

On determination of inductances for circuit models of AC machines*

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Abstract: The paper discuss a problem of determination of inductances for AC machine windings when saturation of magnetic circuit is not neglected. For such cases, computation of magnetic field distribution in the machine magnetic circuit is a starting point for post processing procedures leading to various values, among others the co-energy in a given area and linkage fluxes of windings. This paper shows how to determine winding inductances in a nonlinear magnetic circuit from these two values and also how to compute directly nonlinear inductances. Problem is not trivial because such inductances are not uniquely determined as for linear case. In the paper a definition of nonlinear inductances is proposed which makes the choice unique.

Key words: electric machines, modelling magnetic non-linearity, dynamic inductances, non-linear inductances

1. Introduction

To develop the electrical machine equations by means of the Lagrange's formalism [1] it is sufficient to know the co-energy $E_{mo}(\mathbf{i}, \varphi)$ as a function of all independent currents constituting the vector $\mathbf{i} = [i_1 \dots i_N]$, and the rotation angle φ . The equations take the form

$$\frac{d}{dt} \left(\frac{\partial E_{mo}(\mathbf{i}, \varphi)}{\partial i_n} \right) + R_n \cdot i_n = u_n, \quad n \in \{1, 2, \dots, N\}, \quad (1a)$$

$$J \frac{d^2 \varphi}{dt^2} + D \frac{d\varphi}{dt} = \frac{\partial E_{mo}(\mathbf{i}, \varphi)}{\partial \varphi} + T_m. \quad (1b)$$

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However, very often the electrical equations (1a) are used in form

$$\frac{d}{dt}(\mathbf{L}_n(\mathbf{i}, \varphi) \cdot \mathbf{i}) + \mathbf{R} \cdot \mathbf{i} = \mathbf{u}, \quad (2)$$

which is an extended version of the machine equations assuming a linear magnetic circuit. Elements of the matrix $\mathbf{L}_n(\mathbf{i}, \varphi)$ are nonlinear functions of all currents and angle φ and can be treated as nonlinear inductances. The whole matrix is called the matrix of nonlinear inductances. That matrix must fulfill the relationship

$$\Psi(\mathbf{i}, \varphi) = \mathbf{L}_n(\mathbf{i}, \varphi) \cdot \mathbf{i}, \quad (3)$$

where $\Psi(\mathbf{i}, \varphi)$ is a vector of linkage fluxes

$$\Psi(\mathbf{i}, \varphi) = [\psi_1(\mathbf{i}, \varphi) \ \psi_2(\mathbf{i}, \varphi) \ \cdots \ \psi_N(\mathbf{i}, \varphi)].$$

Its elements can be obtained from the co-energy function

$$\psi_n(\mathbf{i}, \varphi) = \frac{\partial E_{mo}(\mathbf{i}, \varphi)}{\partial i_n}, \quad n \in \{1, 2, \dots, N\}. \quad (4)$$

It is evident that, the individual fluxes $\psi_n(\mathbf{i}, \varphi)$ are determined uniquely from the co-energy but the elements of the matrix $\mathbf{L}_n(\mathbf{i}, \varphi)$ are not.

There are numerous papers related with the problem of determining the inductances of electrical machine windings useful in their circuit models, to mention only [3, 4, 6-9]. Magnetic field computation techniques are employed to estimate these parameters because the circuit models of electrical machines are still very important tools in modelling various operational problems of electric energy generation and utilisation. However, the problem seems to be still open. This paper discusses the problem of determining the elements of the matrix $\mathbf{L}_n(\mathbf{i}, \varphi)$ from field computations.

2. Definition of non-linear inductances

Only the dynamic inductances can be uniquely determined by the linkage fluxes (4) as

$$L_{n,k}^d(\mathbf{i}, \varphi) = \frac{\partial \psi_n(\mathbf{i}, \varphi)}{\partial i_k}, \quad n, k \in \{1, 2, \dots, N\}. \quad (5)$$

These inductances constitute the matrix of dynamic inductances $\mathbf{L}_d(\mathbf{i}, \varphi)$, which is symmetrical as the linkage fluxes must meet the conditions

$$\frac{\partial \psi_n(\mathbf{i}, \varphi)}{\partial i_k} = \frac{\partial \psi_k(\mathbf{i}, \varphi)}{\partial i_n}, \quad n, k \in \{1, 2, \dots, N\}. \quad (6)$$

In order to define the nonlinear inductances $L_{n,k}^n(\mathbf{i}, \varphi)$, constituting the matrix $\mathbf{L}_n(\mathbf{i}, \varphi)$ in (3), an introduction of additional assumption is necessary. It has been proved [2, 5] that it is

sufficient to assume the symmetry of the matrix of nonlinear inductances by the analogy with the inductance matrix for the linear case. Below, the definition of nonlinear inductances is recalled briefly.

