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Stability conditions for linear continuous-time fractional-order state-delayed systems

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Abstract. The stability problem of continuous-time linear fractional order systems with state delay is considered. New simple necessary and sufficient conditions for the asymptotic stability are established. The conditions are given in terms of eigenvalues of the state matrix and time delay. It is shown that in the complex plane there exists such a region that location in this region of all eigenvalues of the state matrix multiplied by delay in power equal to the fractional order is necessary and sufficient for the asymptotic stability. Parametric description of boundary of this region is derived and simple new analytic necessary and sufficient conditions for the stability are given. Moreover, it is shown that the stability of the fractional order system without delay is necessary for the stability of this system with delay. The considerations are illustrated by a numerical example.

Key words: linear system, fractional, continuous-time, state delay, asymptotic stability.

1. Introduction

Dynamical systems described by fractional order differential or difference equations have been investigated in several areas such as viscoelasticity, electrochemistry, diffusion processes, automatic control, power electronic, etc. The problem of analysis and synthesis of dynamical systems described by fractional order differential (or difference) equations has been considered in monographs [1–4], for example.

The problems of stability of linear continuous-time and discrete-time fractional order systems, standard and positive, have been investigated in the above mentioned monographs and in many papers, see [5–12] for example, and references therein.

In last years the stability problem of fractional systems with delays has been considered in [13–19].

The aim of the paper is to give the methods (graphical and analytic) for asymptotic stability checking for fractional order continuous-time linear systems with state delay. The stability problem of such systems has not been considered yet. The stability problem of standard (i.e. non-fractional) continuous-time linear systems with pure delay has been considered in [20–24].

In the paper the following notations is used: $\Re^{n\times m}$ – the set of $n \times m$ real matrices and $\Re^n = \Re^{n\times 1}$; I – the identity matrix.

2. Preliminaries and problem formulation

Consider a continuous-time linear system of fractional order with pure state delay described by the state equation

$$D_t^\alpha x(t) = Ax(t-h) + Bu(t), \quad 0 < \alpha < 2, \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $h \in \Re$ is a delay and

$$D_t^\alpha x(t) = \frac{1}{\Gamma(p-\alpha)} \int_0^t \frac{x^{(p)}(\tau)d\tau}{(t-\tau)^{\alpha+1-p}}, \quad p-1 \le \alpha \le p, \tag{2}$$

is the Caputo definition of the fractional α-order derivative, where $x^{(p)}(t) = d^p x(t)/dt^p$ (p is a natural number) and $\Gamma(\alpha)$ is the Euler gamma function.

The characteristic function (quasi-polynomial in s^α and e^{-sh}) of the system (1) can be computed from the formula

$$\begin{aligned} q(s) &= \det(Is^\alpha - Ae^{-sh}) = \prod_{i=1}^n q_i(s) \\ &= \prod_{i=1}^n (s^\alpha - \lambda_i e^{-sh}), \end{aligned} \tag{3}$$

where

$$q_i(s) = s^\alpha - \lambda_i e^{-sh} \tag{4}$$

and $\lambda_i = u_i + jv_i$ is the i-th eigenvalue of the matrix A ($i = 1, ..., n$).

The fractional system (1) is bounded-input bounded-output (BIBO) stable (shortly stable) if and only if $q(s)$ has no poles with non-negative real parts, i.e.

$$q(s) = \prod_{i=1}^n (s^\alpha - \lambda_i e^{-sh}) \neq 0 \quad \text{for} \quad \text{Re}\, s > 0. \tag{5}$$

The condition (5) can be written in the form of n conditions

$$q_i(s) \neq 0 \quad \text{for} \quad \text{Re}\, s > 0, \quad i = 1, ..., n. \tag{6}$$

The characteristic quasi-polynomial which has no poles with non-negative real parts is called the stable quasi-polynomial.

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The aim of the paper is to give the methods for checking the conditions (6) and (5).

3. The main result

First of all we consider the stability problem for the fractional quasi-polynomial (4), where $\lambda_i = u_i + jv_i$ is a complex number.

