LMI based fuzzy observer design for Takagi-Sugeno models containing vestigial nonlinear terms

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The paper deals with the problem of full order fuzzy observer design for the class of continuous-time nonlinear systems, represented by Takagi-Sugeno models containing vestigial nonlinear terms. On the basis of the Lyapunov stability criterion and the incremental quadratic inequalities, two design conditions for this kind of system model are outlined in the terms of linear matrix inequalities. A numerical example is given to illustrate the procedure and to validate the performances of the proposed approach.

Key words: Takagi-Sugeno models, fuzzy observer, linear matrix inequality, incremental quadratic inequality

1. Introduction

As is well known, observer design is a hot research field owing to its particular importance in the state-space control. The nonlinear system theory principles, using Lipschitz conditions, has emerged as a method capable of use in state estimation for nonlinear systems, although Lipschitz condition is a strong restrictive condition which many classes of systems may not satisfy. Design method for asymptotic observer for nonlinear systems with globally Lipschitz nonlinearities is presented, e.g., in [1], [4], [13], [22], the problem of designing asymptotic observers for the system whose nonlinear time-varying terms satisfy an incremental quadratic inequality, is given in [2], [3].

An alternative to design an observer for nonlinear systems is fuzzy modeling approach, which benefits from the advantages of the approximation techniques approximating nonlinear system model equations [20]. Stability conditions, relying on the feasibility of an associated system of linear matrix inequalities (LMI) and Takagi-Sugeno (TS) fuzzy model based nonlinear state observers, were educed, e.g., in [11], [17], [19].

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Controllers for TS nonlinear systems with time-varying terms, but with the structure exploiting local nonlinear models, were considered, e.g., in [5], [6].

In the paper, TS fuzzy models with vestigial nonlinear terms (VNT) are considered in a nonlinear state observer design task. The paper extends the method given in [15] and using the properties of incremental quadratic constraints for TS models with VNT as well as the Krasovskii theorem, it is demonstrated that an incremental quadratic inequality, parameterized by a multiplier matrix, can be reflected in an extended LMI form of design conditions. Since fewer control rules can be exploited, the proposed method mainly reduce computational burden which is often favorable for implementation.

The paper is sequenced in seven sections. Following the introduction in Section 1, basic nature of the TS fuzzy models is presented in Section 2. The preliminary results, focused on the definition of fuzzy state observers for TS models with VNT and on the incremental quadratic constraint inequality formulation, are presented in Section 3. Section 4 and Section 5 provide the stability analysis of the TS fuzzy state observer by use of LMIs, and explain the observer design conditions. Section 6 illustrates the observer design task by the numerical solution for both type of LMI forms and the last Section 7 draws some conclusion remarks.

Throughout the paper, the notations is narrowly standard in such way that x^T, X^T denotes the transpose of the vector x and matrix X, respectively, $X = X^T > 0$ means that X is a symmetric positive definite matrix, $rank(\cdot)$ remits the rank of a matrix, the symbol I_n indicates the n-th order unit matrix, $I\!R$ denotes the set of real numbers and $I\!R^{n \times r}$ refers to the set of all $n \times r$ real matrices.

2. Takagi-Sugeno fuzzy model

The systems under consideration devolve to the class of MIMO nonlinear dynamic continuous-time systems, described, using TS approach, as follows

$$\dot{q}(t) = \sum_{i=1}^{s} h_i(\theta(t))(A_i q(t) + B_i u(t) + G_i \mathbf{p}(t)), \tag{1}$$

$$y(t) = Cq(t), \tag{2}$$

where $q(t) \in I\!R^n$, $u(t) \in I\!R^r$, $y(t) \in I\!R^m$ are vectors of the state, input, and output variables, respectively, $C \in I\!R^{m \times n}$ $A_i \in I\!R^{n \times n}$, $B_i \in I\!R^{n \times r}$, $G_i \in I\!R^{n \times r_p}$, $i = 1, 2, \ldots, s$, are constant matrices, $t \in I\!R$ is the time variable, $h_i(\theta(t))$ is the weight for i-th rule, satisfying, by definition, the property

$$0 \leqslant h_i(\theta(t)) \leqslant 1, \quad \sum_{i=1}^{s} h_i(\theta(t)) = 1 \text{ for all } i \in \langle 1, \ldots, s \rangle \tag{3}$$

and

$$\theta(t) = \begin{bmatrix} \theta_1(t) & \theta_2(t) & \cdots & \theta_v(t) \end{bmatrix} \tag{4}$$

is the vector of the premise variables, where s, v are the numbers of fuzzy rules and premise variables, respectively. The nonlinear function $\mathbf{p}(t) \in \mathbb{R}^{r_p}$ is continuous and implicitly given by [2]

$$\mathbf{p}(t) = \varphi\big(Vq(t) + W\mathbf{p}(t)\big) \tag{5}$$

and $V \in \mathbb{R}^{m_p \times n}$, $W \in \mathbb{R}^{m_p \times r_p}$ are constant matrices.

