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Dynamic systems with a finite degrees of freedom number

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Taking as a starting point the law of conservation of the total energy of the system, and introducing two basic state functions - the Lagrangian and the Rayleigh function, the general form of the equation of motion for any dynamic system with a finite number of degrees of freedom is derived. The theory is illustrated by considering the rotating - type electromechanical energy converter with six degrees of freedom being the model of all essentially important types of DC and AC machines, including rotating power amplifiers, induction - and synchronous type motors - all of them discussed from both, the steady-state and the transient point of view. In the next part of the paper there is described a simple electric circuit with its model characterized by the holonomic constraints of the velocity-type. Finally, there is presented the kinematics and dynamics of the interesting mechanical system - the gyroscope placed on the rotating Earth.

Key words: analytical dynamics, electromechanical systems, circuit theory

1. Principles

Let us start by recalling the law of conservation of energy

$$\frac{d}{dt}(T+U)=P+\frac{\partial}{\partial t}(T+U) \tag{1}$$

where:

T denotes the kinetic energy of the system,

U is the potential energy of the system and

P is the power developed by non-potential forces.

Eq. (1) states that if the system is free from non-potential forces and its kinetic and potential energy do not depend explicitly on time, then during the motion the total energy of the system will be conserved at a constant, initially gained level.

Introducing the following n-dimensional vectors:

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q – a vector of generalized coordinates,

$\dot{q}$ – a vector of generalized velocities,

p – a vector of generalized momentums,

$\dot{p}$ – a vector of generalized inertia forces,

f – a vector of generalized potential forces,

$\bar{f}$ – a vector of generalized non-potential forces,

the quantities occurring in (1) take the form

$$T = \int_{t_0}^{t} \dot{q}^T \dot{p} dt = \begin{cases} \int_{p_0}^{p} \dot{q}^T(p,q,t)dp & := & T(p,q,t) \\ \dot{q}^T p - \int_0^{\dot{q}} d\dot{q}^T p(\dot{q},q,t) & := & \dot{q}^T p - T^*(\dot{q},q,t) \end{cases} \tag{2}$$

$$U = \int_{\tau_0}^{t} \dot{q}^T f dt = \int_{q_0}^{q} dq^T f(q,t) \; := \; U(q,t) \tag{3}$$

$$P = \dot{q}^T \bar{f} = \dot{q}^T \bar{f}(\dot{q},q,t) \tag{4}$$

where t_0 $(\neq \tau_0)$ is such that $\dot{q}(t_0) = 0$ and the Jacobian matrices of the vector functions $\dot{q}(p)$ and $f(q)$ satisfies the condition of symmetry

$$\frac{\partial \dot{q}}{\partial p} = \left(\frac{\partial \dot{q}}{\partial p}\right)^T, \qquad \frac{\partial f}{\partial q} = \left(\frac{\partial f}{\partial q}\right)^T \tag{5}$$

and $\dot{q}(p)$ is invertible, i.e. $\dot{q} = \dot{q}(p,q,t) \;\Rightarrow\; p = p(\dot{q},q,t)$. Thus, denoting

$$T(p,q,t) + U(q,t) := H(p,q,t) \tag{6}$$

$$T^*(\dot{q},q,t) - U(q,t) := L(\dot{q},q,t) \tag{7}$$

we get

$$H(p,q,t) = \dot{q}^T p - L(\dot{q},q,t). \tag{8}$$

Differentiating both sides of (8) successively with respect to p, q, t and $\dot{q}$, we obtain in addition the following formulae

$$\frac{\partial H}{\partial p} = \dot{q}^T, \qquad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \qquad p = \frac{\partial L}{\partial \dot{q}^T}. \tag{9}$$

According to the law of conservation of energy, (1) can be written in the form

$$D\left(\dot{q}^T \frac{\partial L}{\partial \dot{q}^T} - L\right) = \dot{q}^T \bar{f}(\dot{q},q,t) - \frac{\partial L}{\partial t} \tag{10}$$

where $\frac{d}{dt}$ is replaced by the symbol D.

The function $L(\dot{q},q,t)$ defined by (7) plays a most important role in dynamics and is called the Lagrangian of the system. Its first term $T^*(\dot{q},q,t)$ defined by (2) is called the kinetic coenergy of the system. The total energy, $H = T + U$, expressed as a function of p, q and t is known as the Hamiltonian of the system.

Now, to derive the equation of motion, let us proceed as follows:

$$\begin{aligned} DL &= \ddot{q}^T \frac{\partial L}{\partial \dot{q}^T} + \dot{q}^T \frac{\partial L}{\partial q^T} + \frac{\partial L}{\partial t} \\ &= \left[\ddot{q}^T \frac{\partial L}{\partial \dot{q}^T} + \dot{q}^T D\left(\frac{\partial L}{\partial \dot{q}^T}\right)\right] - \dot{q}^T D\left(\frac{\partial L}{\partial \dot{q}^T}\right) + \dot{q}^T \frac{\partial L}{\partial q^T} + \frac{\partial L}{\partial t} \\ &= D\left(\dot{q}^T \frac{\partial L}{\partial \dot{q}^T}\right) - \dot{q}^T \left[D\left(\frac{\partial L}{\partial \dot{q}^T}\right) - \frac{\partial L}{\partial q^T}\right] + \frac{\partial L}{\partial t}. \end{aligned} \tag{11}$$

Writing the last equation in the equivalent form, we have

$$\begin{aligned} D\left(\dot{q}^T \frac{\partial L}{\partial \dot{q}^T} - L\right) &= \dot{q}^T \left[D\left(\frac{\partial L}{\partial \dot{q}^T}\right) - \frac{\partial L}{\partial q^T}\right] - \frac{\partial L}{\partial t} \\ &= \dot{q}^T \left[D\left(\frac{\partial L}{\partial \dot{q}^T}\right) - \frac{\partial L}{\partial q^T} - \bar{f}(\dot{q},q,t)\right] + \dot{q}^T \bar{f}(\dot{q},q,t) - \frac{\partial L}{\partial t}. \end{aligned} \tag{12}$$

Comparing (10) with (12) and taking into account that they must hold for any $\dot{q}$ we draw a conclusion that

$$D\left(\frac{\partial L}{\partial \dot{q}^T}\right) - \frac{\partial L}{\partial q^T} = \bar{f}(\dot{q},q,t). \tag{13}$$

Since the terms occurring in the last equation represent the generalized forces, (13) describes a dynamic equilibrium of all of them, and as such, is the sought equation of motion [1].

Further, it will be convenient to extract from the vector of generalized non-potential forces $\bar{f}(\dot{q},q,t)$ such a vector, denoted here by $-\hat{f}(\dot{q},q,t)$, which, by the definition, satisfies the following condition of symmetry

$$\frac{\partial \hat{f}}{\partial \dot{q}} = \left(\frac{\partial \hat{f}}{\partial \dot{q}}\right)^T. \tag{14}$$

Denoting the remaining vector by $\widetilde{f}(\dot{q},q,t)$, we have

$$\bar{f}(\dot{q},q,t) = \widetilde{f}(\dot{q},q,t) - \hat{f}(\dot{q},q,t). \tag{15}$$

[1] Observe that by applying (9), (13) can be replaced by the equivalent system $\dot{q} = \frac{\partial H}{\partial p^T}$, $\dot{p} = -\frac{\partial H}{\partial q^T} + \bar{f}[\dot{q}(p,q,t),q,t]$ with the state described by the pair q, p.

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Now, on the basis of (14), we can define the function

$$R(\dot{q},q,t) = \int_0^{\dot{q}} d\dot{q}^T \hat{f}(\dot{q},q,t) \tag{16}$$

and observe that

$$\frac{\partial R}{\partial \dot{q}^T} = \hat{f}(\dot{q},q,t). \tag{17}$$

In result, the equation of motion can be written finally in the form

$$D\left(\frac{\partial L}{\partial \dot{q}^T}\right) - \frac{\partial L}{\partial q^T} + \frac{\partial R}{\partial \dot{q}^T} = \widetilde{f}(\dot{q},q,t). \tag{18}$$

The function $R(\dot{q},q,t)$ is a new state function called the Rayleigh function of the system.

Closing let us notice that, to the contrary of U, which is just a function of the variables q, t, the R-funcion, similarly to T^*-function, is basically dependent on the velocity vector $\dot{q}$. In a simple but rather typical situation, $\hat{f}$ is a linear function of $\dot{q}$

$$\hat{f} = B\dot{q} - u(t) \tag{19}$$

where its homogeneous part, $B\dot{q}$ with $B = B^T$, represents the vector of dissipative forces occurring in the system as a result of the viscous friction or some equivalent effect, and $u(t)$ represents a vector of the applied external forces. In such case R takes a form

$$R = \frac{1}{2}\dot{q}^T B\dot{q} - \dot{q}^T u(t) \tag{20}$$

being the difference of a quadratic and a linear form in $\dot{q}$; the first represents the half of the power dissipated in the system, and the second – the power developed by all external forces.

Finally, let us recall that the equation of motion (18) has been derived under the assumption that a dimensionality of the vector q is equal to the number n of degrees of freedom of the system. There are, however, some problems in dynamics, where it is either necessary or at least useful to describe the system by the vector q of dimensionality bigger than n, say $(n+m)$. In such case the components of vector q are no longer independent – they are tied up by m equations of constraints which generally are reducible to the form

$$G(q,t)\dot{q} + h(q,t) = 0 \qquad \begin{cases} G \in \Re^{m\times(n+m)} \\ h \in \Re^m \end{cases} \tag{21}$$

linear in $\dot{q}$. In particular, for the so-called holonomic systems the constraints formulated originally as

$$e(q,t) = 0, \qquad e \in \Re^m \tag{22}$$

after differentiation $D[e(q,t)] = 0$ are reduced to

$$\frac{\partial e(q,t)}{\partial q}\dot{q} + \frac{\partial e(q,t)}{\partial t} = 0 \tag{23}$$

i.e. to a special case of (21).