A root location of (4) in the open left-half plane is equivalent to the root location in this plane of the quasi-polynomial

$$w_i(z) = z^\alpha e^z - \widetilde{\lambda}_i, \tag{7}$$

where $\widetilde{\lambda}_i = \widetilde{u}_i + j\widetilde{v}_i$, $\widetilde{u}_i = h^\alpha u_i$, $\widetilde{v}_i = h^\alpha v_i$.

We apply the D decomposition method of Nejmark (see [23] for the state of the art and [10] as an example of application) for determining the stability region of (4) in the complex $(\widetilde{u}_i, \widetilde{v}_i)$-plane.

Substituting $z = j\omega$ in (7) and equating to zero we obtain the parametric description of boundary of the stability region in the complex $(\widetilde{u}_i, \widetilde{v}_i)$-plane. This boundary is a part of the curve with the parametric description

$$(j\omega)^\alpha e^{j\omega} = \widetilde{u}_i(\omega) + j\widetilde{v}_i(\omega) \tag{8}$$

for $\omega \in (-\infty, \infty)$.

Taking account that $(j\omega)^\alpha = \omega^\alpha e^{j\alpha\pi/2}$, Eq. (8) for $\omega > 0$ can be written in the form of two real equations

$$\begin{aligned} \widetilde{u}_i(\omega) &= \omega^\alpha \cos(\omega + \alpha\pi/2), \\ \widetilde{v}_i(\omega) &= \omega^\alpha \sin(\omega + \alpha\pi/2). \end{aligned} \tag{9}$$

Since $(-j\omega)^\alpha = \omega^\alpha e^{-j\alpha\pi/2}$ $(\omega > 0)$ one has

$$(-j\omega)^\alpha e^{-j\omega} = \widetilde{u}_i(\omega) - j\widetilde{v}_i(\omega), \quad \omega \in (-\infty, \infty), \tag{10}$$

where $\widetilde{u}_i(\omega)$ and $\widetilde{v}_i(\omega)$ are defined by (9).

This means that the curve described by (8) (or (9), equivalently) for $\omega \in (-\infty, \infty)$ is symmetric with respect to the real axis of the complex $(\widetilde{u}_i, \widetilde{v}_i)$-plane.

From (9) it follows that $\widetilde{v}_i(\omega) = 0$ for $\omega = \pm\omega_b$ where $w_b = \pi - \alpha\pi/2$ and

$$\widetilde{u}_i(\omega_b) = \widetilde{u}_i(-\omega_b) = -\omega_b^\alpha = -[\pi(1 - \alpha/2)]^\alpha. \tag{11}$$

Hence, the curve (8) (or (9)) for $\omega \in [-\omega_b, \omega_b]$ is a closed curve. This curve divides the complex $(\widetilde{u}_i, \widetilde{v}_i)$-plane into two regions, one bounded and one unbounded (see Fig. 1). Denote by $S(\alpha)$ the bounded region.

The quasi-polynomial (7) for $z = j\omega$ can be written in the form $w_i(j\omega) = P_i(\omega) + jQ_i(\omega)$, where $P_i(\omega) = \omega^\alpha \cos(\omega + \alpha\pi/2) - \widetilde{u}_i(\omega)$, $Q_i(\omega) = \omega^\alpha \sin(\omega + \alpha\pi/2) - \widetilde{v}_i(\omega)$. If ω increases in the interval $[-\omega_b, \omega_b]$ from $-\omega_b$ to ω_b then the point $w_i(j\omega)$ moves along the boundary (8) in the positive direction. From this and positivity of the Jacobian

$$J(\omega) = \det \begin{bmatrix} \dfrac{\partial P_i(\omega)}{\partial \widetilde{u}_i(\omega)} & \dfrac{\partial P_i(\omega)}{\partial \widetilde{v}_i(\omega)} \\ \dfrac{\partial Q_i(\omega)}{\partial \widetilde{u}_i(\omega)} & \dfrac{\partial Q_i(\omega)}{\partial \widetilde{v}_i(\omega)} \end{bmatrix} = 1$$

it follows that the stability region lies on the left of the boundary (8) [24]. This means that $S(\alpha)$ is the stability region of the quasi-polynomial (7) and also (4).

