10.1515/acsc-2015-0011

Archives of Control Sciences
Volume 25(LXI), 2015
No. 2*, pages 177–187*

Robust stability of a class of uncertain fractional order linear systems with pure delay

MIKOŁAJ BUSŁOWICZ

The paper considers the robust stability problem of uncertain continuous-time fractional order linear systems with pure delay in the following two cases: a) the state matrix is a linear convex combination of two known constant matrices, b) the state matrix is an interval matrix. It is shown that the system is robustly stable if and only if all the eigenvalues of the state matrix multiplied by delay in power equal to fractional order are located in the open stability region in the complex plane. Parametric description of boundary of this region is derived. In the case a) the necessary and sufficient computational condition for robust stability is established. This condition is given in terms of eigenvalue-loci of the state matrix, fractional order and time delay. In the case b) the method for determining the rectangle with sides parallel to the axes of the complex plane in which all the eigenvalues of interval matrix are located is given and the sufficient condition for robust stability is proposed. This condition is satisfied if the rectangle multiplied by delay in power equal to fractional order lie in the stability region. The considerations are illustrated by numerical examples.

Key words: linear system, fractional, continuous-time, pure delay, robust stability, interval matrix.

1. Introduction

Dynamical systems described by fractional order differential or difference equations have been investigated in several areas such as viscoelasticity, electrochemistry, diffusion processes, automatic control, etc. The problem of analysis and synthesis of such systems has been considered in many books and papers, see [7-10, 13, 15] for example, and references therein.

The problem of stability of linear continuous-time fractional order systems has been investigated in the papers [2, 3, 6, 11, 12, 14].

M. Busłowicz was with Bialystok University of Technology, Faculty of Electrical Engineering, Wiejska 45D, 15-351 Bialystok, Poland. Corresponding person is Andrzej Ruszewski, e-mail: andrusz@pb.bialystok.pl

The work was supported by the Ministry of Science and High Education of Poland under the grant no. S/WE/1/2011.

Received 16.09.2014.

The aim of the paper is to give the methods (graphical and analytical) for robust stability checking of fractional order continuous-time linear systems with pure delay in two cases:

- the state matrix of the system is a linear convex combination of two known constant matrices,
- the state matrix is an interval matrix.

In the paper the following notations will be used: $\Re^{n\times m}$ – the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$; I – the identity matrix.

2. Preliminaries and problem formulation

Consider an uncertain continuous-time linear system of fractional order with pure delay described by the homogeneous state equation

$$D_t^{\alpha}x(t) = A_u x(t-h), \quad 0<\alpha<2, \tag{1}$$

where $x(t)\in\Re^n$, $A_u\in\Re^{n\times n}$, $h\in\Re$ is a delay,

$$D_t^{\alpha}x(t) = \frac{1}{\Gamma(p-\alpha)}\int\limits_0^t \frac{x^{(p)}(\tau)d\tau}{(t-\tau)^{\alpha+1-p}}, \quad p-1\leqslant\alpha\leqslant p, \tag{2}$$

is the Caputo definition of the fractional α-order derivative, where $x^{(p)}(t) = d^p x(t)/dt^p$ (p is a natural number) and $\Gamma(\alpha)$ is the Euler gamma function.

We consider the following uncertain matrices A_u:

- A_u is the convex combination of two known constant matrices $B,C\in\Re^{n\times n}$

$$A_u = A(\gamma) = (1-\gamma)B+\gamma C, \ \gamma\in[0,\ 1], \tag{3}$$

- A_u is the interval matrix

$$A_u = A_I = [B,\ C] = \{A=[a_{ij}],\ b_{ik}\leqslant a_{ik}\leqslant c_{ik},\quad i,k=1,2,...,n\}, \tag{4}$$

  where b_{ik} and c_{ik} are entries of matrices B and C, respectively.

Every element $a_{ik}(\gamma)$ of the matrix (3) is the convex combination $a_{ik}(\gamma) = (1-\gamma)b_{ik}+\gamma c_{ik}$, $\gamma\in[0,1]$ of the entries b_{ik} and c_{ik} of B and C. Assumption that $b_{ik}\leqslant c_{ik}$, $i,k=1,2\ldots,n$ is not necessary.