First, the co-energy function $E_{mo}(\mathbf{i}, \varphi)$ should be analytically expressed. It is a multivariate function of currents and it can be expressed in the form of the Taylor series

$$E_{mo}(\mathbf{i}, \varphi) = \sum_{r=1}^{\infty} \frac{1}{(2r)!} d^{2r} E_{mo}(\mathbf{0}, \varphi), \tag{7}$$

that contains only even derivatives. Then, the linkage fluxes can be express by the Taylor series with only the odd derivatives

$$\psi_n(\mathbf{i}, \varphi) = \sum_{r=1}^{\infty} \frac{1}{(2r-1)!} d^{2r-1} \psi_n(\mathbf{0}, \varphi), \quad n \in \{1, 2, \dots, N\}. \tag{8}$$

The m -th derivative takes the following form

$$d^m \psi_n = \left\{ \sum_{s=1}^N i_s \frac{\partial}{\partial i_s} \right\}^m \psi_n \Big|_0$$

and it will be rewritten further to facilitate the development of definition of the nonlinear inductances

$$d^m \psi_n = \left\{ \left(\sum_{s=1}^N i_s \frac{\partial}{\partial i_s} \right)^{m-1} \left(\sum_{k=1}^N i_k \frac{\partial}{\partial i_k} \right) \right\} \psi_n \Big|_0 = \sum_{k=1}^N \left(\left(\sum_{s=1}^N i_s \frac{\partial}{\partial i_s} \right)^{m-1} \frac{\partial \psi_n}{\partial i_k} \Big|_0 \right) \cdot i_k.$$

Based on that form, the nonlinear inductance $L^n_{n,k}(\mathbf{i}, \varphi)$ – can be defined as [5]

$$L^n_{n,k} \stackrel{\text{def}}{=} \sum_{r=1}^{\infty} \frac{1}{(2r-1)!} \left\{ \sum_{s=1}^N i_s \frac{\partial}{\partial i_s} \right\}^{2(r-1)} \frac{\partial \psi_n}{\partial i_k} \Big|_0. \tag{9}$$

The nonlinear inductances constitute the matrix $\mathbf{L}_n(\mathbf{i}, \varphi)$. It is symmetric, which is implied by the relationship (6). The matrices $\mathbf{L}_n(\mathbf{i}, \varphi)$ and $\mathbf{L}_d(\mathbf{i}, \varphi)$ can be written as follows

$$\mathbf{L}_n(\mathbf{i}, \varphi) = \sum_{r=1}^{\infty} \mathbf{L}_{n,(2r-1)}(\mathbf{i}, \varphi) \quad \mathbf{L}_d(\mathbf{i}, \varphi) = \sum_{r=1}^{\infty} \mathbf{L}_{d,(2r-1)}(\mathbf{i}, \varphi), \tag{10}$$

where

$$\mathbf{L}_{d,m}(\mathbf{i}, \varphi) = \frac{1}{m!} \begin{bmatrix} \frac{\partial(d^m \psi_1)}{\partial i_1} & \dots & \frac{\partial(d^m \psi_1)}{\partial i_N} \\ \vdots & & \vdots \\ \frac{\partial(d^m \psi_N)}{\partial i_1} & \dots & \frac{\partial(d^m \psi_N)}{\partial i_N} \end{bmatrix},$$

$$\mathbf{L}_{n,m}(\mathbf{i}, \varphi) = \frac{1}{m!} \cdot \left\{ \sum_{s=1}^N i_s \frac{\partial}{\partial i_s} \right\}^{m-1} \left[\begin{array}{ccc} \frac{\partial \psi_1}{\partial i_1} & \dots & \frac{\partial \psi_1}{\partial i_N} \\ \vdots & & \vdots \\ \frac{\partial \psi_N}{\partial i_1} & \dots & \frac{\partial \psi_N}{\partial i_N} \end{array} \right]_0.$$

Relationships between those matrices can be determined by recalling that

$$\frac{\partial (d^m \psi_n)}{\partial i_k} = \left\{ \sum_{s=1}^N i_s \frac{\partial}{\partial i_s} \right\}^{m-1} \frac{\partial \psi_n}{\partial i_k} \Big|_0$$

$$\mathbf{L}_{d,m}(\mathbf{i}, \varphi) = m \cdot \mathbf{L}_{n,m}(\mathbf{i}, \varphi).$$