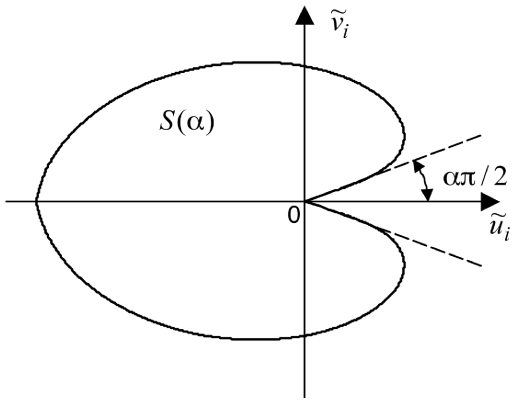

Fig. 1. Stability region $S(\alpha)$ for $0 < \alpha < 1$

From the above we have the following lemma.

Lemma 1. The quasi-polynomial (7) (and (4)) is stable if and only if the complex number $\lambda_i = u_i + jv_i$ multiplied by h^α (i.e. $\widetilde{\lambda}_i = \widetilde{u}_i + j\widetilde{v}_i$ with $\widetilde{u}_i = h^\alpha u_i$, $\widetilde{v}_i = h^\alpha v_i$) lies in the complex $(\widetilde{u}_i, \widetilde{v}_i)$-plane in the stability region $S(\alpha)$ with the boundary (8) for $\omega \in [-\omega_b, \omega_b]$.

From (9) for $\omega = 0$ we have $\widetilde{u}_i(0) = \widetilde{v}_i(0) = 0$ and

$$\tan\left|\frac{\widetilde{v}_i(0)}{\widetilde{u}_i(0)}\right| = \tan\frac{\alpha\pi}{2}. \tag{12}$$

This means that for any point $\widetilde{\lambda}_i = \widetilde{u}_i + j\widetilde{v}_i$ in the stability region $S(\alpha)$ the following condition holds

$$|\arg\widetilde{\lambda}_i| > \alpha\pi/2, \tag{13}$$

where $\arg\widetilde{\lambda}_i \in (-\pi, \pi]$ denotes the main argument of the complex number $\widetilde{\lambda}_i$.

The main argument of $\lambda_i = u_i + jv_i$ can be computed from the formula

$$\arg\lambda_i = \operatorname{sgn}(v_i) \cdot \arccos(u_i/|\lambda_i|). \tag{14}$$

The stability regions $S(\alpha)$ in the complex $(\widetilde{u}_i, \widetilde{v}_i)$-plane are shown in Figs. 1 and 2 for $0 < \alpha < 1$ and for $1 < \alpha < 2$, respectively. Figure 3 shows the stability regions $S(\alpha)$ for a few values of fractional order $0 < \alpha < 2$. For $\alpha \geq 2$ the stability regions $S(\alpha)$ are empty sets.