It is supposed in the next that all premise variables are measurable and independent on $u(t)$ (more details can be found, e.g., in [14], [21]).

3. Preliminary results

Definition 3 *Considering* (1), (2), *and using the same set of membership function, the nonlinear state estimator is defined as*

$$\dot{q}_e(t) = \sum_{i=1}^{s} h_i(\theta(t)) \left(A_i q_e(t) + B_i u(t) + G_i \mathbf{p}_e(t)\right) + J_i(y(t) - y_e(t))), \tag{6}$$

$$y_e(t) = C q_e(t), \tag{7}$$

$$\mathbf{p}_e(t) = \varphi\big(V q_e(t) + W \mathbf{p}_e(t) + L(y(t) - y_e(t))\big), \tag{8}$$

where $q_e(t) \in \mathbb{R}^n$ is the estimation of the system state vector, $\mathbf{p}_e(t) \in \mathbb{R}^{r_p}$ is the estimation of the nonlinear function $\mathbf{p}(t)$ and $J_i \in \mathbb{R}^{n \times m}$, $i = 1, 2, \ldots, s$, and $L \in \mathbb{R}^{m_p \times m}$ is the set of the observer gain matrices, which has to be so designed that the observer is stable.

Proposition 1 *(incremental quadratic constraint) If a matrix $M \in \mathcal{M}$, where $\mathcal{M}$ is the set of real incremental multiplier matrices of dimension $(m_p + r_p) \times (m_p + r_p)$, then for given matrices $V \in \mathbb{R}^{m_p \times n}$, $W \in \mathbb{R}^{m_p \times r_p}$, $L \in \mathbb{R}^{m_p \times m}$ and $C \in \mathbb{R}^{m \times n}$ the incremental quadratic constraint is*

$$\begin{bmatrix} e^T(t) & \delta\mathbf{p}^T(t) \end{bmatrix} N \begin{bmatrix} e(t) \\ \delta\mathbf{p}(t) \end{bmatrix} \geqslant 0, \tag{9}$$

where

$$N = \begin{bmatrix} (V - LC)^T & 0 \\ 0 & I_{r_p} \end{bmatrix} Q^T M Q \begin{bmatrix} V - LC & 0 \\ 0 & I_{r_p} \end{bmatrix}, \tag{10}$$

$$Q = \begin{bmatrix} I_{m_p} & W \\ 0 & I_{r_p} \end{bmatrix} \tag{11}$$

and $I_{m_p} \in \mathbb{R}^{m_p \times m_p}$, $I_{r_p} \in \mathbb{R}^{r_p \times r_p}$ are the identity matrices.

Proof (compare, e.g., [3]) Defining the state estimate error

$$e(t) = q(t) - q_e(t), \tag{12}$$

(8) can be written as

$$\mathbf{p}_e(t) = \varphi\left(V(q(t) - e(t)) + W\mathbf{p}_e(t) + L(Cq(t) - C(q(t) - e(t)))\right) = $$
$$= \varphi\left(Vq(t) + W\mathbf{p}_e(t) - (V - LC)e(t)\right). \tag{13}$$

Introducing the variables

$$z_1(t) = Vq(t) + W\mathbf{p}(t), \tag{14}$$

$$z_2(t) = Vq(t) + W\mathbf{p}_e(t) - (V - LC)e(t), \tag{15}$$

it yields

$$\delta z(t) = z_1(t) - z_2(t) = (V - LC)e(t) + W\delta\mathbf{p}(t), \tag{16}$$

where

$$\delta\mathbf{p}(t) = \mathbf{p}(t) - \mathbf{p}_e(t). \tag{17}$$

Since now (5), (13) implies

$$\delta\mathbf{p}(t) = \mathbf{p}(t) - \mathbf{p}_e(t) = \varphi(z_1(t)) - \varphi(z_2(t)) = \delta\varphi(t), \tag{18}$$