As a consequence, the relationships between the matrices $\mathbf{L}_n(\mathbf{i}, \varphi)$ and $\mathbf{L}_d(\mathbf{i}, \varphi)$ take the form

$$\mathbf{L}_n(\mathbf{i}, \varphi) = \sum_{r=1}^{\infty} \frac{1}{(2r-1)} \cdot \mathbf{L}_{d,(2r-1)}(\mathbf{i}, \varphi), \quad (11a)$$

$$\mathbf{L}_d(\mathbf{i}, \varphi) = \sum_{r=1}^{\infty} (2r-1) \cdot \mathbf{L}_{n,(2r-1)}(\mathbf{i}, \varphi). \quad (11b)$$

3. Algorithms leading to nonlinear inductances – case study for two windings

Both dynamic and nonlinear inductances can only be determined from data obtained from computation of magnetic field distribution over the magnetic circuit of a machine. The post-processing procedures of many finite element packages allow evolution of the values of the co-energy as well as fluxes for each individual machine winding. This can be used as a starting point for the algorithms determining the required inductances.

Below, the three such algorithms are described and an example is provided regards two windings, with the co-energy, linkage fluxes and inductances approximated by the Taylor series, taking the first nonlinear term only into account.

Algorithm starting from the co-energy

Values of the co-energy obtained from field computation at a given value of angle φ should be used to calculate the coefficients of the co-energy function of currents (i_1, i_2) approximated with the first two terms of the Taylor series

$$\begin{aligned}
 E_{mo}(i_1, i_2) &= \frac{1}{(2)!} d^2 E_{mo}(0,0) + \frac{1}{(4)!} d^4 E_{mo}(0,0) = \\
 &= \frac{1}{2} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} E_{mo}(i_1, i_2) \Big|_0 + \frac{1}{24} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^4 \right\} E_{mo}(i_1, i_2) \Big|_0.
 \end{aligned}$$

Substituting respective partial derivatives by constant coefficients the co-energy function can be written in form

$$\begin{aligned}
 E_{mo}(i_1, i_2) &= \frac{1}{2} (A_{2,0} \cdot i_1^2 + 2A_{1,1} \cdot i_1 \cdot i_2 + A_{0,2} \cdot i_2^2) + \\
 &+ \frac{1}{4} (A_{4,0} \cdot i_1^4 + 4A_{3,1} \cdot i_1^3 \cdot i_2 + 6A_{2,2} \cdot i_1^2 \cdot i_2^2 + 4A_{1,3} \cdot i_1 \cdot i_2^3 + A_{0,4} \cdot i_2^4)
 \end{aligned} \quad (12)$$

Based on that function, the linkage fluxes can be found

$$\begin{aligned}
 \psi_1(i_1, i_2) &= \frac{\partial E_{mo}(i_1, i_2)}{\partial i_1} = (A_{2,0} \cdot i_1 + A_{1,1} \cdot i_2) + \\
 &+ (A_{4,0} \cdot i_1^3 + 3A_{3,1} \cdot i_1^2 \cdot i_2 + 3A_{2,2} \cdot i_1 \cdot i_2^2 + A_{1,3} \cdot i_2^3),
 \end{aligned} \quad (13a)$$

$$\begin{aligned}
 \psi_2(i_1, i_2) &= \frac{\partial E_{mo}(i_1, i_2)}{\partial i_2} = (A_{1,1} \cdot i_1 + A_{0,2} \cdot i_2) + \\
 &+ (A_{3,1} \cdot i_1^3 + 3A_{2,2} \cdot i_1^2 \cdot i_2 + 3A_{1,3} \cdot i_1 \cdot i_2^2 + A_{0,4} \cdot i_2^3).
 \end{aligned} \quad (13b)$$

Given fluxes, the matrix of dynamic inductances $\mathbf{L}_d(\mathbf{i}, \varphi)$ can uniquely be determined

$$\begin{aligned}
 \mathbf{L}_d(\mathbf{i}, \varphi) &= \begin{bmatrix} \frac{\partial \psi_1}{\partial i_1} & \frac{\partial \psi_1}{\partial i_2} \\ \frac{\partial \psi_2}{\partial i_1} & \frac{\partial \psi_2}{\partial i_2} \end{bmatrix} = \begin{bmatrix} A_{2,0} & A_{1,1} \\ A_{1,1} & A_{0,2} \end{bmatrix} + \\
 &+ 3 \cdot \begin{bmatrix} A_{4,0} \cdot i_1^2 + 2A_{3,1} \cdot i_1 \cdot i_2 + A_{2,2} \cdot i_2^2; & A_{3,1} \cdot i_1^2 + 2A_{2,2} \cdot i_1 \cdot i_2 + A_{1,3} \cdot i_2^2 \\ A_{3,1} \cdot i_1^2 + 2A_{2,2} \cdot i_1 \cdot i_2 + A_{1,3} \cdot i_2^2; & A_{2,2} \cdot i_1^2 + 2A_{1,3} \cdot i_1 \cdot i_2 + A_{0,4} \cdot i_2^2 \end{bmatrix}.
 \end{aligned} \quad (14)$$