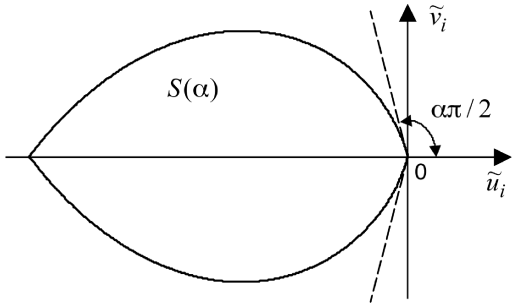

Fig. 2. Stability region $S(\alpha)$ for $1 < \alpha < 2$

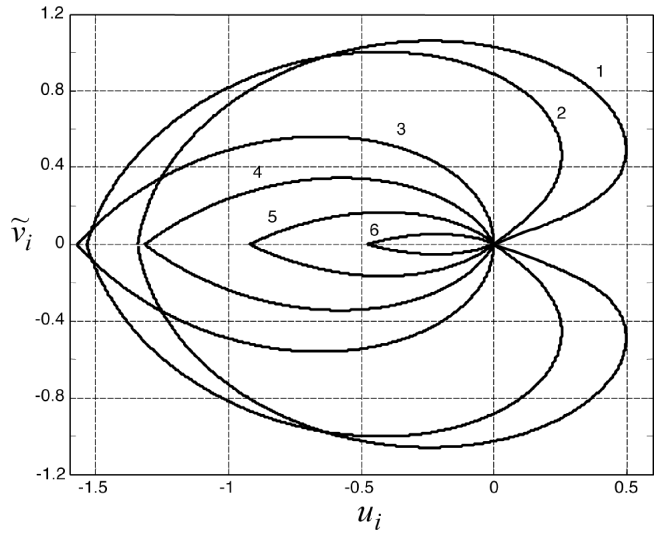

Fig. 3. Stability regions $S(\alpha)$ for a few values of α: $\alpha = 0.3$ (boundary 1); $\alpha = 0.5$ (boundary 2); $\alpha = 1$ (boundary 3); $\alpha = 1.2$ (boundary 4); $\alpha = 1.4$ (boundary 5); $\alpha = 1.6$ (boundary 6)

For $\alpha = 1$ the stability region $S(\alpha)$ is the asymptotic stability region of the natural order system with pure state delay $\dot{x}(t) = Ax(t-h)$ [20]. This region is shown in Fig. 3 as the region with boundary denoted by number 3.

From (3) and the above consideration we have the following theorem and remarks.

Theorem 1. The fractional system (1) with pure state delay is stable if and only if all eigenvalues λ_i, $i = 1, ..., n$, of the matrix A multiplied by h^α (i.e. $\widetilde{\lambda}_i = \widetilde{u}_i + j\widetilde{v}_i$) lie in the open region $S(\alpha)$ in the complex $(\widetilde{u}_i, \widetilde{v}_i)$-plane with the boundary (8) (or (9)) for $\omega \in [-\omega_b,\ \omega_b]$.

Recall that satisfaction of (13) for $i = 1, ..., n$, is the necessary and sufficient condition for stability of the system (1) without delay, that is of the system $D_t^\alpha x(t) = Ax(t) + Bu(t)$, $0 < \alpha < 2$ (here $\widetilde{\lambda}_i = \lambda_i = u_i + jv_i$ is the i-th eigenvalue of A).

Remark 1. Stability of the fractional system (1) without delay is necessary for stability of this system with delay.

Remark 2. Asymptotic stability of the matrix A (all eigenvalues have negative real parts) is necessary for stability of the system (1) for $1 < \alpha < 2$. For $0 < \alpha < 1$ eigenvalues of A may have positive real parts. Moreover, the system for $0 < \alpha < 1$ may be stable when all eigenvalues of A are complex conjugate with positive real parts.

Remark 3. The fractional system (1) is unstable if the matrix A has at least one non-negative real eigenvalue. In particular, this holds if $\det A = 0$.

Remark 4. If the fractional system (1) is stable then real eigenvalues of A are negative and greater than $-[\pi(1 - \alpha/2)]^\alpha / h^\alpha$.

The condition of Theorem 1 can be written in the analytic form as follows.

Theorem 2. The fractional system (1) with pure state delay is stable if and only if for all eigenvalues $\lambda_i = u_i + jv_i$, $i = 1, 2, ..., n$, of the matrix A the following two conditions hold

$$|\arg \lambda_i| > \alpha\pi/2, \tag{15}$$

$$h^\alpha |\lambda_i| < |\omega_{0i}|^\alpha, \tag{16}$$

where $\arg \lambda_i \in (-\pi,\ \pi]$ denotes the main argument of the eigenvalue λ_i and

$$\omega_{0i} = \arg \lambda_i - sgn(v_i) \cdot \alpha\pi/2. \tag{17}$$

Proof. Since $\varphi_i = \arg \lambda_i = \arg \lambda_i h^\alpha = \arg \widetilde{\lambda}_i$ (see Fig. 4), satisfaction of (15) for $i = 1, 2, ..., n$ is necessary for the stability of the fractional system (1).