The fractional system (1) is robustly bounded-input bounded-output (BIBO) stable (shortly robustly stable) if and only if characteristic quasi-polynomial

$$q(s) = \det(Is^{\alpha}-A_u e^{-sh}) \tag{5}$$

has no poles with non-negative real parts, i.e. $q(s) \neq 0$ for $\text{Re}\, s > 0$. The above condition can be written the form

$$\det(Is^{\alpha} - [(1-\gamma)B + \gamma C]e^{-sh}) \neq 0, \quad \text{for } \text{Re}\, s > 0, \ \forall \gamma \in [0,\ 1], \tag{6}$$

for the system (1) with the state matrix (3) and

$$\det(Is^{\alpha} - Ae^{-sh}) \neq 0, \quad \text{for } \text{Re}\, s > 0, \ \forall A \in A_I = [B,\ C] \tag{7}$$

for the system (1) with the state matrix (4).

The conditions for robust stability checking of interval systems with pure delay of natural order have been proposed in [1] for continuous-time systems and in [4], [16] for fractional interval discrete-time linear systems.

The aim of the paper is to give the methods for robust stability checking of the system (1) with uncertain matrix A_u of the forms (3) and (4), i.e. checking the conditions (6) and (7).

3. The main result

If $A_u \equiv A$ (i.e. A_u is a constant known matrix) then the system (1) has the form

$$D_t^{\alpha} x(t) = Ax(t-h), \quad 0 < \alpha < 2. \tag{8}$$

In [3] the following condition for stability of (8) has been proved.

Lemma 6 *The fractional system* (8) *with pure delay is stable if and only if all the eigenvalues* $\lambda_i = u_i + jv_i$, *(*$i = 1, \ldots, n$*) of the matrix A multiplied by* h^{α} *(i.e.* $\widetilde{\lambda}_i = \widetilde{u}_i + j\widetilde{v}_i$ *with* $\widetilde{u}_i = h^{\alpha} u_i$, $\widetilde{v}_i = h^{\alpha} v_i$*) lie in the complex* $(\widetilde{u}_i, \widetilde{v}_i)$*-plane in the stability region* $S(\alpha)$*. Boundary of this region has parametric description*

$$(j\omega)^{\alpha} e^{j\omega} = \widetilde{u}_i(\omega) + j\widetilde{v}_i(\omega), \quad \omega \in [-\omega_b,\ \omega_b] \tag{9}$$

where $\omega_b = \pi - \alpha\pi/2$.

The boundary (9) crosses negative real axis of the complex $(\widetilde{u}_i, \widetilde{v}_i)$-plane in point $-[\pi(1-\alpha/2)]^{\alpha}$ for $\omega = \pm\omega_b$.

Boundaries of stability region $S(\alpha)$ for few values of fractional order $\alpha \in (0, 2)$ are shown in Fig. 1. For $\alpha \geqslant 2$ the stability region $S(\alpha)$ is empty.

For $0 < \alpha < 1$ a part of the stability region $S(\alpha)$ lies in right half-plane whereas for $1 \leqslant \alpha < 2$ the stability region entirely lies in the open left-half plane.

For any point $\widetilde{\lambda}_i = \widetilde{u}_i + j\widetilde{v}_i$ in the stability region $S(\alpha)$ the following condition holds [3]