The effective method of dealing with holonomic [1] and non-holonomic systems, provided the last one is confined to the case [3]:

$$G(q,t) \equiv G(q), \qquad h(q,t) \equiv 0$$

is to modify the original Rayleigh function given by (16) to the form

$$R = \int_0^{\dot{q}} d\dot{q}^T \hat{f}(\dot{q},q,t) - [G(q,t)\dot{q} + h(q,t)]^T \lambda \tag{24}$$

and, finally, by removing the term independent of $\dot{q}$, to the following one

$$R = \int_0^{\dot{q}} d\dot{q}^T [\hat{f}(\dot{q},q,t) - G^T(q,t)\lambda] \tag{25}$$

where $\lambda \in \Re^m$ is the Lagrange multipliers' vector containing m undetermined functions of time.

The above procedure enables us to regard all $(n+m)$ components of the vector q as entirely independent and, in consequence, to apply the equation of motion in its unchanged form given by (18), which together with the equation of constraints in its original form (21) or (22) make a complete system of $(n+2m)$ scalar equations for the unknown functions of time: $(n+m)$ components of q and m components of λ. So, for R given by (25), the equation of motion (18) takes a form

$$D\left(\frac{\partial L}{\partial \dot{q}^T}\right) - \frac{\partial L}{\partial q^T} + \hat{f}(\dot{q},q,t) - G^T(q,t)\lambda = \widetilde{f}(\dot{q},q,t)$$

and finally

$$D\left(\frac{\partial L}{\partial \dot{q}^T}\right) - \frac{\partial L}{\partial q^T} = \bar{f}(\dot{q},q,t) + G^T(q,t)\lambda. \tag{26}$$

Let us now return to the law of conservation of energy (10), and by applying (15) and (17), let us rewrite it in the form

$$D\left(\dot{q}^T\frac{\partial L}{\partial \dot{q}^T} - L\right) = \dot{q}^T\left[\widetilde{f}(\dot{q},q,t) - \frac{\partial R}{\partial \dot{q}^T}\right] - \frac{\partial L}{\partial t}. \tag{27}$$

Assuming next that R is given by (25), the last equation, similarly to (18), holds for any system with constraints of the type (21). In consequence, we get

$$D\left(\dot{q}^T \frac{\partial L}{\partial \dot{q}^T} - L\right) = \dot{q}^T \left[\bar{f}(\dot{q}, q, t) + G^T(q, t)\lambda\right] - \frac{\partial L}{\partial t}$$

and

$$D\left(\dot{q}^T \frac{\partial L}{\partial \dot{q}^T} - L\right) = \dot{q}^T \bar{f}(\dot{q}, q, t) - \frac{\partial L}{\partial t} - h^T(q, t)\lambda. \tag{28}$$

Comparison of (13) with (26) and (10) with (28), shows how the constraints modify the equation of motion of the system and the law of conservation of its energy.

2. Example 1: The rotating-type electromechanical energy converter

To illustrate the theory, let us consider the general model of the rotating-type electromechanical energy converter which is characterized by p pairs of salient poles having for each pair two orthogonal stator windings with the number of turns equal to N_a and N_b, respectively, and three, symmetrically distributed rotor windings α, β and γ, each with the same number of turns equal to N, as shown in Fig. 1. Here $p\varphi$ denotes the so-called electrical angle of the rotor's position, where φ is the angle of its geometric position. The current and the flux linkage of each coil is denoted by the $\dot{q}$ and ψ, respectively, with the subscript a or b indicating the stator's coil, and the subscript α, β or γ indicating the rotor's coil. The remaining symbols are standard and their meaning is clear from context.

The power of the magnetic field of three rotor's coils is given by

$$p = \dot{q}_\alpha \dot{\psi}_\alpha + \dot{q}_\beta \dot{\psi}_\beta + \dot{q}_\gamma \dot{\psi}_\gamma \tag{29}$$

or writing the last expression in the vector-matrix notation, we get

$$p = \begin{bmatrix} \dot{q}_\alpha \\ \dot{q}_\beta \\ \dot{q}_\gamma \end{bmatrix}^T \begin{bmatrix} \dot{\psi}_\alpha \\ \dot{\psi}_\beta \\ \dot{\psi}_\gamma \end{bmatrix}. \tag{30}$$

Introducing next the following two linear orthonormal transformations of the current and voltage vectors [2]

$$\begin{bmatrix} \dot{q}_\alpha \\ \dot{q}_\beta \\ \dot{q}_\gamma \end{bmatrix}, \begin{bmatrix} u_\alpha \\ u_\beta \\ u_\gamma \end{bmatrix}, \begin{bmatrix} \dot{\psi}_\alpha \\ \dot{\psi}_\beta \\ \dot{\psi}_\gamma \end{bmatrix} = H\left(\begin{bmatrix} \dot{q}_A \\ \dot{q}_B \\ \dot{q}_0 \end{bmatrix}, \begin{bmatrix} u_A \\ u_B \\ u_0 \end{bmatrix}, \begin{bmatrix} \dot{\psi}_A \\ \dot{\psi}_B \\ \dot{\psi}_0 \end{bmatrix}\right), \tag{31}$$

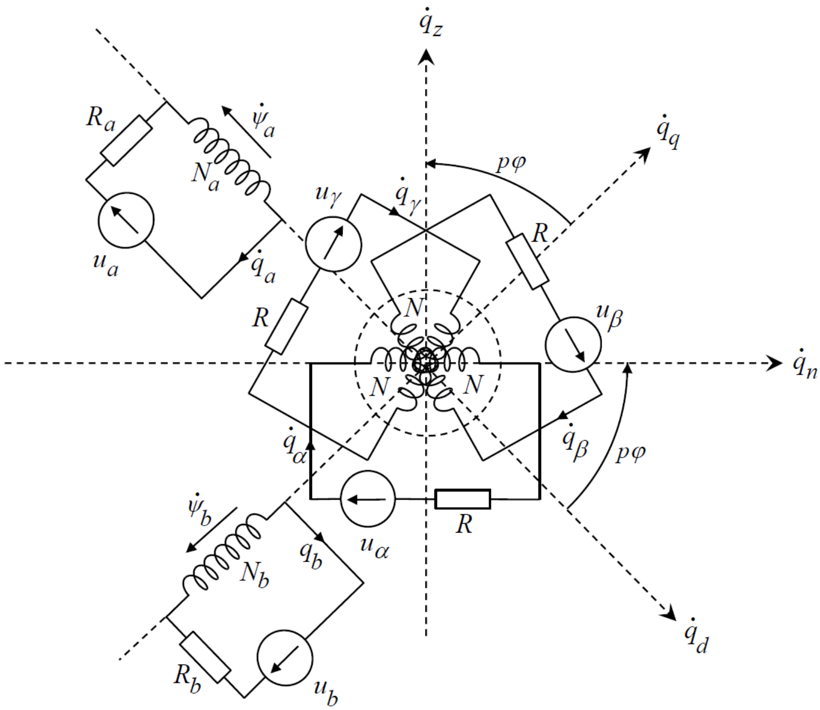

Figure 1. Model of the rotating - type electromechanical energy converter.

$$\begin{bmatrix} \dot{\psi}_A \\ \dot{\psi}_B \\ \dot{\psi}_0 \end{bmatrix}, \begin{bmatrix} \dot{q}_A \\ \dot{q}_B \\ \dot{q}_0 \end{bmatrix} \begin{bmatrix} u_A \\ u_B \\ u_0 \end{bmatrix} = G(\varphi) \left(\begin{bmatrix} \dot{\psi}_d \\ \dot{\psi}_q \\ \dot{\psi}_0 \end{bmatrix}, \begin{bmatrix} \dot{q}_d \\ \dot{q}_q \\ \dot{q}_0 \end{bmatrix} \begin{bmatrix} u_d \\ u_q \\ u_0 \end{bmatrix} \right) \tag{32}$$

where

$$H = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 & \frac{\sqrt{2}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad H^{-1} = H^T \tag{33}$$

$$G(\varphi) = \begin{bmatrix} \cos p\varphi & \sin p\varphi & 0 \\ -\sin p\varphi & \cos p\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G^{-1}(\varphi) = G^T(\varphi) \tag{34}$$

expression (30) can be modified to

$$p = \begin{bmatrix} \dot{q}_A \\ \dot{q}_B \\ \dot{q}_0 \end{bmatrix}^T \begin{bmatrix} \dot{\psi}_A \\ \dot{\psi}_B \\ \dot{\psi}_0 \end{bmatrix} = \begin{bmatrix} \dot{q}_A \\ \dot{q}_B \\ \dot{q}_0 \end{bmatrix}^T G(\varphi) \begin{bmatrix} \dot{\psi}_d \\ \dot{\psi}_q \\ \dot{\psi}_0 \end{bmatrix}. \tag{35}$$

So, the time integral

$$T_{ROT} = \int_{t_0}^{t} \begin{bmatrix} \dot{q}_A \\ \dot{q}_B \\ \dot{q}_0 \end{bmatrix}^T \begin{bmatrix} \cos p\varphi & \sin p\varphi & 0 \\ -\sin p\varphi & \cos p\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi}_d \\ \dot{\psi}_q \\ \dot{\psi}_0 \end{bmatrix} dt \tag{36}$$

$$= \int_{\psi_d(t_0)}^{\psi_d} (\dot{q}_A \cos p\varphi - \dot{q}_B \sin p\varphi)\, d\psi_d + \int_{\psi_q(t_0)}^{\psi_q} (\dot{q}_A \sin p\varphi + \dot{q}_B \cos p\varphi)\, d\psi_q + \int_{\psi_0(t_0)}^{\psi_0} \dot{q}_0 d\psi_0$$

represents the magnetic energy stored within the three rotor's windings.