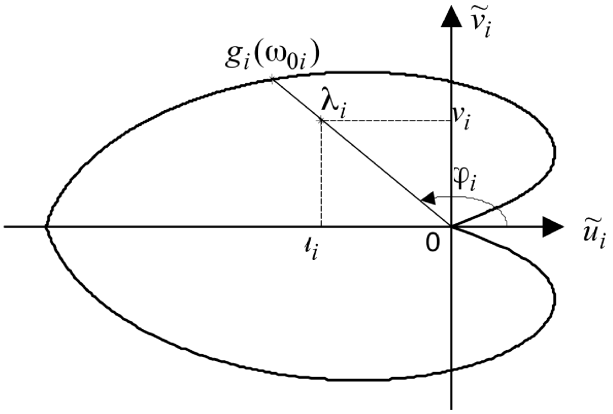

Fig. 4. Plot for the proof of Theorem 2

From Fig. 4 it follows that the point $\widetilde{\lambda}_i = \widetilde{u}_i + j\widetilde{v}_i = h^\alpha |\lambda_i| e^{j\varphi_i}$ lies in $S(\alpha)$ if and only if $h^\alpha |\lambda_i| < |g_i(\omega_{0i})|$, where ω_{0i} is defined by (17) and $g(\omega_{0i}) = (j\omega_{0i})^\alpha e^{j\omega_{0i}}$. Because $|g(\omega_{0i})| = |(j\omega_{0i})^\alpha e^{j\omega_{0i}}| = |\omega_{0i}|^\alpha$ the condition (16) must be satisfied. This completes the proof.

Remark 5. It is interesting to note that the form of the stability criterion (15), (16) is quite similar to the stability criterion of Ref. [9], in that it includes both the phase/argument condition of Matignon [5] and, additionally, the modulus condition.

From Theorem 2 we have the following corollary.

Corollary 1. If the fractional system (1) with pure state delay is stable for $h = 0$ then it is stable for all $h \in [0,\ h_0)$, $h_0 = \min_i\{h_i\}$, where

$$h_i = \exp\left(\frac{\ln(|\omega_{0i}|^\alpha / |\lambda_i|)}{\alpha}\right), \quad i = 1, 2, ..., n. \tag{18}$$

Proof. From (16) it follows that $h_i^\alpha = |\omega_{0i}|^\alpha / |\lambda_i|$. Computing h_i from this equality one obtains (18).

Remark 6. If $\lambda_i = u_i$ is real and negative then $|\lambda_i| = |u_i|$ and $|\omega_{0i}| = [\pi(1 - \alpha/2)]$.

Remark 7. It is sufficient to check the stability conditions given in Theorems 1 and 2 only for real and complex eigenvalues of A with positive imaginary parts. This follows from the fact that complex eigenvalues of A are pair-wise conjugate and the stability region $S(\alpha)$ is symmetric with respect to real axis.

If $\alpha = 1$ then the stability region $S(\alpha)$ has the form shown in Fig. 3 (boundary 3). From (15) and (16) we obtain, respectively, $|\arg\lambda_i| > \pi/2$, which is equivalent to $u_i < 0$, and $h|\lambda_i| < |\omega_{0i}|$. From (14) for $v_i \geq 0$ it follows that $\arg\lambda_i = \arctan(|v_i|/|u_i|)$ and by (17) $|\omega_{0i}| = |\arctan(|v_i|/|u_i|) - \pi/2| = \arctan(|u_i|/|v_i|)$.

Corollary 2. If $\alpha = 1$ then the system (1) is asymptotically stable if and only if for all eigenvalues $\lambda_i = u_i + jv_i$ $(i = 1, ..., n)$ of the matrix A the following conditions holds: $u_i < 0$ and $h|\lambda_i| < \arctan(|u_i|/|v_i|)$.

Corollary 3. If the system (1) with $\alpha = 1$ is asymptotically stable for $h = 0$, then this system is asymptotically stable for all $h \in [0,\ h_0)$, $h_0 = \min_i\{h_i\}$, where

$$h_i = \frac{1}{|\lambda_i|}\arctan\frac{|u_i|}{|v_i|}, \quad i = 1, 2, ..., n. \tag{19}$$

The above results have been obtained in [20].

4. Illustrative example

Consider the fractional system (1) with $h = 1$ and the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2.3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{20}$$

The matrix A has the following eigenvalues: $\lambda_{1,2} = -0.4629 \pm j0.7165$ and $\lambda_3 = -1.3741$.

Eigenvalues of A (denoted by 'o') and the stability regions $S(\alpha)$ for a few values of α are shown in Fig. 5. From this figure and Theorem 1 it follows that the system is stable for $\alpha = 0.4$ and $\alpha = 0.8$, and it is unstable for $\alpha = 0.3$ and $\alpha = 0.9$. Moreover, from Fig. 3 we conclude that for $\alpha \in [0.4,\ 0.8]$ the system is stable and it is unstable for $\alpha \in [0.9,\ 2)$.

