem", *Appl. Math. Modelling* 35, 4367–4378 (2011).

[9] B. Johansson, D. Lesnic, and T. Reeve, "Numerical approximation of the one-dimensional inverse Cauchy-Stefan problem using a method of fundamental solutions", *Inverse Probl. Sci. Eng.* 19, 659–677 (2011).

[10] D. Słota, "Restoring boundary conditions in the solidification of pure metals", *Comput. & Structures* 89, 48–54 (2011).

[11] C.-S. Liu, "Solving two typical inverse Stefan problems by using the Lie-group shooting method", *Int. J. Heat Mass Transfer* 54, 1941–1949 (2011).

[12] M. Marois, M. Désilets, and M. Lacroi, "What is the most suitable fixed grid solidification method for handling time-varying inverse Stefan problems in high temperature industrial furnaces?", *Int. J. Heat Mass Transfer* 55, 5471–5478 (2012).
[13] E. Hetmaniok, I. Nowak, D. Słota, and A. Zielonka, "Determination of optimal parameters for the immune algorithm used for solving inverse heat conduction problems with and without a phase change", *Numer. Heat Transfer* B 62, 462–478 (2012).
[14] E. Hetmaniok, D. Słota, and A. Zielonka, "Experimental verification of immune recruitment mechanism and clonal selection algorithm applied for solving the inverse problems of pure metal solidification", *Int. Comm. Heat & Mass Transf.* 47, 7–14 (2013).
[15] B. Johansson, D. Lesnic, and T. Reeve, "A meshless method for an inverse two-phase one-dimensional linear Stefan problem", *Inverse Probl. Sci. Eng.* 21, 17–33 (2013).
[16] D. Słota, "The application of the homotopy perturbation method to one-phase inverse Stefan problem", *Int. Comm. Heat & Mass Transf.* 37, 587–592 (2010).
[17] D. Słota, "Homotopy perturbation method for solving the two-phase inverse Stefan problem", *Numer. Heat Transfer* A 59, 755–768 (2011).
[18] R. Grzymkowski and D. Słota, "One-phase inverse Stefan problems solved by Adomian decomposition method", *Comput. Math. Appl.* 51, 33–40 (2006).
[19] D. Słota, "Direct and inverse one-phase Stefan problem solved by variational iteration method", *Comput. Math. Appl.* 54, 1139–1146 (2007).
[20] S. Liao, "Homotopy analysis method: a new analytic method for nonlinear problems", *Appl. Math. Mech. – Engl. Ed.* 19, 957–962 (1998).
[21] S. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman and Hall – CRC Press, Boca Raton, 2003.
[22] S. Liao, "On the homotopy analysis method for nonlinear problems", *Appl. Math. Comput.* 147, 499–513 (2004).
[23] S. Liao, *Homotopy Analysis Method in Nonlinear Differential Equations*, Springer Education Press, Berlin, 2012.
[24] T. Fan and X. You, "Optimal homotopy analysis method for nonlinear differential equations in the boundary leyer", *Numer. Algor.* 62, 337–354 (2013).
[25] S. Liao, "An optimal homotopy-analysis approach for strongly nonlinear differential equations", *Commun. Nonlinear Sci. Numer. Simulat.* 15, 2003–2015 (2010).
[26] A. Shidfar, A. Babaei, A. Molabahrami, and M. Alinejadmofrad, "Approximate analytical solutions of the nonlinear reaction-diffusion-convection problems", *Math. Comput. Modelling* 53, 261–268 (2011).
[27] R.A. Van Gorder, "Control of error in the homotopy analysis of semi-linear elliptic boundary value problems", *Numer. Algor.* 61, 613–629 (2012).
[28] S. Abbasbandy, "Homotopy analysis method for heat radiation equations", *Int. Comm. Heat & Mass Transf.* 34, 380–387 (2006).
[29] D. Chauhan, R. Agrawal, and P. Rastogi, "Magnetohydrodynamic slip flow and heat transfer in a porous medium over a stretching cylinder: homotopy analysis method", *Numer. Heat Transfer* A 62, 136–157 (2012).
[30] E. Hetmaniok, M. Pleszczyński, D. Słota, and A. Zielonka, "Usage of the homotopy analysis method for determining the temperature in the casting-mould system", *Hutnik* 81 (1), 50–54 (2014).
[31] E. Hetmaniok, D. Słota, R. Wituła, and A. Zielonka, "Solution of the one-phase inverse Stefan problem by using the homotopy analysis method", *Appl. Math. Modelling* (2015), (to be published).
[32] O. Abdulaziz, A. Bataineh, and I. Hashim, "On convergence of homotopy analysis method and its modification for fractional modified KdV equations", *J. Appl. Math. Comput.* 33, 61–81 (2010).
[33] M. Zurigat, S. Momani, Z. Odibat, and A. Alawneh, "The homotopy analysis method for handling systems of fractional differential equations", *Appl. Math. Modelling* 34, 24–35 (2010).
[34] D.W. Brzeziński and P. Ostalczyk, "High-accuracy numerical integration methods for fractional order derivatives and integrals computations", *Bull. Pol. Ac.: Tech.* 62 (4), 723–733 (2014).
[35] W. Mitkowski and P. Skruch, "Fractional-order models of the supercapacitors in the form of RC ladder networks", *Bull. Pol. Ac.: Tech.* 61 (3), 581–587 (2013).
[36] M. Sowa, "A subinterval-based method for circuits with fractional order elements", *Bull. Pol. Ac.: Tech.* 62 (3), 449–454 (2014).
[37] S. Abbasbandy and E. Shivanian, "A new analytical technique to solve Fredholm's integral equations", *Numer. Algor.* 56, 27–43 (2011).
[38] H. Vosughi, E. Shivanian, and S. Abbasbandy, "A new analytical technique to solve Volterra's integral equations", *Math. Methods Appl. Sci.* 34, 1243–1253 (2011).
[39] E. Hetmaniok, D. Słota, T. Trawiński, and R. Wituła, "Usage of the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind", *Numer. Algor.* 67 (2014), 163–185.
[40] M. Araghi and S. Behzadi, "Numerical solution of nonlinear Volterra-Fredholm integro-differential equations using homotopy analysis method", *J. Appl. Math. Comput.* 37, 1–12 (2011).
[41] M. Ghoreishi, A. Ismail, and A. Alomari, "Comparison between homotopy analysis method and optimal homotopy asymptotic method for nth-order integro-differential equation", *Math. Methods Appl. Sci.* 34, 1833–1842 (2011).
[42] E. Hetmaniok, D. Słota, T. Trawiński, and R. Wituła, "An analytical technique for solving general linear integral equations of the second kind and its application in analysis of flash lamp control circuit", *Bull. Pol. Ac.: Tech.* 62 (3), 413–421 (2014).
[43] Z. Odibat, "A study on the convergence of homotopy analysis method", *Appl. Math. Comput.* 217, 782–789 (2010).
[44] M. Turkyilmazoglu, "Convergence of the homotopy analysis method", *arXiv* 1006.4460v1, 1–12 (2010).
[45] M. Turkyilmazoglu, "Some issues on HPM and HAM: a convergence scheme", *Math. Comput. Modelling* 53, 1929–1936 (2011).
[46] K. Yabushita, M. Yamashita, and K. Tsuboi, "An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method", *J. Phys. A: Math. Theor.* 40, 8403–8416 (2007).
[47] F. Akyildiz and K. Vajravelu, "Magnetohydrodynamic flow of a viscoelastic fluid", *Phys. Lett.* A 372, 3380–3384 (2008).