Let us observe that by treating the current $\dot{q}$ as a generalized velocity and the flux linkage ψ as a generalized momentum, expression (36) represents the kinetic energy of the considered subsystem. Thus, according to (2), the corresponding coenergy takes a form

$$\begin{aligned} T^*_{ROT} = & \int_0^{\dot{q}_A \cos p\varphi - \dot{q}_B \sin p\varphi} \psi_d d\left(\dot{q}_A \cos p\varphi - \dot{q}_B \sin p\varphi\right) \\ & + \int_0^{\dot{q}_A \sin p\varphi + \dot{q}_B \cos p\varphi} \psi_q d\left(\dot{q}_A \sin p\varphi + \dot{q}_B \cos p\varphi\right) + \int_0^{\dot{q}_0} \psi_0 d\dot{q}_0. \end{aligned} \tag{37}$$

Adding to it the kinetic coenergy (=energy) of the mechanical subsystem $(\frac{1}{2}J\dot{\varphi}^2)$ as well as two simple expressions describing the magnetic coenergy related with the stator coils, and taking into account that the discussed model is free from any potential energy, the sought Lagrangian takes a form

$$\begin{aligned} \mathcal{L}(\varphi,\dot{\varphi},\dot{q}_a,\dot{q}_b,\dot{q}_A,\dot{q}_B,\dot{q}_0) = & \frac{1}{2}J\dot{\varphi}^2 + \int_0^{\dot{q}_a} \psi_a d\dot{q}_a + \int_0^{\dot{q}_b} \psi_b d\dot{q}_b + \int_0^{\dot{q}_0} \psi_0 d\dot{q}_0 \\ & + \int_0^{\dot{q}_A \cos p\varphi - \dot{q}_B \sin p\varphi} \psi_d d(\dot{q}_A \cos p\varphi - \dot{q}_B \sin p\varphi) \\ & + \int_0^{\dot{q}_A \sin p\varphi + \dot{q}_B \cos p\varphi} \psi_q d(\dot{q}_A \sin p\varphi + \dot{q}_B \cos p\varphi) \end{aligned} \tag{38}$$

where each flux linkage is a given function of the currents.

Assuming next that, besides J – the moment of inertia of the rotor, the remaining mechanical part of the model is characterized by B – the parameter of its viscous friction and $f(t)$ – the external torque applied to the rotor, and that its electrical part is characterized by the rotor's variables transformed according to (31), the Rayleigh function takes a form (cf. (20))

$$\begin{aligned}\mathcal{R}(\dot\varphi,\dot q_a,\dot q_b,\dot q_A,\dot q_B,\dot q_0,t)&=\frac{1}{2}\left[B\dot\varphi^2+R_a\dot q_a^2+R_b\dot q_b^2+R\left(\dot q_A^2+\dot q_B^2+\dot q_0^2\right)\right]\\&-\left[\dot\varphi f(t)+\dot q_a u_a(t)+\dot q_b u_b(t)+\dot q_A u_A(t)+\dot q_B u_B(t)+\dot q_0 u_0(t)\right].\end{aligned} \tag{39}$$

So, introducing $\mathcal{L}$ and $\mathcal{R}$ given by (38) and (39) into the equation of motion (18), and taking into account that $\tilde f\dot q,q,t)\equiv 0$ and there are no constraints imposed on the coordinates and velocities, we get the model which represents the dynamic system with six degrees of freedom described by the following set of equations

$$D\psi_a+R_a\dot q_a=u_a(t) \tag{40}$$

$$D\psi_b+R_b\dot q_b=u_b(t) \tag{41}$$

$$\begin{bmatrix}\cos p\varphi & \sin p\varphi\\ -\sin p\varphi & \cos p\varphi\end{bmatrix}\begin{bmatrix}D\psi_d\\ D\psi_q\end{bmatrix}-p\dot\varphi\begin{bmatrix}\sin p\varphi & -\cos p\varphi\\ \cos p\varphi & \sin p\varphi\end{bmatrix}\begin{bmatrix}\psi_d\\ \psi_q\end{bmatrix}+$$

$$+R\begin{bmatrix}\dot q_A\\ \dot q_B\end{bmatrix}=\begin{bmatrix}u_A(t)\\ u_B(t)\end{bmatrix} \tag{42, 43}$$

$$D\psi_0+R\dot q_0=u_0(t) \tag{44}$$

$$J\ddot\varphi+B\dot\varphi-p[\psi_q(\dot q_A\cos p\varphi-\dot q_B\sin p\varphi)-\psi_d(\dot q_A\sin p\varphi+\dot q_B\cos p\varphi)]=f(t) \tag{45}$$

and which after pre-multiplying both sides of its two equations (42) and (43) by the orthonormal matrix $\begin{bmatrix}\cos p\varphi & -\sin p\varphi\\ \sin p\varphi & \cos p\varphi\end{bmatrix}$ and introducing the notation $\dot q:=i$ and $\dot\varphi:=\Omega$, we obtain in the light of (32), the following final result

$$D\psi_a+R_a i_a=u_a(t) \tag{46}$$

$$D\psi_b+R_b i_b=u_b(t) \tag{47}$$

$$D\psi_d+Ri_d+p\psi_q\Omega=u_d(t) \tag{48}$$

$$D\psi_q+Ri_q-p\psi_d\Omega=u_q(t) \tag{49}$$

$$D\psi_0+Ri_0=u_0(t) \tag{50}$$

$$(JD+B)\Omega-p(\psi_q i_d-\psi_d i_q)=f(t) \tag{51}$$

where the flux linkages are monotonically increasing functions of their arguments

$$\left.\begin{aligned}\psi_a&=mf_{ad}(mi_a+i_d)+f_a(i_a)\\ \psi_d&=f_{ad}(mi_a+i_d)+f_d(i_d)\end{aligned}\right\}\quad \frac{\partial\psi_a}{\partial i_d}=m\frac{\partial f_{ad}}{\partial i_d}=\frac{\partial\psi_d}{\partial i_a} \tag{52, 53}$$

$$\left.\begin{aligned}\psi_b &= n f_{bq}\left(n i_b + i_q\right) + f_b(i_b)\\ \psi_q &= f_{bq}\left(n i_b + i_q\right) + f_q(i_q)\end{aligned}\right\} \quad \frac{\partial\psi_b}{\partial i_q} = n\frac{\partial f_{bq}}{\partial i_q} = \frac{\partial\psi_q}{\partial i_b} \tag{54, 55}$$

$$\psi_0 = f_0(i_0) \tag{56}$$

with

$$m := \frac{N_a}{N}, \quad n := \frac{N_b}{N}. \tag{57}$$

The obtained system of (46)-(57) together with (31)-(34) forms a basis for studying the static and dynamic properties of all, essentially important, types of electrical machines with their magnetic characteristics both linear and nonlinear. For the linear case, i.e. when (52)-(56) are reduced to

$$\begin{bmatrix}\psi_a\\ \psi_b\\ \psi_d\\ \psi_q\\ \psi_0\end{bmatrix} = \begin{bmatrix} L_a & 0 & M_a & 0 & 0\\ 0 & L_b & 0 & M_b & 0\\ M_a & 0 & L_d & 0 & 0\\ 0 & M_b & 0 & L_q & 0\\ 0 & 0 & 0 & 0 & L_0\end{bmatrix}\begin{bmatrix} i_a\\ i_b\\ i_d\\ i_q\\ i_0\end{bmatrix}. \tag{58}$$

Equations (46)-(51) take a form

$$\begin{bmatrix} L_aD+R_a & 0 & M_aD & 0 & 0\\ 0 & L_bD+R_b & 0 & M_bD & 0\\ M_aD & pM_b\Omega & L_dD+R & pL_q\Omega & 0\\ -pM_a\Omega & M_bD & -pL_d\Omega & L_qD+R & 0\\ 0 & 0 & 0 & 0 & L_0D+R\end{bmatrix}\begin{bmatrix} i_a\\ i_b\\ i_d\\ i_q\\ i_0\end{bmatrix} = \begin{bmatrix} u_a(t)\\ u_b(t)\\ u_d(t)\\ u_q(t)\\ u_0(t)\end{bmatrix} \tag{59, 60}$$

$$(JD+B)\,\Omega - p\left[\left(M_b i_b + L_q i_q\right) i_d - \left(M_a i_a + L_d i_d\right) i_q\right] = f(t). \tag{61}$$

This means that for the steady-state operation of the machine, i.e. for

$$\Omega = \text{const.} \tag{62}$$

the electrical part of the model is described by the system of four linear differential equations with constant coefficients and with four unknown functions of time, two stator currents i_a and i_b and two rotor currents: i_d colinear with i_a and i_q colinear with i_b, while the fifth equation with unknown i_0 is completely independent from the remaining equations, and where L_0 denotes the leakage inductance of the rotor's coil.

In (61), which evolves from (51) and which physically expresses the equilibrium of all torques acting on the rotor, besides the standard terms, there is a term of special importance denoted by the symbol

$$f_e := p\left(\psi_q i_d - \psi_d i_q\right) = p\left[\left(M_b i_b + L_q i_q\right) i_d - \left(M_a i_a + L_d i_d\right) i_q\right] \tag{63}$$

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which represents the mechanical torque produced by the interaction of the stator's and rotor's currents.