$$|\arg \widetilde{\lambda}_i| > \alpha\pi/2, \tag{10}$$

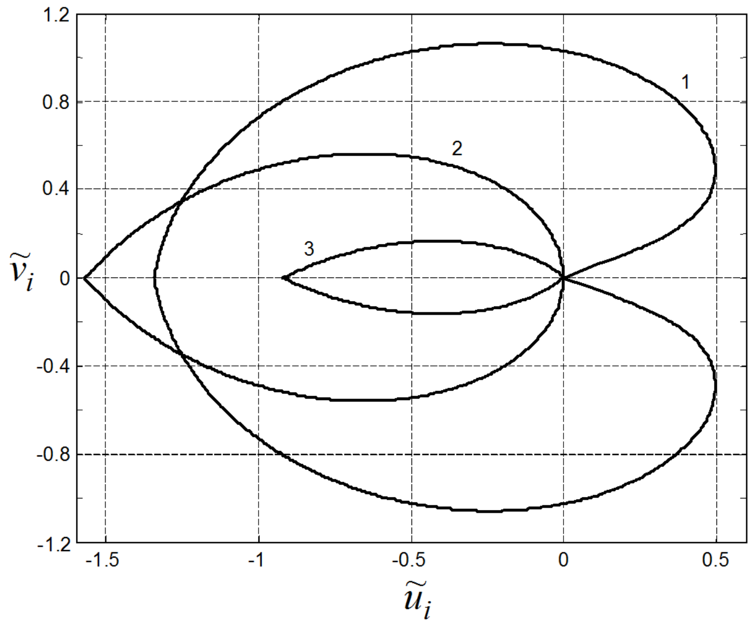

Figure 1: Boundaries of stability region $S(\alpha)$ for $\alpha = 0.3$ (boundary 1); $\alpha = 1$ (boundary 2) and for $\alpha = 1.6$ (boundary 3).

where $\arg\widetilde{\lambda}_i \in (-\pi,\ \pi]$ denotes the main argument of complex number $\widetilde{\lambda}_i$.

The main argument of complex number $\lambda_i = u_i + jv_i$ can be computed from the formula

$$\arg\lambda_i = \operatorname{sgn}(v_i)\cdot\arccos(u_i/|\lambda_i|) \tag{11}$$

The condition (10) for $i = 1,\ldots,n$ is the necessary and sufficient condition for stability of the system (8) without delay, that is of the system $D_t^\alpha x(t) = Ax(t)$, $0 < \alpha < 2$ (here $\widetilde{\lambda}_i \equiv \lambda_i = u_i + jv_i$ is the ith eigenvalue of A).

The condition of Lemma 1 can be written in the analytic form as follows [3].

Lemma 7 *The fractional system* (8) *is stable if and only if for all the eigenvalues* $\lambda_i = u_i + jv_i$, *(*$i = 1,2,\ldots,n$*) of the matrix A the following two conditions hold*

$$|\arg\lambda_i| > \alpha\pi/2, \tag{12}$$

$$h^\alpha|\lambda_i| < |\omega_{0i}|^\alpha, \tag{13}$$

where

$$\omega_{0i} = \arg\lambda_i - \operatorname{sgn}(v_i)\cdot\alpha\pi/2. \tag{14}$$

Lemma 8 *If the fractional system* (8) *is stable for* $h = 0$ *then it is stable for all* $h \in [0,\ h_0)$, $h_0 = \min_i\{h_i\}$, *where*

$$h_i = \exp\left(\frac{\ln(|\omega_{0i}|^\alpha/|\lambda_i|)}{\alpha}\right), \quad i = 1, 2, \ldots, n. \tag{15}$$

Complex eigenvalues of A are pair-wise conjugate and stability region $S(\alpha)$ is symmetric with respect to real axis. Therefore it is sufficient to check the conditions given in Lemmas 1, 2 and 3 only for real eigenvalues of A and for complex eigenvalues with positive imaginary parts.

Remark 1 If the fractional system (8) is stable then all the real eigenvalues of A are negative and greater than $-[\pi(1-\alpha/2)]^\alpha/h^\alpha$.

3.1. Robust stability of the system with the state matrix (3)

The fractional system (1) with the state matrix (3) is robustly stable if and only if all the eigenvalues of this matrix multiplied by h^α lie in the stability region $S(\alpha)$ for all $\gamma \in [0,\ 1]$. This holds if and only if all the systems (8), corresponding to all $\gamma \in [0,\ 1]$ in (3), are stable.

Remark 2 Since stability of fractional system (8) without delay is necessary for stability of this system with delay [3], robust stability of the system (1), (3) (and also (1), (4)) for $h = 0$ is necessary for robust stability of this system for $h > 0$.

It easy to see that stability of the system (8) with the matrices $A = B$, $A = C$ and $A_c = (B+C)/2$, for example, is necessary for robust stability of the system (1), (3). Hence we have the following simple necessary condition.

Lemma 9 *If all the eigenvalues of B or C or $A_c = (B+C)/2$ multiplied by h^α do not lie in the open region $S(\alpha)$ (the eigenvalues do not satisfy the conditions of Lemma* 2*) then the system* (1), (3) *is not robust stable.*

It is easy to see that the condition of Lemma 4 holds if at least one real eigenvalue of B or C or A_c is not negative.