writing (16), (18) compactly as

$$\begin{bmatrix} \delta z(t) \\ \delta\varphi(t) \end{bmatrix} = \begin{bmatrix} V - LC & W \\ 0 & I_{r_p} \end{bmatrix} \begin{bmatrix} e(t) \\ \delta\mathbf{p}(t) \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} \delta z(t) \\ \delta\varphi(t) \end{bmatrix} = \begin{bmatrix} I_{m_p} & W \\ 0 & I_{r_p} \end{bmatrix} \begin{bmatrix} V - LC & 0 \\ 0 & I_{r_p} \end{bmatrix} \begin{bmatrix} e(t) \\ \delta\mathbf{p}(t) \end{bmatrix}, \tag{20}$$

respectively, then (20) for a symmetric $M \in \mathcal{M}$ gives

$$\begin{bmatrix} \delta z^T(t) & \delta\varphi^T(t) \end{bmatrix} M \begin{bmatrix} \delta z(t) \\ \delta\varphi(t) \end{bmatrix} = \begin{bmatrix} e^T(t) & \delta\mathbf{p}^T(t) \end{bmatrix} N \begin{bmatrix} \delta e(t) \\ \delta\mathbf{p}(t) \end{bmatrix} \geqslant 0, \tag{21}$$

where, evidently, N takes the structure (10). This concludes the proof. $\qquad\square$

Note, if the nonlinear term $p(t)$ does not depends on the derivative of a state variable, W is the zero matrix.

4. Fuzzy observer design

To provide an asymptotic estimate of the system state, the design objective is to give conditions on the observer gain matrices J_i, $i = 1, 2, \ldots, s$, and L, which result in asymptotic decaying the estimate error (12).

Theorem 9 *The observer (6)-(8) is asymptotically stable if there exist symmetric positive definite matrices $P \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{m_p \times m_p}$, $Y \in \mathbb{R}^{r_p \times r_p}$ and matrices $Z \in \mathbb{R}^{m_p \times m}$, $Z_j \in \mathbb{R}^{n \times m}$ such that*

$$P = P^T > 0, \ X = X^T > 0, \ Y = Y^T > 0, \tag{22}$$

$$\begin{bmatrix} PA_i - Z_iC + A_i^T P - C^T Z_i^T & * & * \\ G_i^T P & -Y & * \\ XV - ZC & XW & -X \end{bmatrix} < 0, \tag{23}$$

for all $i \in \langle 1, 2, \ldots s \rangle$.

If the above conditions hold, the observer gain matrices can be found as

$$L = X^{-1}Z, \ J_i = P^{-1}Z_i, \ i = 1, 2, \ldots, s. \tag{24}$$

Here, and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof If the Lyapunov function is defined as

$$v(e(t)) = e^T(t)Pe(t) > 0, \tag{25}$$

where $P = P^T > 0$, $P \in \mathbb{R}^{n \times n}$, then evaluation of the time derivative of $v(e(t))$ along a observer trajectory leads to the result

$$\dot{v}(e(t)) = \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) < 0 \tag{26}$$

and using (1), (2) and (6), (7) it yields

$$\begin{aligned} \dot{e}(t) &= \sum_{i=1}^{s} h_i(\theta(t))(A_iq(t) + B_iu(t) + G_i\mathbf{p}(t)) - \\ &- \sum_{i=1}^{s} h_i(\theta(t))(A_iq_e(t) + B_iu(t) + G_i\mathbf{p}_e(t)) + J_i(y(t) - y_e(t)), \end{aligned} \tag{27}$$

$$\dot{e}(t) = \sum_{i=1}^{s} h_i(\theta(t))\big((A_i - J_iC)e(t) + G_i\delta\mathbf{p}(t)\big), \tag{28}$$

respectively. Substituting (28) in (26) results in

$$\begin{aligned} \dot{v}(e(t)) &= \sum_{i=1}^{s} h_i(\theta(t))\big((A_i - J_iC)e(t) + G_i\delta\mathbf{p}(t)\big)^T Pe(t) + \\ &+ \sum_{i=1}^{s} h_i(\theta(t))e^T(t)P\big((A_i - J_iC)e(t) + G_i\delta\mathbf{p}(t)\big) \end{aligned} \tag{29}$$

and with the notation

$$e^{\circ T}(t) = \begin{bmatrix} e^T(t) & \delta \mathbf{p}^T(t) \end{bmatrix} \tag{30}$$

the time derivative of $v(e(t))$ can be written as

$$\dot{v}(e(t)) = \sum_{i=1}^{s} \mathrm{h}_i e^{\circ T}(t) T_i^{\circ} e^{\circ}(t), \tag{31}$$

where

$$T_i^{\circ} = \begin{bmatrix} P(A_i - J_i C) + (A_i - J_i C)^T P & P G_i \\ G_i^T P & 0 \end{bmatrix}. \tag{32}$$