Let us observe that if we are treating the voltages $u_d(t)$ and $u_q(t)$ not as a result of the formal transformation defined by (31) and (32) but just as the DC voltages applied directly to the pair of brushes, being in contact with the commutator, and located at the rotor's d and q axes colinear with the stator's a and b axes, respectively (see Fig. 1), then the system of (59), (61) describes directly the wide spectrum of the ideal DC machines – both motors and generators. Let us also note that for DC machines the voltage $u_0(t) \equiv 0$ and, in consequence, the current $i_0 = 0$, too.

At this point, the following remark is necessary. Comparison of the real DC machine with its mathematical description represented by (59) and (61) indicates that the model is in full agreement with the origin, provided the inductances pM_a, pM_b, pL_d, pL_q and pM_{q1} (see Fig. 2) are replaced by the more precise and experimentally proved quantities known as the speed coefficients K_a, K_b, K_d, K_q and K_1. So, for the real DC machine (59) and (61) should be modified to the form

$$\begin{bmatrix} L_aD+R_a & 0 & M_aD & 0 \\ 0 & L_bD+R_b & 0 & M_bD \\ M_aD & K_b\Omega & L_dD+R & K_q\Omega \\ -K_a\Omega & M_bD & -K_d\Omega & L_qD+R \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_d \\ i_q \end{bmatrix} = \begin{bmatrix} u_a(t) \\ u_b(t) \\ u_d(t) \\ u_q(t) \end{bmatrix} \tag{64}$$

$$(JD+B)\,\Omega - [(K_bi_b+K_qi_q)\,i_d - (K_ai_a+K_di_d)\,i_q] = f(t). \tag{65}$$

As an example of this type of machines, let us consider at first the simplest situation when the stator's circuit "a" does not exist and the rotor's circuit "q" is open, then the model described by (64), (65) reduces to the form

$$\begin{bmatrix} L_bD+R_b & 0 \\ K_b\Omega(t) & L_dD+R \end{bmatrix} \begin{bmatrix} i_b \\ i_d \end{bmatrix} = \begin{bmatrix} u_b(t) \\ u_d(t) \end{bmatrix} \tag{66}$$

$$(JD+B)\,\Omega\,(t) - K_bi_bi_d = f(t) \tag{67}$$

which for $\Omega(t) = \Omega = \text{const.}$, $u_{in} = u_b(t)$, $u_{out} = u_d(t)$ and $i_{out} = -i_d$ describes the single-stage rotating power amplifier, and for $i_b = I_b = \text{const.}$, $u_{in} = u_d(t)$, $\Omega_{out} = \Omega(t)$ and $f(t) < 0$ – the armature controlled DC motor. Both of them connected by their armature circuits form one of the most popular driving system with its input $u_{in}(t) = u_b(t)$ being the voltage applied to the stator's coil of the generator and its output $\Omega_{out}(t)$ – the angular velocity of the motor.

To describe more advanced DC machines, for example the rotating power amplifier known as the amplidyne with its diagram shown in Fig. 2, let us at first modify (58) to

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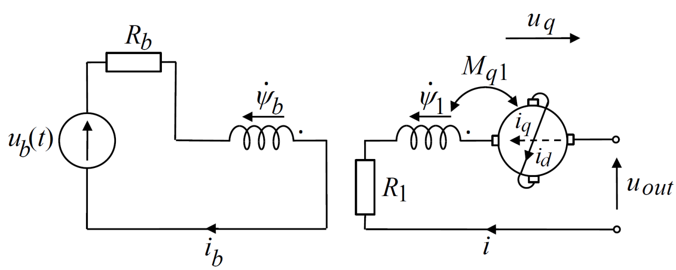

Figure 2. Schematic diagram of the amplidyne.

the form

$$\begin{bmatrix}\psi_d\\ \psi_q\\ \psi_1\\ \psi_b\end{bmatrix}=\begin{bmatrix}L_d & 0 & 0 & 0\\ 0 & L_q & M_{q1} & M_{qb}\\ 0 & M_{q1} & L_1 & M_{1b}\\ 0 & M_{qb} & M_{1b} & L_b\end{bmatrix}\begin{bmatrix}i_d\\ i_q=-i\\ i\\ i_b\end{bmatrix}=\begin{bmatrix}L_d & 0 & 0\\ 0 & M_{q1}-L_q & M_{qb}\\ 0 & L_1-M_{q1} & M_{1b}\\ 0 & M_{1b}-M_{qb} & L_b\end{bmatrix}\begin{bmatrix}i_d\\ i\\ i_b\end{bmatrix} \tag{68}$$

and taking $\Omega(t)=-\Omega=\text{const.}$, let us observe (Fig. 2) that

$$u_b(t)=u_{in},\quad u_d(t)=0,\quad u_q(t)=u_{out}+\dot{\psi}_1+R_1 i. \tag{69}$$

So, applying (47)-(49) and (69), we get

$$D\psi_b+R_b i_b=u_{in} \tag{70}$$

$$D\psi_d+Ri_d+p\psi_q\Omega=0 \tag{71}$$

$$D\psi_q-Ri-p\psi_d\Omega=u_{out}+D\psi_1+R_1 i. \tag{72}$$

Representing next the four flux linkages occurring in (70)-(72) by the right-hand side of (68) and replacing the inductances (multiplied by p) by the corresponding speed coefficients and, finally, writing the result in specially convenient matrix form, we obtain

$$\begin{bmatrix}1 & K_d\Omega & (M_{1b}-M_{qb})D\\ 0 & L_dD+R & k_b\Omega\\ 0 & 0 & L_bD+R_b\end{bmatrix}\begin{bmatrix}u_{out}\\ i_d\\ i_b\end{bmatrix}=\begin{bmatrix}0 & (L_1-2M_{q1}+L_q)D+(R_1+R)\\ 0 & (K_1-K_q)\Omega\\ 1 & (M_{1b}-M_{qb})D\end{bmatrix}\begin{bmatrix}u_{in}\\ -i\end{bmatrix} \tag{73}$$

From the third equation of the system (73) it follows that to eliminate the reaction of the output circuit on its input it is sufficient to connect in series with the output terminals the additional coil denoted by the subscript "1" which is colinear with the b-axis of the model and which satisfies the condition

$$M_{1b}=M_{qb}. \tag{74}$$

In result, the transfer function of the amplidyne denoted by $T(s)$ and its output impedance denoted by $Z(s)$ are, as follows from (73), (74), of the form

$$\begin{aligned} u_{out}(s) &= \frac{K_d K_b \Omega^2}{(L_d s + R)(L_b s + R_b)} u_{in}(s) + \\ &- \left[(L_1 - 2M_{q1} + L_q) s + (R_1 + R) - \frac{(K_1 - K_q) K_d \Omega^2}{L_d s + R} \right] i(s) := T(s) u_{in}(s) - Z(s) i(s). \end{aligned} \tag{75}$$

Let us observe that the voltage gain is proportional to the square of the angular velocity - the property being a result of placing two amplifying stages in one unit - first from b to d, and the second from d to q.

Now, let us turn our attention to AC machines. To be in agreement with practice, let us interchange the role of the stator and the rotor, i.e. the three-phase winding let us identify with the stator and the two-phase winding with the rotor. It means that in all equations applied up to (61) the angle φ must be replaced by $-\varphi$ and, in consequence, $\dot{\varphi} = \Omega$ by $\dot{\varphi} = -\Omega$. Let us further remind that any asymmetrical three-phase system of voltages and currents is equivalent to the superposition of the following three:

- The symmetrical three-phase positive system, i.e. producing torque in the expected direction,
- the symmetrical three-phase negative system, i.e. producing torque in opposite direction,
- the one-phase system producing no torque.

Since the action of the first and the second is similar, but just opposite, let us confine to the first and the third. So, if the vector of input voltages applied to the three stator's coils is of the form

$$\begin{bmatrix} u_\alpha \\ u_\beta \\ u_\gamma \end{bmatrix} = \sqrt{\frac{2}{3}} |U| \begin{bmatrix} \cos \omega t \\ \cos\left(\omega t - \frac{2\pi}{3}\right) \\ \cos\left(\omega t + \frac{2\pi}{3}\right) \end{bmatrix} + \frac{1}{\sqrt{3}} U_0 \sin(\omega t + \varepsilon) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{76}$$

then, according to formula (31), the vector $[u_A, u_B, u_0]^T$ is given by

$$\begin{bmatrix} u_A \\ u_B \\ u_0 \end{bmatrix} = |U| \begin{bmatrix} \cos \omega t \\ \sin \omega t \\ 0 \end{bmatrix} + U_0 \sin(\omega t + \varepsilon) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{77}$$

and, finally, by applying the transformation (32) with φ replaced by $-\varphi$, we get

$$\begin{bmatrix} u_d \\ u_q \\ u_0 \end{bmatrix} = |U| \begin{bmatrix} \cos(\omega t - p\varphi) \\ \sin(\omega t - p\varphi) \\ 0 \end{bmatrix} + U_0 \sin(\omega t + \varepsilon) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{78}$$

At first, let us observe that the unbalanced three-phase system produces the voltage of the form $u_0 = U_0 \sin(\omega t + \varepsilon)$ which has no effect on the rotor's torque but, due to a rather small value of the leakage inductance L_0 results in a relatively big value of the current i_0 (cf. (60)) and, in consequence, in dangerous heating of the stator's windings.