Eigenvalues of (3) depend on uncertain parameter $\gamma \in [0,\ 1]$. Therefore, this matrix has an infinite number of eigenvalues, which form the eigenvalue-loci in the complex plane as parameter γ grows from 0 to 1.

From the above and Lemmas 1 and 2 we obtain the following theorem.

Theorem 5 *The fractional system* (1), (3) *is robustly stable if and only if the eigenvalue-loci of the matrix* (3) *multiplied by h^α lie in the stability region $S(\alpha)$ or equivalently, they satisfy the conditions of Lemma* 2.

The conditions of Theorem 1 can be checked by graphical or analytical verification if all the eigenvalues of (3) (calculated with a sufficiently small step $\Delta\gamma$) multiplied by h^{α} lie in the stability region $S(\alpha)$.

Example 1 *Consider the fractional system* (1), (3) *with* $h = 1$ *and the matrices*

$$B = \begin{bmatrix} 0 & -2 & -0.1 \\ 0.1 & 0.2 & 4 \\ 0 & -0.1 & -0.9 \end{bmatrix}, \quad C = \begin{bmatrix} -0.5 & -1 & 0 \\ 0 & 0 & 1 \\ 0.1 & -1 & -1.9 \end{bmatrix}. \tag{16}$$

Eigenvalues of the matrices B and C are as follows:

$$\lambda_{1,2}(B) = -0.0811 \pm j0.5712; \quad \lambda_3(B) = -0.5379$$

$$\lambda_{1,2}(C) = -0.6125 \pm j0.3681; \quad \lambda_3(C) = -1.1750$$

For $h = 1$ *and* $\alpha = 0.1$ *boundary of stability region* $S(\alpha)$ *crosses negative real axis of the complex* $(\widetilde{u}_i, \widetilde{v}_i)$*-plane in point* $-[\pi(1-\alpha/2)]^{\alpha} = -1.1155 > \lambda_3(C)$. *From Remark* 1 *it follows that the system* (8) *with* $A = C$ *is unstable. This means, according to Lemma* 4 *that the uncertain system* (1), (3) *with the matrices* (16) *is not robustly stable for* $\alpha = 0.1$.

Eigenvalues of B (denoted by '$\triangleright$'), C (denoted by 'Δ'), the eigenvalue-loci of (3), (16) *determined with the step* $\Delta\gamma = 0.025$ *(denoted by '.') and stability regions* $S(\alpha)$ *for few values of* α *are shown in Fig.* 2.

From Fig. 2 *and Theorem* 1 *it follows that the system is robustly stable for* $\alpha = 0.2$ *and* $\alpha = 0.6$ *but it is not robustly stable for* $\alpha = 0.7$. *Moreover, from the above considerations and Fig.* 2 *we conclude that the system is not robustly stable for* $\alpha = 0.1$, *is robustly stable for* $\alpha \in [0.2,\ 0.6]$ *and again is not robustly stable for* $\alpha \in [0.7,\ 2]$.

From Fig. 2 *for* $\alpha = 0.7$ *it also follows that stability of the system* (1) *for* $A_u = B$ *and* $A_u = C$ *is not sufficient for robust stability of the system* (1), (3).

3.2. Robust stability of the system with the state matrix (4)

The necessary conditions for robust stability given in Remarks 1 and 2 and in Lemma 4 are also true for the system (1), (4).

Robust stability of the system (1), (4) is equivalent to location of all the eigenvalues multiplied by h^{α} of the interval matrix (4) in the stability region $S(\alpha)$. Each entry of the interval matrix is an uncertain independent parameter, in general. Therefore, it is impossible to check the above condition exhaustively.