Since (32) is not of full structure, writing (9) and (30) as

$$e^{\circ T}(t) N e^{\circ}(t) \geqslant 0 \tag{33}$$

and using the Krasovskii theorem (see, e.g., [8], [12]), then (31) can be defined as

$$\dot{v}(e(t)) = \sum_{i=1}^{s} \mathrm{h}_i(\theta(t)) e^{\circ T}(t) T_i^{\circ} e^{\circ}(t) \leqslant -e^{\circ T}(t) N e^{\circ}(t) < 0, \tag{34}$$

which implies

$$v(e(t)) \leqslant \sum_{i=1}^{s} \mathrm{h}_i(\theta(t)) e^{\circ T}(t) (T_i^{\circ} + N) e^{\circ}(t) < 0, \tag{35}$$

$$T_i^{\circ} + N < 0 \text{ for all } i, \tag{36}$$

respectively.

Defining the incremental multiplier matrix as

$$M = \mathrm{diag} \begin{bmatrix} X & -Y \end{bmatrix}, \tag{37}$$

where $X = X^T > 0, X \in \mathbb{R}^{m_p \times m_p}, Y = Y^T > 0, Y \in \mathbb{R}^{r_p \times r_p}$, are symmetric positive definite matrices, then (10) implies

$$\begin{aligned} N &= \begin{bmatrix} (V - LC)^T & 0 \\ W^T & I_{r_p} \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \begin{bmatrix} V - LC & W \\ 0 & I_{r_p} \end{bmatrix} = \\ &= \begin{bmatrix} (V - LC)^T X (V - LC) & (V - LC)^T X W \\ W^T X (V - LC) & W^T X W - Y \end{bmatrix}, \end{aligned} \tag{38}$$

$$N = \begin{bmatrix} (V - LC)^T \\ W^T \end{bmatrix} X \begin{bmatrix} V - LC & W \end{bmatrix} - \begin{bmatrix} 0 \\ I_{r_p} \end{bmatrix} Y \begin{bmatrix} 0 & I_{r_p} \end{bmatrix}, \tag{39}$$

respectively. Introducing the matrix variables

$$Z = XL, \qquad Z_i = PJ_i, \tag{40}$$

(36) can be written as

$$
T_i + N = \begin{bmatrix} (XV - ZC)^T \\ W^T X \end{bmatrix} X^{-1} \begin{bmatrix} XV - ZC & XW \end{bmatrix} +
$$
$$
+ \begin{bmatrix} PA_i - Z_i C + A_i^T P - C^T Z_i^T & PG_i \\ G_i^T P & -Y \end{bmatrix} < 0
\tag{41}
$$

and applying Schur complement property, (41) implies (23). This concludes the proof. □

5. Enhanced design condition

In the previous section, was detailed how to find the fuzzy observer design condition ensuring its asymptotic stability. To extend the affine TS fuzzy model principle by introducing the slack matrix variables into LMIs, so the system matrices are decoupled from the Lyapunov matrix [10].

Theorem 10 *The observer* (6)-(8) *is asymptotically stable if there exist symmetric positive definite matrices* $P, S \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{m_p \times m_p}$, $Y \in \mathbb{R}^{r_p \times r_p}$, *matrices* $Z \in \mathbb{R}^{m_p \times m}$, $Z_j \in \mathbb{R}^{n \times m}$ *and a positive scalar* $\gamma > 0$, $\gamma \in \mathbb{R}$, *such that*

$$P = P^T > 0, \quad S = S^T > 0, \quad \gamma > 0, \tag{42}$$

$$X = X^T > 0, \qquad Y = Y^T > 0, \tag{43}$$

$$
\begin{bmatrix}
SA_i - Z_i C + A_i^T S - C^T Z_i^T & * & * & * \\
P - S + \gamma SA_i - \gamma Z_i C & -2\gamma S & * & * \\
G_i^T S & \gamma G_i^T S & -Y & * \\
XV - ZC & 0 & XW & -X
\end{bmatrix} < 0,
\tag{44}
$$

for all $i \in \langle 1, 2, \ldots s \rangle$.