To estimate the nonlinear inductances according the definition (9) the formulas (13a, b) for the linkage fluxes should be modified as follows

$$\psi_1(i_1, i_2) = L_{11}^n(i_1, i_2) \cdot i_1 + L_{12}^n(i_1, i_2) \cdot i_2,$$

$$\psi_2(i_1, i_2) = L_{21}^n(i_1, i_2) \cdot i_1 + L_{22}^n(i_1, i_2) \cdot i_2,$$

where $L_{21}^n(i_1, i_2) = L_{12}^n(i_1, i_2)$. To accomplish that, the linkage fluxes should be approximated by the first two terms of the Taylor series

$$\begin{aligned}\psi_1(i_1, i_2) &= \frac{1}{(1)!} d^1 \psi_1(0,0) + \frac{1}{(3)!} d^3 \psi_1(0,0), \\ \psi_2(i_1, i_2) &= \frac{1}{(1)!} d^1 \psi_2(0,0) + \frac{1}{(3)!} d^3 \psi_2(0,0).\end{aligned}\quad (15)$$

The formula for linkage flux $\psi_1(i_1, i_2)$ can be evaluated as follows

$$\begin{aligned}\psi_1(i_1, i_2) &= \left\{ i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right\} \psi_1(i_1, i_2) \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^3 \right\} \psi_1(i_1, i_2) \Big|_0 = \\ &= \left\{ i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right\} \psi_1(i_1, i_2) \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \cdot \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right) \right\} \psi_1(i_1, i_2) \Big|_0 = \\ &= \left(\frac{\partial \psi_1(i_1, i_2)}{\partial i_1} \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} \frac{\partial \psi_1(i_1, i_2)}{\partial i_1} \Big|_0 \right) \cdot i_1 + \\ &+ \left(\frac{\partial \psi_1(i_1, i_2)}{\partial i_2} \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} \frac{\partial \psi_1(i_1, i_2)}{\partial i_2} \Big|_0 \right) \cdot i_2,\end{aligned}$$

which defines the nonlinear inductances $L^n_{11}(i_1, i_2)$ and $L^n_{12}(i_1, i_2)$

$$L^n_{11}(i_1, i_2) = \left(\frac{\partial \psi_1(i_1, i_2)}{\partial i_1} \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} \frac{\partial \psi_1(i_1, i_2)}{\partial i_1} \Big|_0 \right), \quad (16a)$$

$$L^n_{12}(i_1, i_2) = \left(\frac{\partial \psi_1(i_1, i_2)}{\partial i_2} \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} \frac{\partial \psi_1(i_1, i_2)}{\partial i_2} \Big|_0 \right). \quad (16b)$$

Analogously, the formula for linkage flux $\psi_2(i_1, i_2)$ can be written as

$$\begin{aligned}\psi_2(i_1, i_2) &= \left(\frac{\partial \psi_2(i_1, i_2)}{\partial i_1} \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} \frac{\partial \psi_2(i_1, i_2)}{\partial i_1} \Big|_0 \right) \cdot i_1 + \\ &+ \left(\frac{\partial \psi_2(i_1, i_2)}{\partial i_2} \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} \frac{\partial \psi_2(i_1, i_2)}{\partial i_2} \Big|_0 \right) \cdot i_2 = \\ &= L^n_{21}(i_1, i_2) \cdot i_1 + L^n_{22}(i_1, i_2) \cdot i_2,\end{aligned}$$

which defines the nonlinear inductances $L^n_{21}(i_1, i_2)$ and $L^n_{22}(i_1, i_2)$ as

$$L^n_{21}(i_1, i_2) = \left(\frac{\partial \psi_2(i_1, i_2)}{\partial i_1} \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} \frac{\partial \psi_2(i_1, i_2)}{\partial i_1} \Big|_0 \right), \quad (17a)$$

$$L^n_{22}(i_1, i_2) = \left(\frac{\partial \psi_2(i_1, i_2)}{\partial i_2} \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^2 \right\} \frac{\partial \psi_2(i_1, i_2)}{\partial i_2} \Big|_0 \right). \quad (17b)$$