To robust stability checking we apply the method based on determination of the region in the complex plane in which the eigenvalues of the interval matrix are located. This region can be determined by generalization of the Gershgorin's theorem to the interval matrix case or applying the method based on the matrix measure. These methods

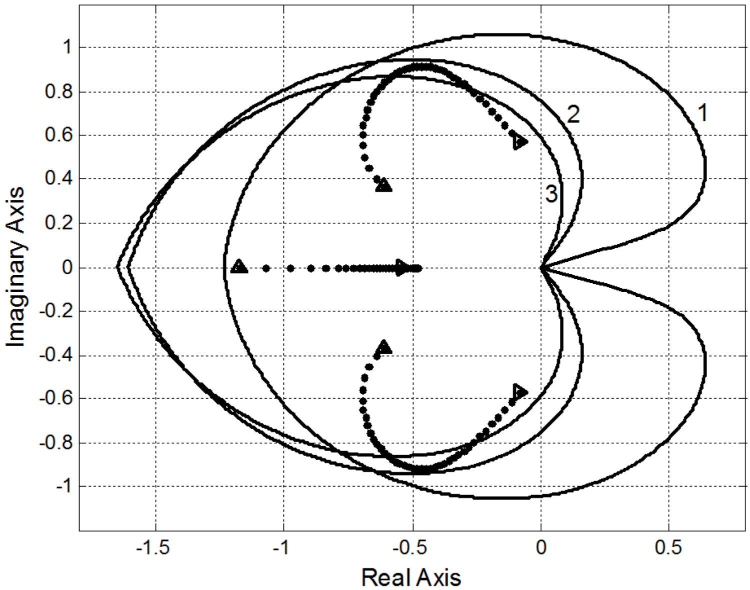

Figure 2: Eigenvalues of B ('$\triangleright$'), C ('Δ'), the eigenvalue-loci of (3), (16) ('.') and stability regions for few values of α: $\alpha = 0.2$ (boundary 1); $\alpha = 0.6$ (boundary 2) and $\alpha = 0.7$ (boundary 3).

have been proposed in the paper [1]. The method based on the matrix measure is recalled in the Appendix.

Let R_I denotes the rectangle with sides parallel to the axes of the complex plane determined by the method described in Appendix. In this rectangle the eigenvalues of the interval matrix (4) are located.

Theorem 6 *If the rectangle R_I multiplied by h^α lies in the open region $S(\alpha)$, then the interval system* (1)*,* (4) *is robustly stable.*

The rectangle R_I with the vertices $V_1 = u_l + jv$, $V_2 = u_r + jv$, $V_3 = u_r - jv$, $V_4 = u_l - jv$ (u_l, u_r and v are computed from (A.4)–(A.6)) is symmetric with respect to the real axis.

If $u_r < 0$ then R_I entirely lies in open left half-plane and the fractional system (1), (4) with the given value of delay h and fractional order α is robustly stable if the vertices V_1 and V_2 multiplied by h^α are in the region $S(\alpha)$. To check this we can apply the conditions (12) and (13) of Lemma 2 for $\lambda_i = V_1$ and $\lambda_i = V_2$.

Moreover, using Lemma 3 we find that the vertices V_1 andV_2 multiplied by h^α are in the region $S(\alpha)$ for all $h \in [0, h_0)$, where $h_0 = \min\{h_1, h_2\}$ and h_1 and h_2 are computed from (15) for $\lambda_i = V_1$ and $\lambda_i = V_2$, respectively.

Example 2 *Consider the fractional system* (1), (4) *with* $\alpha = 0.8$, $h = 0.5$ *and the matrices*

$$B = \begin{bmatrix} -1.5 & -0.3 & 0 \\ -0.2 & -1.2 & -0.3 \\ 0.3 & -0.1 & -1.2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0.2 & 0.5 \\ 0.2 & -1 & 0.3 \\ 0.5 & 0.1 & -1 \end{bmatrix}. \tag{17}$$

From (A.4)–(A.6) *one obtains:* $u_l = -2.2$, $u_r = -0.2$, $v = 0.8$.

Using Lemma 3 *we find that the vertices* $V_1 = u_l + jv$ *and* $V_2 = u_r + jv$ *multiplied by* h^α *lie in the open region* $S(\alpha)$ *for all* $h \in [0, h_1 = 0.5305)$ *and* $h \in [0, h_2 = 0.7115)$, *respectively. Hence,* $h_0 = \min\{h_1, h_2\} = 0.5305$ *and the uncertain fractional order system* (1), (4) *with the matrices* (17) *is robustly stable for all* $h \in [0, h_0)$, $h_0 = 0.5305$.