If the above conditions hold, the observer gain matrices can be found as

$$L = X^{-1} Z, \quad J_i = S^{-1} Z_i, \ i = 1, 2, \ldots s. \tag{45}$$

Proof Since (28) implies

$$\sum_{i=1}^{s} h_i(\theta(t)) \big((A_i - J_i C) e(t) + G_i \delta \mathbf{p}(t) - \dot{e}(t) \big) = 0, \tag{46}$$

then, with arbitrary symmetric positive definite matrices $S_1, S_2 \in \mathbb{R}^{n \times n}$, it yields

$$\begin{bmatrix} e^T(t)S_1 & \dot{e}^T(t)S_2 \end{bmatrix} \sum_{i=1}^{s} h_i(\theta(t))\left((A_i - J_iC)e(t) + G_i\delta\mathbf{p}(t) - \dot{e}(t)\right) = 0. \tag{47}$$

Adding (47) as well as its transposition to (26) gives

$$\dot{v}(e(t)) = \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) +$$
$$+ \begin{bmatrix} e^T(t)S_1 & \dot{e}^T(t)S_2 \end{bmatrix} \sum_{i=1}^{s} h_i(\theta(t))\left((A_i - J_iC)e(t) + G_i\delta\mathbf{p}(t) - \dot{e}(t)\right) +$$
$$+ \sum_{i=1}^{s} h_i(\theta(t))\left((A_i - J_iC)e(t) + G_i\delta\mathbf{p}(t) - \dot{e}(t)\right)^T \begin{bmatrix} S_1e(t) & S_2\dot{e}(t) \end{bmatrix} < 0 \tag{48}$$

and with the notation

$$e^{\bullet T}(t) = \begin{bmatrix} e^T(t) & \dot{e}^T(t) & \delta\mathbf{p}^T(t) \end{bmatrix} \tag{49}$$

the time derivative of $v(e(t))$ can be written as

$$\dot{v}(e(t)) = \sum_{i=1}^{s} h_i(\theta(t))e^{\bullet T}(t)T_i^{\bullet}e^{\bullet}(t), \tag{50}$$

where

$$T_i^{\bullet} = \begin{bmatrix} S_1(A_i - J_iC) + (A_i - J_iC)^T S_1 & * & * \\ P - S_1 + S_2(A_i - J_iC) & -2S_2 & * \\ G_i^T S_1 & G_i^T S_2 & 0 \end{bmatrix}. \tag{51}$$

Using the incremental multiplier matrix (37), then (9) with respect to (49) implies

$$e^{\bullet T}(t)N^{\bullet}e^{\bullet}(t) \geqslant 0, \tag{52}$$

where

$$N^{\bullet} = \begin{bmatrix} (V - LC)^T \\ 0 \\ W^T \end{bmatrix} X \begin{bmatrix} V - LC & 0 & W \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ I_{r_p} \end{bmatrix} Y \begin{bmatrix} 0 & 0 & I_{r_p} \end{bmatrix} \tag{53}$$

and, analogously to (35), it has to be

$$v(e(t)) \leqslant \sum_{i=1}^{s} h_i e^{\bullet T}(t)(T_i^{\bullet} + N^{\bullet})e^{\bullet}(t) < 0, \tag{54}$$

$$T_i^{\bullet} + N^{\bullet} < 0, \tag{55}$$

$$
\begin{bmatrix}
S_1(A_i - J_iC) + (A_i - J_iC)^T S_1 & * & * & * \\
P - S_1 + S_2(A_i - J_iC) & -2S_2 & * & * \\
G_i^T S_1 & G_i^T S_2 & -Y & * \\
XV - XLC & 0 & XW & -X
\end{bmatrix} < 0,
\tag{56}
$$

respectively. Nominating the matrix variables

$$
S_1 = S, \ S_2 = \gamma S, \ Z = XL, \ Z_i = SJ_i
\tag{57}
$$

where $\gamma > 0$, $\gamma \in \mathbb{R}$, (56) implies (44). This concludes the proof. $\qquad\square$

The importance of Theorem 10 is that it separates the matrix P from the system matrices A_i, B_i, i.e. there are no terms containing product of P and any of them. This enables to derive less conservative design conditions with respect to natural affine properties of TS models.