Now, let us concentrate on the steady-state operation of the discussed model, i.e. when $\dot{\varphi} = \Omega = \text{const.}$, so that

$$\begin{aligned} p\varphi &= p\Omega t - \delta \\ \omega t - p\varphi &= \left(1 - \frac{p\Omega}{\omega}\right)\omega t + \delta := s\omega t + \delta \end{aligned} \tag{79}$$

$$s = 1 - \frac{p\Omega}{\omega}. \tag{80}$$

Parameters δ and s are so-called torque angle and the slip, respectively. In connection with the last notion, we must distinguish between two types of AC machines or, more precisely, two types of AC motors:

1. induction type motors with both rotor coils short-circuited leading to $s \in (0, 1)$,
2. synchronous type motors with one rotor coil short-circuited and the other supplied from the DC voltage source forcing $s = 0$.

For the first type, the following constraints are typical:

$$\begin{matrix} R_a = R_b & L_a = L_q & u_a(t) = 0 \\ L_a = L_b & M_a = M_b & u_b(t) = 0 \end{matrix} \tag{81}$$

and the voltages $u_d(t)$ and $u_q(t)$ given by (78) will be presented in their exponential form

$$\begin{bmatrix} u_d(t) \\ u_q(t) \end{bmatrix} = Re\left\{ \begin{bmatrix} 1 \\ -j \end{bmatrix} U e^{js\omega t} \right\} \quad \text{with} \quad U := |U| e^{j\delta} \tag{82}$$

so that the steady-state currents will be sought in the exponential form too

$$\begin{bmatrix} i_a(t) \\ i_b(t) \\ i_d(t) \\ i_q(t) \end{bmatrix} = Re\left\{ \begin{bmatrix} I_a \\ I_b \\ I_d \\ I_q \end{bmatrix} e^{js\omega t} \right\} \tag{83}$$

where their complex amplitudes are unknown.

Introducing this trial vector (83) into (59) with Ω replaced by $-\Omega$ and representing in its right-hand side $u_a(t)$ and $u_b(t)$ by zeroes, and $u_d(t)$, $u_q(t)$ by expression (82), we get

$$\begin{bmatrix} R_a + jsX_a & 0 & jsX_{ad} & 0 \\ 0 & R_a + jsX_a & 0 & jsX_{ad} \\ jsX_{ad} & -(1-s)X_{ad} & R + jsX_d & -(1-s)X_d \\ (1-s)X_{ad} & jsX_{ad} & (1-s)X_d & R + jsX_d \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_d \\ I_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -j \end{bmatrix} U \tag{84}$$

where

$$X_a = X_b = \omega L_a, \qquad X_d = X_q = \omega L_d, \qquad X_{ad} = X_{bq} = \omega M_a. \tag{85}$$

Then, after pre-multiplying the current and the voltage vector of the last equation by the unitary 4×4 matrix

$$T := \text{block diag}\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \right\} \tag{86}$$

and at the same time performing the unitary transformation on the coefficient matrix, i.e. pre-multiplying it by T and post-multiplying by $T^{-1} = \bar{T}^T := T^*$ we obtain the equivalent system of equations in the form

$$\begin{bmatrix} R_a + jsX_a & 0 & jsX_{ad} & 0 \\ 0 & R_a + jsX_a & 0 & jsX_{ad} \\ j(2s-1)X_{ad} & 0 & R + j(2s-1)X_d & 0 \\ 0 & jX_{ad} & 0 & R + jX_d \end{bmatrix} \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} I_a - jI_b \\ I_a + jI_b \\ I_d - jI_q \\ I_d + jI_q \end{bmatrix} \right\}$$
$$= \sqrt{2} U \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{87}$$

Looking at it, we see at once that

$$\begin{bmatrix} I_a - jI_b \\ I_d - jI_q \end{bmatrix} = 0 \tag{88}$$

and from the remaining two equations we get the following result

$$\begin{bmatrix} I_a + jI_b \\ I_d + jI_q \end{bmatrix} = \frac{2U}{\Delta} \begin{bmatrix} -jX_{ad}s \\ R_a + jX_a s \end{bmatrix} \tag{89}$$

where $\Delta = R_aR - \left(X_aX_d - X_{ad}^2\right)s + j(R_aX_d + RX_as)$. So, we have

$$\begin{bmatrix} I_a \\ I_b \\ I_d \\ I_q \end{bmatrix} = \frac{U}{\Delta}\begin{bmatrix} -jX_{ad}s \\ -X_{ad}s \\ R_a + jX_as \\ -j(R_a + jX_as) \end{bmatrix} \tag{90}$$

Next, let us find the torque produced by the discussed model, or strictly speaking, its average value denoted by F_e. On the basis of (63), (81) and (85), we can write

$$F_e = (pX_{ad}/2\omega)\, Re\,(\bar{I}_bI_d - \bar{I}_aI_q) \tag{91}$$

and, finally, by replacing the currents by the right-hand side of (90), we get

$$F_e = \frac{-\left(p/\omega\right) R_aX_{ad}^2|U|^2s}{\left[R_aR - \left(X_aX_d - X_{ad}^2\right)s\right]^2 + (R_aX_d + RX_as)^2} \tag{92}$$

i.e the expression describing the torque v. the slip (or Ω) – the fundamental characteristic of the induction type motor.

Let us now return to the torque equation in its general form (51). As is well known from practice, the transient processes for which the responsibility is taken by the mechanical part of the motor are much slower than those caused by its electrical part. In result, in (51) we can replace the instantaneous torque f_e, cf. (63), by its average value F_e. So, it means that for the induction type motors (51) with Ω replaced by $(-\Omega)$ can be simplified to the form

$$(JD + B)\,(-\Omega) - F_e\,(\Omega) = F_L \tag{93}$$

where $F_e(\Omega)$ is given by (92) with s replaced by the right-hand side of (80) and, for uniqueness, it is assumed that the external torque $f(t) = F_L = \text{const.}$.

Writing the last equation as

$$J\dot{\Omega} = -\left[F_e\,(\Omega) + B\Omega\right] - F_L \tag{94}$$

and taking the typical graph of the function $F_e(\Omega)$, the dynamics of the motor is demonstrated by the so-called phase trajectory shown in Fig. 3, where $\bar{\Omega}$ is for the given load F_L the asymptotically stable (indicated by arrows) angular velocity of the motor.

Finally, let us turn our attention to the synchronous type motors characterized by the slip $s = 0$ and, additionally, by the negligibly small resistance of their stator's coils $(R \approx 0)$ and by the saliency of the rotor's poles, so that their parameters do not satisfy the constraints formulated earlier in (81). Moreover, one of the two rotor's coils is supplied from the DC voltage source, say $u_b(t) = E$, while the other remains short-circuited. The formulae (82) and (83) are still valid under the condition that s is equated to zero. In

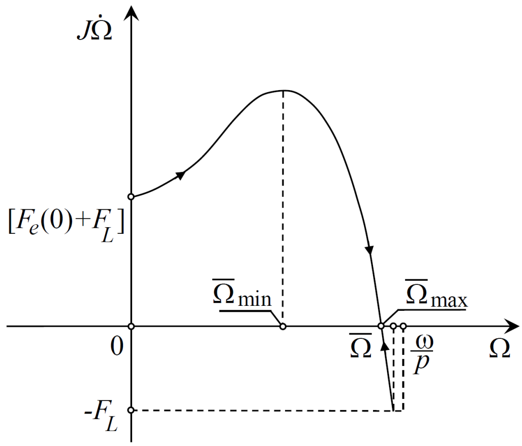

Figure 3. Induction type motor – the phase trajectory.

consequence, (59) with Ω replaced by $(-\Omega)$ takes a form

$$\begin{bmatrix} R_a & 0 & 0 & 0 \\ 0 & R_b & 0 & 0 \\ 0 & -X_{bq} & 0 & -X_q \\ X_{ad} & 0 & X_d & 0 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_d \\ I_q \end{bmatrix} \begin{bmatrix} 0 \\ E \\ |U|\cos\delta \\ |U|\sin\delta \end{bmatrix} \tag{95}$$

from which we get at once

$$\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} 0 \\ E/R_b \end{bmatrix}, \quad \begin{bmatrix} I_d \\ I_q \end{bmatrix} = \begin{bmatrix} (1/X_d)\,|U|\sin\delta \\ -(1/X_q)\,|U|\cos\delta - \left(X_{bq}/X_q\right)(E/R_b) \end{bmatrix} \tag{96}$$

and introducing this result to the modified form of (63)

$$F_e = \left({}^{p}\!/_{\omega}\right)\left[\left(X_{bq}I_b + X_qI_q\right)I_d - \left(X_{ad}I_a + X_dI_d\right)I_q\right] \tag{97}$$

we obtain finally

$$F_e = -\left({}^{p}\!/_{\omega}\right)\left[\frac{X_{bq}}{X_q}\frac{E|U|}{R_b}\sin\delta + \frac{1}{2}\frac{X_d - X_q}{X_dX_q}|U|^2\sin 2\delta\right] := -\left(A\sin\delta + A_0\sin 2\delta\right) \tag{98}$$

i.e. the fundamental characteristic linking the torque with its angle.