The region $S(\alpha)$ *for* $\alpha = 0.8$ *and the rectangle* R_I *multiplied by* h^α *for few values of delay h are shown in Fig.* 3. *From this figure it follows that all the rectangles lie in the stability region and, according to Theorem* 2, *the system is robustly stable for* $h \in [0,\ 0.5305)$.

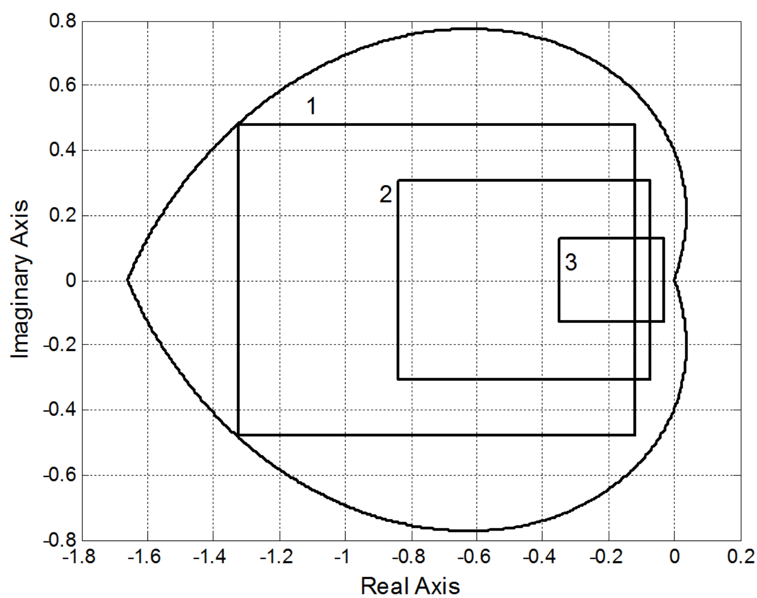

Figure 3: Boundary of $S(\alpha)$ for $\alpha = 0.8$ and rectangle R_I for few values of h: $h = h_0 = 0.5305$ (rectangle 1); $h = 0.3$ (rectangle 2); $\alpha = 0.1$ (rectangle 3).

From the above considerations and Example 2 it follows that the sufficient condition given in Theorem 2 may be satisfied in the case of diagonally dominant matrices B and C with negative diagonal entries.

Recall that a matrix is diagonally dominant if for every row (or column), the magnitude of the diagonal entry in a row (column) is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row (column).

4. Concluding remarks

The robust stability problem of uncertain continuous-time linear fractional order systems (1) with pure delay with an uncertain state matrix of the form (3) (linear convex combination of two known constant matrices) and (4) (interval matrix) has been considered. It has been shown that the system is robustly stable if and only if all the eigenvalues of uncertain state matrix multiplied by delay in power equal to fractional order are located in the open stability region $S(\alpha)$ in the complex plane.

In the case of state matrix (3) the necessary and sufficient condition for robust stability has been established in Theorem 1 and in the case of the state matrix (4) the sufficient condition for robust stability has been given in Theorem 2.

Appendix. Determination of the eigenvalue-region of the interval matrix (4) by the method based on the matrix measure

Using the method based on the matrix measure we find the rectangle R_I with sides parallel to the axes of the complex plane in which are located all the eigenvalues of interval matrix (4) [1].

Let $L = [l_{ip}]$ and $R = [r_{ip}]$, $i, p = 1, 2, \ldots, n$ be constant matrices defined by: $l_{ii} = b_{ii}$, $r_{ii} = c_{ii}$ for $i = 1, 2, \ldots, n$ and

$$l_{ip} = r_{ip} = \max\{|b_{ip}|, |c_{ip}|\}, \quad i, p = 1, 2, \ldots, n, \quad i \neq p, \tag{A.1}$$

where b_{ip} and c_{ip} are the entries of the matrices B and C of the interval matrix (4).