6. Illustrative example

As an illustrative system model, the nonlinear dynamics of the ball-and-beam system, represented by the nonlinear fourth order state-space model, was adopted from [7] in the form

$$
\begin{aligned}
\dot{q}_1(t) &= q_2(t), & z(t) &= q_1(t), \\
\dot{q}_2(t) &= a\big(q_1(t)q_4^2(t) - g\sin(q_3(t))\big), & y_1(t) &= q_1(t), \\
\dot{q}_3(t) &= q_4(t), & y_2(t) &= q_4(t), \\
\dot{q}_4(t) &= u(t), & y_3(t) &= q_3(t),
\end{aligned}
$$

where the input variable $u(t)$ is the angular acceleration of the beam [rad/s^2], the output variable $y(t)$ is equal $q_1(t)$ and the measured variables are $q_1(t)$, $q_4(t)$ and $q_3(t)$, while $q_1(t)$ is the position of the ball [m], $q_2(t)$ is the velocity of the ball [m/s], $q_3(t)$ is the angle of the beam [rad] and $q_4(t)$ is the velocity of the beam [rad/s].

The nonlinear model parameters are

$$
a = \frac{m}{m + \frac{J}{r^2}} = 0.7143, \qquad g = 9.81,
$$

where

m - the mass of the ball $\qquad$ 0.11 kg,
J - the inertia mom. of the ball $\quad$ $1.76 * 10^{-5}$ kgm^2,
r - the radius of the ball $\qquad$ 0.02 m,
g - the gravitational constant $\quad$ 9.81 m/s^2.

Introducing the premise variable

$$\theta(t) = q_1(t)q_4(t)$$

which is bounded in the prescribed sector $q_1(t)q_4(t) \in \langle -d, d \rangle = \langle -5, 5 \rangle$, the associated sector functions, as well as the normalized membership functions, are

$$w_2(q_1(t)q_4(t)) = h_2(\theta(t)) = \begin{cases} 1, & \theta(t) \geqslant d, \\ \frac{1}{d}\theta(t), & 0 < \theta(t) < d, \\ 0, & \theta(t) \leqslant 0, \end{cases}$$

$$w_3(q_1(t)q_4(t)) = h_3(\theta(t)) = \begin{cases} 0, & \theta(t) \geqslant d, \\ -\frac{1}{d}\theta(t), & -d < \theta(t) < 0, \\ 1, & \theta(t) \leqslant -d, \end{cases}$$

$$w_1(q_1(t)q_4(t)) = h_1(\theta(t)) = 1 - h_2(\theta(t)) - h_3(\theta(t))$$

and the nonlinear function $\mathbf{p}(t)$ is given as

$$\mathbf{p}(t) = \sin(q_3(t)) = \sin\left(\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} q(t)\right) = \sin(Vq(t) + W\mathbf{p}(t)),$$

where

$$q^T(t) = \begin{bmatrix} q_1(t) & q_2(t) & q_3(t) & q_4(t) \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad W = 0.$$

Evidently, since both $q_1(t)$, $q_4(t)$ are measured, the premise variable $\theta(t) = q_1(t)q_4(t)$ can be computed.

Consequently, the representation of the nonlinear differential equations of the system in a TS fuzzy system model gives

$$\dot{q}(t) = \sum_{i=1}^{3} h_i(\theta(t))(A_i q(t) + bu(t) + g\mathbf{p}(t)),$$

$$y(t) = Cq(t), \quad z(t) = c_z^T q(t),$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & ad \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -ad \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$c_z^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad g^T = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note, the pairs (A_i, C), $i = 1, 2, 3$, are observable.

Now, the obtained observer design conditions of Theorem 9 and Theorem 10 are applied.

Using SeDuMi package for Matlab [9], [18] and solving (22), (23) for the matrix variables P, X, Y, Z, Z_i, $i = 1, 2, 3$, the task was feasible and the parameters were following

$$P = \begin{bmatrix} 0.7503 & -0.2604 & 0.0000 & 0.0000 \\ -0.2604 & 0.2523 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.6280 & -0.0848 \\ 0.0000 & 0.0000 & -0.0848 & 0.7598 \end{bmatrix},$$