Next, let us concentrate on dynamic properties of the synchronous type motor assuming that its slip s is equal to zero not only in the steady-state but as well in its transient operation. In consequence, the torque angle δ introduced in (79) as a fixed parameter should be treated now as a time-dependent variable. So, we have

$$\omega t - p\varphi = \delta(t) \tag{99}$$

and

$$\left.\begin{aligned} \varphi &= \left(\tfrac{\omega}{p}\right) t - \left(\tfrac{1}{p}\right) \delta(t) \\ \dot{\varphi} &= \tfrac{\omega}{p} - \left(\tfrac{1}{p}\right) \dot{\delta}(t) \\ \ddot{\varphi} &= - \left(\tfrac{1}{p}\right) \ddot{\delta}(t) \end{aligned}\right\} \tag{100}$$

It means that for the synchronous type motor (51) with Ω replaced by $(-\dot{\varphi})$, i.e.

$$J(-\ddot{\varphi}) + B(-\dot{\varphi}) - F_e = f(t) \tag{101}$$

is, in the light of (98) and (100), of the form

$$\left(\frac{1}{p}\right)\left(J\ddot{\delta} + B\dot{\delta}\right) + A\sin\delta + A_0 \sin 2\delta = B\left(\frac{\omega}{p}\right) + f(t) \tag{102}$$

representing mathematically the nonlinear differential equation of the second order with $\delta(t)$ being its unknown function of time. Assuming next that the external torque $f(t) = F_L =$ const. and the frictional torque $B\dot{\delta}$ is negligibly small in comparison with the remaining components of (102), we get

$$\left(\frac{J}{pA}\right)\ddot{\delta} + \sin\delta + \left(\frac{A_0}{A}\right)\sin 2\delta = \frac{B\omega}{pA} + \frac{F_L}{A} \tag{103}$$

and finally, after introducing the new scale for the time, and defining the two new parameters $\mathcal{A}$ and $\mathcal{F}$:

$$t := \sqrt{\frac{J}{pA}}\tau, \quad \frac{A_0}{A} := \mathcal{A}, \quad \frac{B\omega}{pA} + \frac{F_L}{A} := \mathcal{F} \tag{104}$$

we obtain the equation of the form

$$\frac{d^2\delta}{d\tau^2} + \sin\delta + \mathcal{A}\sin 2\delta = \mathcal{F}. \tag{105}$$

Integrating all terms of the last equation with respect to δ, we get[2]

$$\frac{1}{2}\dot{\delta}^2 - \left(\cos\delta + \frac{1}{2}\mathcal{A}\cos 2\delta + \mathcal{F}\delta\right) = \mathcal{E}, \quad \text{where now} \quad \dot{\delta} := \frac{d\delta}{d\tau}. \tag{106}$$

Identifying next $\frac{1}{2}\dot{\delta}^2 := \mathcal{T}$ as the kinetic energy of the system and the term

$$-\left(\cos\delta + \frac{1}{2}\mathcal{A}\cos 2\delta + \mathcal{F}\delta\right) := \mathcal{U} \tag{107}$$

[2] $\int \frac{d^2\delta}{d\tau^2} d\delta = \int \left[\frac{d}{d\tau}\left(\frac{d\delta}{d\tau}\right)\right]\left(\frac{d\delta}{d\tau}\right) d\tau = \int \left(\frac{d\delta}{d\tau}\right) d\left(\frac{d\delta}{d\tau}\right) = \frac{1}{2}\left(\frac{d\delta}{d\tau}\right)^2 := \frac{1}{2}\dot{\delta}^2$

as its potential energy, (106) expresses in mathematical language the law of conservation of energy. The graph $\dot{\delta}$ as a function of δ drawn for various values of the total energy $\mathcal{E}$ shown in Fig. 4 is known as the phase portrait of the system. Here, it is necessary to point out that the equilibria indicated in Fig. 4 as the centers around which there exist undamped oscillations are, in fact, asymptotically stable focuses – the result of unavoidable viscous friction represented by the small but, in any case, positive parameter B.

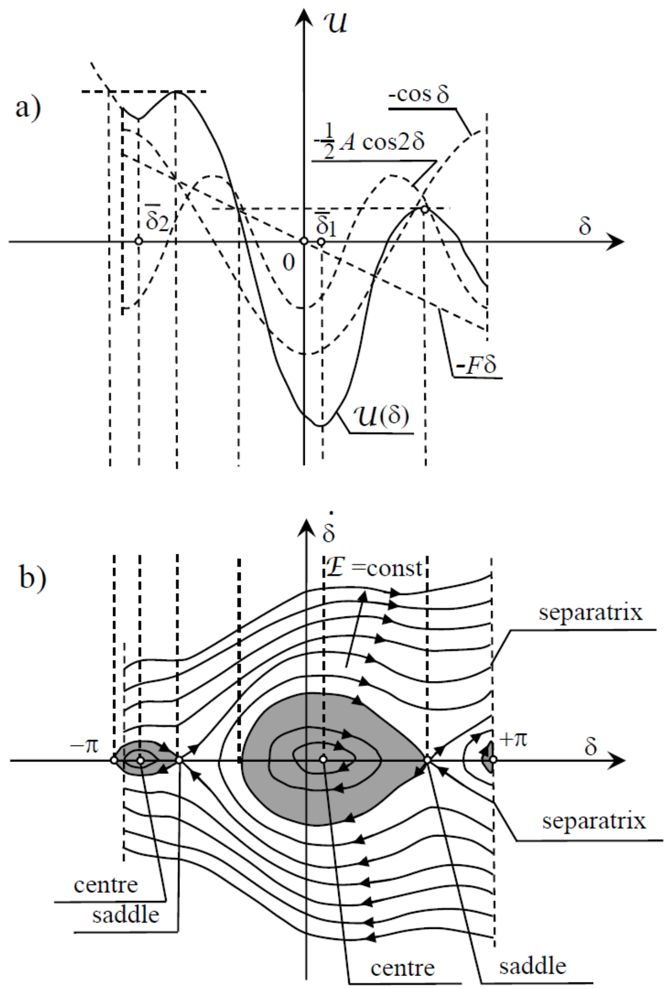

Figure 4. Synchronous type motor: a) the potential energy as a function of the torque angle, b) the phase portrait.

3. Example 2: – The electric circuit

Let us concentrate now on the simple electric circuit whose diagram is shown in Fig. 5. Here $S = 1/C$ and $j(t)$ is representing the inverse of the capacitance and the current source, respectively. The remaining symbols are standard.

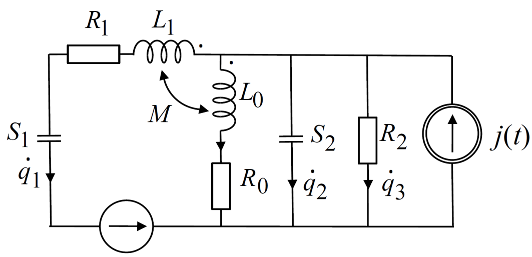

Figure 5. Illustrative electric circuit.

At first, let us observe that the system of Fig. 5 as consisting of four independent loops and the current source, is the system with three degrees of freedom. For the clarity of exposition, it will be, however, useful to describe it by the four branch currents, viz. $\dot{q}_1$, $\dot{q}_0$,$\dot{q}_2$ and $\dot{q}_3$, which as follows from the diagram of Fig. 5 are connected by the equation of constraints

$$\dot{q}_1 + \dot{q}_0 + \dot{q}_2 + \dot{q}_3 - j(t) = 0 \tag{108}$$

being directly in the form (23) characteristic for holonomic systems.

Identifying next magnetic energy (=coenergy) stored in inductances and electric energy stored in capacitances as the kinetic and potential energy, respectively, we get, according to (7), the following expression for the Lagrangian

$$\mathcal{L}(\dot{q}_1,\dot{q}_0,q_1,q_2) = \frac{1}{2}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_0 \end{bmatrix}^T \begin{bmatrix} L_1 & M \\ M & L_O \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_0 \end{bmatrix} - \frac{1}{2}S_1 q_1^2 - \frac{1}{2}S_2 q_2^2. \tag{109}$$

Similarly, applying (20), (24) and (108), we get the Rayleigh function in the form

$$\mathcal{R}(\dot{q}_1,\dot{q}_0,\dot{q}_2,\dot{q}_3,\lambda,t) = \frac{1}{2}R_1\dot{q}_1^2 + \frac{1}{2}R_0\dot{q}_0^2 + \frac{1}{2}R_2\dot{q}_3^2 - \dot{q}_1 u(t) - [\dot{q}_1 + \dot{q}_0 + \dot{q}_2 + \dot{q}_3 - j(t)]\lambda \tag{110}$$

Introducing $\mathcal{L}$ and $\mathcal{R}$ given by (109) and (110) into the equation of motion (18) and noting that $\widetilde{f} \equiv 0$, we get the following result

$$L_1\ddot{q}_1 + M\ddot{q}_0 + S_1 q_1 + R_1\dot{q}_1 - u(t) - \lambda = 0 \tag{111}$$

$$M\ddot{q}_1 + L_0\ddot{q}_0 + R_0\dot{q}_0 - \lambda = 0 \tag{112}$$

$$S_2q_2 - \lambda = 0 \tag{113}$$

$$R_2\dot{q}_3 - \lambda = 0 \tag{114}$$

which together with (108) represents a complete mathematical model of the system of Fig. 5. To simplify it, let us proceed as follows. At first, observe that

$$\lambda = S_2q_2 \tag{115}$$

$$\dot{q}_3 = (1/R_2)S_2q_2 := G_2S_2q_2 \tag{116}$$

and denote

$$\dot{q}_1 := i_1 \tag{117}$$

$$\dot{q}_0 := i_0. \tag{118}$$

Then, the system of (117), (108), (111) and (112) written in this sequence has in matrix notation the following form

$$\begin{bmatrix} D & 0 & -1 & 0 \\ 0 & D+G_2S_2 & 1 & 1 \\ S_1 & -S_2 & L_1D+R_1 & MD \\ 0 & -S_2 & MD & L_0D+R_0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ i_1 \\ i_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ j(t) \end{bmatrix} \tag{119}$$

which by replacing the capacitors' charges by the corresponding voltages, i.e. by putting

$$\begin{aligned} q_1 &= C_1u_1 \\ q_2 &= C_2u_2 \end{aligned} \tag{120}$$

gives, finally, the following simple and clear result

$$\begin{bmatrix} C_1D & 0 & -1 & 0 \\ 0 & C_2D+G_2 & 1 & 1 \\ 1 & -1 & L_1D+R_1 & MD \\ 0 & -1 & MD & L_0D+R_0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ j(t) \end{bmatrix}. \tag{121}$$

Let us observe that as long as the inductances satisfy the condition

$$L_1L_0 - M^2 > 0 \tag{122}$$

the system is of order four with its column of unknown functions being the state vector of the system.