Lemma 10 *All the eigenvalues $\lambda_i(A_I)$ of the interval matrix (4) are located in the rectangle R_I determined by the following inequalities*

$$u_l \leqslant \mathrm{Re}\lambda_i(A_I) \leqslant u_r, \tag{A.2}$$

$$-v \leqslant \mathrm{Im}\lambda_i(A_I) \leqslant v, \tag{A.3}$$

where

$$u_l = -\min\{\mu_1(-L), \mu_\infty(-L)\}, \tag{A.4}$$

$$u_r = \min\{\mu_1(R), \mu_\infty(R)\}, \tag{A.5}$$

$$v = \min\{\mu_1(jR), \mu_\infty(jR)\}. \tag{A.6}$$

$\mu_1(X)$ and $\mu_\infty(X)$ denote the measures of the complex matrix X defined by [5]

$$\mu_1(X) = \max_j \left[\mathrm{Re}(x_{jj}) + \sum_{i=1, i\neq j}^{n} |x_{ij}| \right], \tag{A.7}$$

$$\mu_\infty(X) = \max_i \left[\mathrm{Re}(x_{ii}) + \sum_{j=1, j\neq i}^{n} |x_{ij}| \right]. \tag{A.8}$$

References

[1] M. BUSŁOWICZ: Asymptotic stability of dynamical interval systems with pure delay. *Scientific J. Bialystok University of Technology*, Technical Sciences, Electricity, **83**(11), (1992), 61-77.

[2] M. BUSŁOWICZ: Stability of linear continuous-time fractional order systems with delays of the retarded type. *Bull. Pol. Acad. Sci. Techn.*, **56**(4), (2008), 319-324.

[3] M. BUSŁOWICZ: Stability conditions of continuous-time fractional order linear systems with pure delay. Submitted for publication in *Bull. Pol. Acad. Sci. Techn.*, (2014).

[4] M. BUSŁOWICZ and A. RUSZEWSKI: Robust stability check of fractional discrete-time linear systems with interval uncertainties. In: K.J. LATAWIEC et al. (Eds.): Advances in Modeling and Control of Non-integer Order Systems (*Lecture Notes in Electrical Engineering*, **320**), Springer, 2015, 199-208.

[5] C.A. DESOER and M. VIDYASAGAR: Feedback Systems: Input-output Properties. Acad. Press, New York, 1975.

[6] A.R. FIORAVANTI, C. BONNET, H. ÖZBAY and S. NICULESCU: A numerical method for stability windows and unstable root-locus calculation for linear fractional time-delay systems. *Automatica*, **48**(11), (2012), 2824-2830.

[7] T. KACZOREK: Selected Problems of Fractional Systems Theory. Springer, Berlin, 2011.

[8] T. KACZOREK: Determination of positive realizations with reduced numbers of delays or without delays for discrete-time linear systems. *Archives of Control Sciences*, **22**(4), (2012), 371-384.

[9] T. KACZOREK and Ł. SAJEWSKI: Realization Problem for Positive and Fractional Systems. Springer, Heidelberg, 2014.

[10] T. KACZOREK and K. ROGOWSKI: Fractional Linear Systems and Electrical Circuits. Oficyna Wydawnicza Politechniki Białostockiej, Białystok, 2014.

[11] E. KASLIK and S. SIVASUNDARAM: Analytical and numerical methods for the stability analysis of linear fractional delay differential equations. *J. of Computational and Applied Mathematics*, **236**(16), (2012), 4027-4041.

[12] H. LI, S.M. ZHONG and H.B. LI: Stability analysis of fractional order systems with time delay. *Int. J. of Mathematical, Computational Science and Engineering*, **8**(4), (2014), 400-403.

[13] Y. LUO and Y.-Q. CHEN: Fractional Order Motion Controls. John Wiley & Sons Ltd, Chichester, 2013.

[14] A. MESBAHI and M. HAERI: Stability of linear time invariant fractional delay systems of retarded type in the space of delay parameters. *Automatica*, **49**(5), (2013), 1287-1294.

[15] C.A. MONJE, Y.Q. CHEN, B.M. VINAGRE, D. XUE and V. FELIU-BATLLE: Fractional-order Systems and Controls Fundamentals and Applications, Springer, London, 2010.

[16] A. RUSZEWSKI: Practical stability and asymptotic stability of interval fractional discrete-time linear state-space system. In: R. Szewczyk et al. (Eds.): Recent Advances in Automation, Robotics and Measuring Techniques (*Advances in Intelligent Systems and Computing*, **267**) Springer, 2014, 217-227.