$$X = 0.8258, \quad Y = 0.8258, \quad L = \begin{bmatrix} 0.0000 & 0.0046 & 0.9457 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 2.3545 & 0.0000 & 0.0000 \\ 4.9549 & 0.0000 & 0.0000 \\ 0.0000 & 0.5301 & 0.7720 \\ 0.0000 & 0.5471 & 0.4632 \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 2.3551 & 0.9264 & 0.0010 \\ 4.9342 & 4.2713 & 0.0136 \\ -0.0798 & 0.5314 & 0.7719 \\ -0.5577 & 0.5601 & 0.4628 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 2.3551 & -0.9264 & -0.0010 \\ 4.9342 & -4.2713 & -0.0136 \\ 0.0798 & 0.5314 & 0.7719 \\ 0.5577 & 0.5601 & 0.4628 \end{bmatrix},$$

by which the stable global observer was obtained with the sets of stable eigenvalue spectrum of subsystems

$$\rho(A_{e1}) = \{-0.6595 \pm 0.4528i \ -1.1773 \pm 1.8892i\},$$

$$\rho(A_{e2}) = \rho(A_{e3}) = \{-0.6771 \pm 0.4268i \ -1.1665 \pm 2.0191i\},$$

where $A_{ei} = A_i - J_i C$, $i = 1, 2, 3$.

The comparison among both design conditions is necessary. By the enhanced design conditions, i.e., by solving (42)–(44) for LMI variables P, S, X, Y, Z, Z_i, $i = 1, 2, 3$ and given $\delta = 2$, the following results were obtained

$$P = \begin{bmatrix} 0.6236 & -0.2111 & 0.0000 & 0.0000 \\ -0.2111 & 0.2747 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.3570 & -0.0317 \\ 0.0000 & 0.0000 & -0.0317 & 0.5061 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.1050 & -0.0649 & 0.0000 & 0.0000 \\ -0.0649 & 0.1531 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1610 & 0.0021 \\ 0.0000 & 0.0000 & 0.0021 & 0.1441 \end{bmatrix},$$

$$X = 0.3715, \quad Y = 0.3715, \quad L = \begin{bmatrix} 0.0000 & 0.0000 & 1.0000 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 3.0108 & 0.0000 & 0.0000 \\ 1.3983 & 0.0000 & 0.0000 \\ 0.0000 & 0.8134 & 0.2251 \\ 0.0000 & 1.1038 & -0.1398 \end{bmatrix}, \quad J_2 = J_3 = \begin{bmatrix} 3.0117 & 0.0000 & 0.0000 \\ 1.3987 & 0.0000 & 0.0000 \\ 0.0000 & 0.8134 & 0.2254 \\ 0.0000 & 1.1032 & -0.1398 \end{bmatrix},$$

$$\rho(A_{e1}) = \{-0.1964 - 0.5737 - 1.1326 - 2.4371\},$$

$$\rho(A_{e2}) = \rho(A_{e3}) = \{-0.1966 - 0.5737 - 1.1319 - 2.4380\}.$$

Comparing with the first approach, the enhanced method tends to produce the same subsystem observer gain matrices, which can radically reduce the fuzzy observer structures, since the result tends to be a linear observer for the nonlinear system. Moreover, the solution from Theorem 10 is less conservative than an equivalent solution from Theorem 9 and gives a strictly aperiodically observer state variables response.

It should be pointed out that the proposed technique, using TS models with VNT, might give more conservative results than the existing ones in some cases, but the advantage of them consists of designing a fuzzy observer with fewer rules and less computational burden. Compared with the standard algorithms [11], the number of premise variables in this example is one smaller and the number of rules was reduced from six to three.

7. Concluding remarks

Newly extended nonlinear fuzzy observer design principle, based on the TS state-space models with VNT, is presented in the paper. This is achieved by application of an enhanced Lyapunov inequality, reflecting the incremental quadratic constraint parameterized by a symmetric multiplier matrix, the Krasovskii theorem, as well as the Lyapunov matrix decoupling principle realized by using symmetric slack matrices and a tuning parameter. Since the stability conditions based on the standard form of the Lyapunov inequality are very conservative as a common symmetric positive definite matrix verifying all Lyapunov inequalities is required, the presented principle, naturally exploiting the affine properties of TS fuzzy models, strictly decouples Lyapunov matrix and the system parameter matrices in the resulting LMIs, and significantly reduces the conservativeness in the fuzzy observer design, comparing with the standard Lyapunov inequality approach.

In the presented version, the observer stability problem is solved considering premise variables determined from the set of measurable state variables and the main aim was to reduce the number of premise variables, such as fuzzy rules.

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