Now, let us assume that the inductances, instead of inequality (122) satisfy the equation

$$L_1L_0 - M^2 = 0 \tag{123}$$

which physically means that both coils are coupled with the same magnetic flux. So, we can write

$$\begin{aligned} L_1 &= N_1^2 \Lambda \\ L_0 &= N_0^2 \Lambda \\ M &= N_1 N_0 \Lambda \end{aligned} \tag{124}$$

where Λ is the common magnetic conductance, and N_1 and N_0 are the number of turns of the respective coils. Thus, denoting

$$N_0/N_1 := \vartheta \tag{125}$$

the inductances L_0 and M can be expressed as

$$\begin{aligned} L_0 &= \vartheta^2 L_1 \\ M &= \vartheta L_1 . \end{aligned} \tag{126}$$

Introducing them into (121) and pre-multiplying its both sides by the matrix

$$T = T^{-1} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \vartheta & -1 \end{bmatrix} \tag{127}$$

we get finally the following result

$$\begin{bmatrix} C_1 D & 0 & -1 & 0 \\ 0 & C_2 D + G_2 & 1 & 1 \\ 1 & -1 & L_1 D + R_1 & \vartheta L_1 D \\ \vartheta & 1-\vartheta & \vartheta R_1 & -R_0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ \vartheta & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ j(t) \end{bmatrix} \tag{128}$$

characterized by no operator D in the last row of the coefficient matrix and of its presence outside the main diagonal just at the position (3,4). It means that for the case defined by (123) the considered system is reduced to order three but its complete solution requires the knowledge of four initial conditions - two for the voltages and two for the currents.

To explain this difference, let us post-multiply the coefficient matrix of (128) by T^T and, pre-multiply by it the vector of its unknown functions. In result, we get

$$\begin{bmatrix} C_1 D & 0 & -1 & -\vartheta \\ 0 & C_2 D + G_2 & 1 & -(1-\vartheta) \\ 1 & -1 & L_1 D + R_1 & \vartheta R_1 \\ \vartheta & 1-\vartheta & \vartheta R_1 & \vartheta^2 R_1 + R_0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ i_1 + \vartheta i_0 \\ -i_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ \vartheta & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ j(t) \end{bmatrix} \tag{129}$$

i.e. the system of equations characterized by the coefficient matrix free from the operator D not only in the last row but as well in the last column. It means that its solution requires just three initial conditions, two imposed on voltages u_1 and u_2 and one – on the linear combination, $i_1 + \vartheta i_0$, of the currents. So, the first three components of the column of unknown functions, viz. u_1, u_2 and $i_1 + \vartheta i_0$, are defining the state vector of the system.

4. Example 3: – The gyroscope placed on the Earth

The schematic diagram of the gyroscope with two degrees of freedom is shown in Fig. 6. Principally, it consists of two gimbals (shown as frames), inner and outer, with their axes mutually perpendicular, and the solid disc forced to rotate with the constant angular velocity $\dot\psi = \Omega$ around its axis x_1 perpendicular to the axis of the inner frame. The centre of mass of each of its three components is located at point 0 identified with the origin of any of the applied coordinate systems.

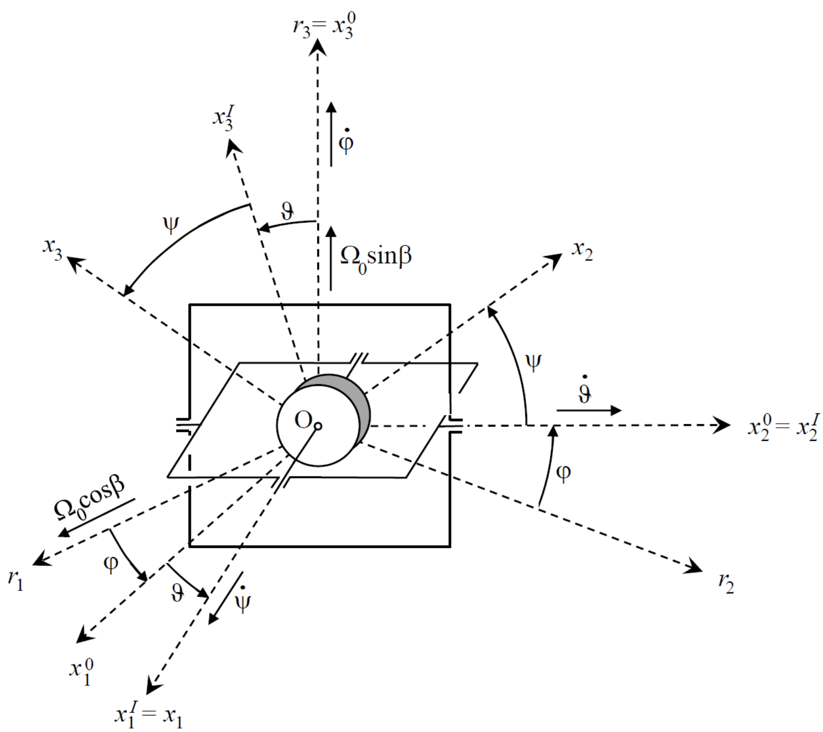

Figure 6. Schematic diagram of the gyroscope.

When the gyroscope is placed on the northern hemi-sphere of the Earth, the axis r_3 of the outer frame is perpendicular to the idealized spherical surface of the Earth and

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directed out of its interior, see Fig. 7. The remaining two axes r_1 and r_2 of the orthogonal system of coordinates are directed as follows: r_1 – to the north and r_2 – to the west, forming in result the right-hand side system of coordinates. Assuming next that the angular velocity vector Ω_0 representing the rotation of the Earth around its kinematic axis is directed as shown in Fig. 7, then its components in r-coordinates take the following form: $r_1 = \Omega_0 \cos\beta$, $r_2 = 0$, $r_3 = \Omega_0 \sin\beta$, where β is the corresponding Latitude.

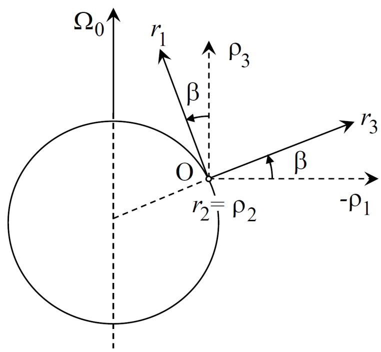

Figure 7. Placing the gyroscope on the rotating Earth.

Let us now consider the case when the inner frame is fixed at the right angle to the outer frame, so that $\vartheta = 0$ (see Fig. 6), and the gyroscope reduces to the system with one degree of freedom described by the angle φ as the only generalized coordinate. The corresponding kinematics is illustrated in Fig. 8, where Ω_0, $\dot{\varphi}$ and ψ understood as angular velocity vectors are projected, at first, on the auxiliary system of axes x_1^I, x_2^I, x_3^I rigidly connected with the inner frame, and then on the system x_1, x_2, x_3 fixed in the rotating disc. In result, its angular velocity vector denoted by ω^R takes a form

$$\omega^R = \begin{bmatrix} \dot{\psi} + (\cos\beta\cos\varphi)\,\Omega_0 \\ (\sin\psi)\,\dot{\varphi} + (\sin\beta\sin\psi - \cos\beta\sin\varphi\cos\psi)\,\Omega_0 \\ (\cos\psi)\,\dot{\varphi} + (\sin\beta\cos\psi + \cos\beta\sin\varphi\sin\psi)\,\Omega_0 \end{bmatrix} \tag{130}$$

Introducing it into the expression

$$T^R = \frac{1}{2}\left[J_1^R\left(\omega_1^R\right)^2 + J_2^R\left(\omega_2^R\right)^2 + J_3^R\left(\omega_3^R\right)^2\right] \tag{131}$$

which represents the kinetic energy of the rotating disc, where $J_1^R > J_2^R = J_3^R$ are its principal moments of inertia, and then removing from the result all terms proportional to Ω_0^2 as being negligibly small, and replacing $\dot{\psi}$ by Ω we get, finally, T^R in the form

$$T^R = \frac{1}{2}\left\{J_1^R\left[\Omega^2 + 2\,(\cos\beta\cos\varphi)\,\Omega_0\Omega\right] + J_3^R\left[\dot{\varphi}^2 + 2\,(\sin\beta)\,\Omega_0\dot{\varphi}\right]\right\} \tag{132}$$