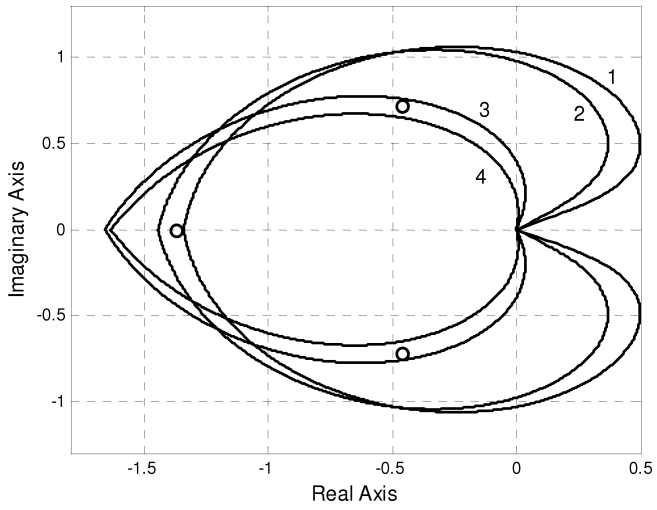

Fig. 5. Eigenvalues of A ('o') and stability regions for a few values of α: $\alpha = 0.3$ (boundary 1); $\alpha = 0.4$ (boundary 2); $\alpha = 0.8$ (boundary 3) and $\alpha = 0.9$ (boundary 4)

If, for example, $\alpha = 0.8$ then one has $|\arg\lambda_{1,2}| = 2.1444$ and $|\arg\lambda_3| = \pi$. This means that $|\arg\lambda_i| > \alpha\pi/2$ and the system with $\alpha = 0.8$ is stable for $h = 0$.

From Corollary 1 one has $h_{1,2} = 1.0828$, $h_3 = 1.2670$ and $h_0 = \min\{h_1, h_2, h_3\} = 1.0828$. This means that if $\alpha = 0.8$ then the fractional system (1), (20) is stable if and only if $h \in [0,\ 1.0828)$.

Figure 6 shows step responses $y(t) = x_1(t)$ of the system for $\alpha = 0.8$ and a few values of delay. The plots confirm the above result that the system is stable for $h < h_0$ and unstable for $h > h_0$.

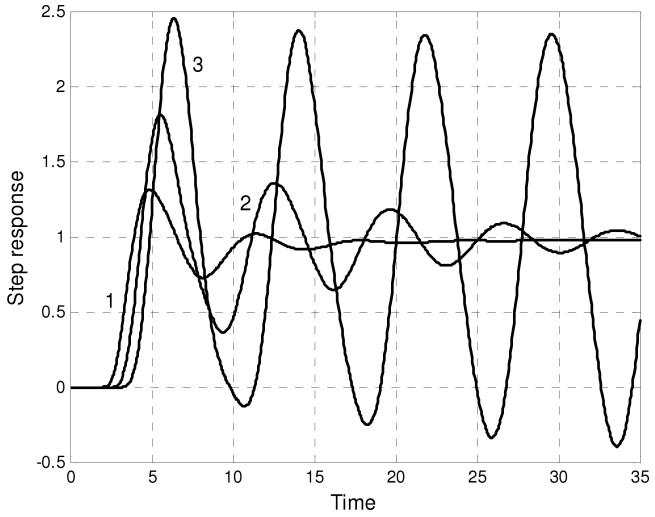

Fig. 6. Step responses for $\alpha = 0.8$ and few values of delay: $h = 0.7$ (plot 1); $h = 0.9$ (plot 2); $h = 1.1$ (plot 3)

5. Concluding remarks

The stability problem for continuous-time linear fractional order systems with state delay (1) has been considered. It has been shown that the system is stable if and only if all eigenvalues of the state matrix multiplied by delay in power equal to a fractional order lie in the stability region in the complex plane (Theorem 1). The new simple analytic condition for the stability are given in Theorem 2. Moreover, a new simple analytic method for computation of values of delay for which the system is stable, is derived in Corollary 1.

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