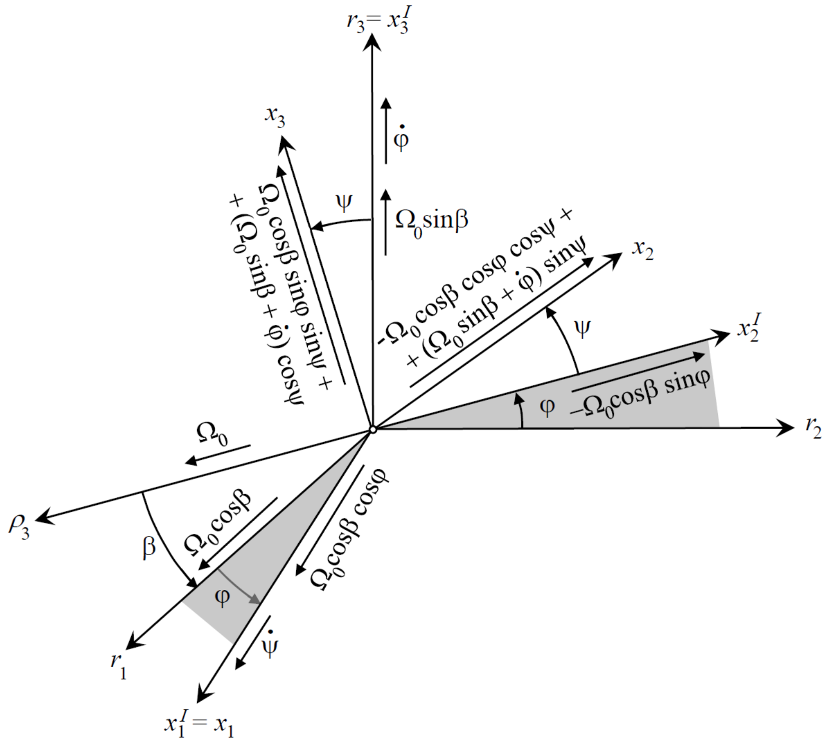

Figure 8. The kinematics of gyroscope applied to indicate the Meridian.

Putting next in the right-hand side of expression (130) $\psi = \dot{\psi} = 0$, we obtain the angular velocity vector

$$\omega^I = \omega^0 = \begin{bmatrix} (\cos\beta\cos\varphi)\,\Omega_0 \\ -(\cos\beta\sin\varphi)\,\Omega_0 \\ \dot{\varphi} + (\sin\beta)\,\Omega_0 \end{bmatrix} \tag{133}$$

common for the two rigidly connected frames. So, neglecting, as before, the terms proportional to Ω_0^2, the kinetic energy of each frame takes a form

$$\left.\begin{aligned} T^I &= \tfrac{1}{2} J_3^I \\ T^0 &= \tfrac{1}{2} J_3^0 \end{aligned}\right\} \left[\dot{\varphi}^2 + 2(\sin\beta)\,\Omega_0 \dot{\varphi}\right] \tag{134}$$

where J_3^I and J_3^0 are the principal moments of inertia of the inner and outer frame evaluated with respect to $x_3^I (= x_3^0)$ – axis (see Fig. 6).

Adding then the right-hand sides of (132) and (134), and neglecting in the sum any term free from φ or proportional to $\dot{\varphi}$, as being immaterial for the result, and observing that the potential energy of the considered system is equal to zero, the sought Lagrangian

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takes finally the following form

$$\mathcal{L}\left(\dot{\varphi},\varphi\right) = T^R + T^I + T^0 = \frac{1}{2}\left[\left(J_3^R + J_3^I + J_3^0\right)\dot{\varphi}^2 + 2J_1^R\Omega_0\Omega\cos\beta\cos\varphi\right]. \tag{135}$$

To complete the model, let us formulate as well the Rayleigh function which in the considered case is just of the form

$$\mathcal{R}(\dot{\varphi}) = \frac{1}{2}B_0\dot{\varphi}^2 \tag{136}$$

where B_0 is the coefficient of the viscous friction in the bearings of the outer frame.

So, introducing the two state functions $\mathcal{L}(\dot{\varphi},\varphi)$ and $\mathcal{R}(\dot{\varphi})$ into the equation of motion in its general form (18) with $\widetilde{f} \equiv 0$, we obtain finally the following result

$$\left(J_3^R + J_3^I + J_3^0\right)\ddot{\varphi} + B_0\dot{\varphi} + \left(J_1^R\Omega_0\Omega\cos\beta\right)\sin\varphi = 0. \tag{137}$$

Thus, the system is characterized by two states of equilibrium – the stable one corresponding to $\varphi = 0$ i.e. indicating the Meridian, and the unstable related with $\varphi = \pi$. Depending on the value of B_0, the point of stable equilibrium is either a focus or a node.

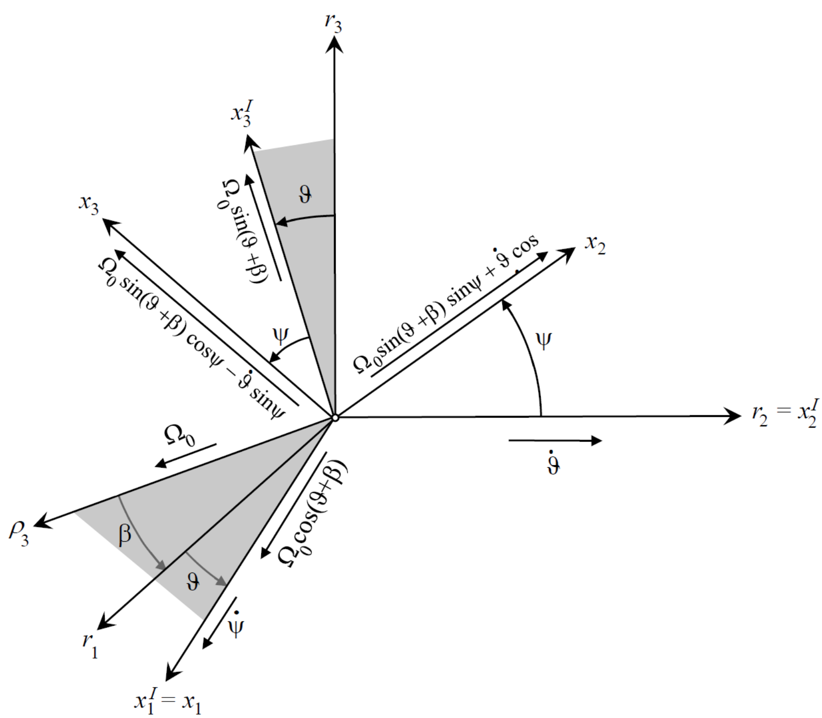

Figure 9. The kinematics of gyroscope indicating the Latitude.

Now, let us concentrate on the second special case, when $\varphi = 0$ and the gyroscope reduces once more to the system with one degree of freedom described, to the contrary of the previous case, by the angle ϑ as the only generalized coordinate, see Fig. 6. The corresponding kinematics is illustrated in Fig. 9, where the modified set of three angular velocity vectors, viz. Ω_0, $\dot{\vartheta}$ and $\dot{\psi} = \Omega$ is projected similarly as before, at first on the auxiliary system of axes x_1^I, x_2^I, x_3^I rigidly connected with the inner frame, and then on the system x_1, x_2, x_3 fixed in the rotating disc. In result, the angular velocity vector of the rotating disc takes a form

$$\omega^R = \begin{bmatrix} \Omega + \Omega_0 \cos(\vartheta + \beta) \\ (\cos\psi)\,\dot{\vartheta} + \Omega_0 \sin\psi \sin(\vartheta + \beta) \\ -(\sin\psi)\,\dot{\vartheta} + \Omega_0 \cos\psi \sin(\vartheta + \beta) \end{bmatrix} \tag{138}$$

from which by putting $\psi = 0$, we obtain at once the angular velocity vector Ω^I of the inner frame

$$\omega^I = \begin{bmatrix} \Omega_0 \cos(\vartheta + \beta) \\ \dot{\vartheta} \\ \Omega_0 \sin(\vartheta + \beta) \end{bmatrix}. \tag{139}$$

The remaining vector ω^0, due to the fixed position of the outer frame, viz. $\varphi = 0$, is equal to zero, too.

Applying the same procedure and the same approximations as before, let us find the corresponding Lagrangian and the Rayleigh function and, finally, the sought equation of motion. In result, we obtain

$$\mathcal{L}(\dot{\vartheta}, \vartheta) = T^R + T^I = \frac{1}{2}\left[\left(J_2^R + J_2^I\right)\dot{\vartheta}^2 + 2J_1^R \Omega_0 \Omega \cos(\vartheta + \beta)\right] \tag{140}$$

$$\mathcal{R}(\dot{\vartheta}) = \frac{1}{2} B_I \dot{\vartheta}^2 \tag{141}$$

$$\left(J_2^R + J_2^I\right)\ddot{\vartheta} + B_I \dot{\vartheta} + J_1^R \Omega_0 \Omega \sin(\vartheta + \beta) = 0 \tag{142}$$

where B_I is the coefficient of the viscous friction in the bearings of the inner frame.

Let us observe that now, similarly to the previously considered case, the system is characterized by two states of equilibrium – the stable one corresponding to $\vartheta = -\beta$ i.e. indicating the Latitude and the unstable related to $\vartheta = \pi - \beta$. For the stable case, the axis x_1 of the rotating disc is parallel to the axis of the rotating Earth and both have the same direction, see Fig. 9.

At the end of this section it is necessary to add that in practice the role of the solid disc is played by the rotor of the high-speed induction type motor, while the role of the inner frame is played by its stator.

By positioning the stator horizontally with possibility of its free rotation around the vertical axis, we obtain the instrument for pointing out the Meridian. By positioning the axis of the rotor in the plane of the Meridian with the possibility of free rotation of the stator around the horizontal axis, we get the instrument for pointing out the Latitude.

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