It should be noticed that $L^n_{21}(i_1, i_2) = L^n_{12}(i_1, i_2)$ as

$$\frac{\partial \psi_1(i_1, i_2)}{\partial i_2} = \frac{\partial \psi_2(i_1, i_2)}{\partial i_1}.$$

Application of those definitions to formulas (13a, b) leads to the symmetric matrix of non-linear inductances in form

$$\mathbf{L}_n(\mathbf{i}, \varphi) = \begin{bmatrix} A_{2,0} & A_{1,1} \\ A_{1,1} & A_{0,2} \end{bmatrix} + \begin{bmatrix} A_{4,0} \cdot i_1^2 + 2A_{3,1} \cdot i_1 \cdot i_2 + A_{2,2} \cdot i_2^2; A_{3,1} \cdot i_1^2 + 2A_{2,2} \cdot i_1 \cdot i_2 + A_{1,3} \cdot i_2^2 \\ A_{3,1} \cdot i_1^2 + 2A_{2,2} \cdot i_1 \cdot i_2 + A_{1,3} \cdot i_2^2; A_{2,2} \cdot i_1^2 + 2A_{1,3} \cdot i_1 \cdot i_2 + A_{0,4} \cdot i_2^2 \end{bmatrix}. \quad (18)$$

It takes exactly the same form as the matrix of dynamic inductances $\mathbf{L}_d(\mathbf{i}, \varphi)$ and only the digit "3" following the second matrix should be omitted. This has been proved in a previous section.

Generally, the problem is reduced to determining the three coefficients $A_{2,0}$, $A_{1,1}$, $A_{0,2}$ for the first term of the co-energy function, and the five $A_{4,0}$, $A_{3,1}$, $A_{2,2}$, $A_{1,3}$, $A_{0,4}$, for the second one.

Algorithm based on the linkage fluxes

When starting with values of linkage fluxes ψ_1 and ψ_2 obtained from field computations, the coefficients of the Taylor series for function $\psi_1(i_1, i_2)$ and $\psi_2(i_1, i_2)$ should be estimated, while only the first two terms are considered

$$\psi_1(i_1, i_2) = \left\{ i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right\} \psi_1(i_1, i_2) \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^3 \right\} \psi_1(i_1, i_2) \Big|_0, \quad (19a)$$

$$\psi_2(i_1, i_2) = \left\{ i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right\} \psi_2(i_1, i_2) \Big|_0 + \frac{1}{6} \left\{ \left(i_1 \cdot \frac{\partial}{\partial i_1} + i_2 \cdot \frac{\partial}{\partial i_2} \right)^3 \right\} \psi_2(i_1, i_2) \Big|_0. \quad (19b)$$

However, those two functions are not independent as they must to fulfil condition

$$\frac{\partial \psi_1(i_1, i_2)}{\partial i_2} = \frac{\partial \psi_2(i_1, i_2)}{\partial i_1}.$$

It means that some partial derivatives in (19a, b) should be mutually related. For two functions only this can relatively easy be done, but when the number of linkage fluxes functions

grows it is not straightforward to find these relationships. In the considered case the formulas for the linkage fluxes (13a, b) fulfil the required condition because

$$\frac{\partial E_{\text{mo}}(i_1, i_2)}{\partial i_1 \partial i_2} = \frac{\partial E_{\text{mo}}(i_1, i_2)}{\partial i_2 \partial i_1}.$$

It means that the linkage fluxes should be approximated by the formulas

$$\psi_1(i_1, i_2) = (A_{2,0} \cdot i_1 + A_{1,1} \cdot i_2) + (A_{4,0} \cdot i_1^3 + 3A_{3,1} \cdot i_1^2 \cdot i_2 + 3A_{2,2} \cdot i_1 \cdot i_2^2 + A_{1,3} \cdot i_2^3) \quad (20a)$$

$$\psi_2(i_1, i_2) = (A_{1,1} \cdot i_1 + A_{0,2} \cdot i_2) + (A_{3,1} \cdot i_1^3 + 3A_{2,2} \cdot i_1^2 \cdot i_2 + 3A_{1,3} \cdot i_1 \cdot i_2^2 + A_{0,4} \cdot i_2^3) \quad (20b)$$

next steps are similar. The matrix of dynamic inductance $\mathbf{L}_d(\mathbf{i}, \varphi)$ in form (14) can be developed using the matrix of nonlinear inductances $\mathbf{L}_n(\mathbf{i}, \varphi)$ in form (16). Also, the co-energy function in form (12) can be completed because all necessary coefficients appear in formulas (20a, b).

Algorithm for direct computation of nonlinear inductances

The matrix of nonlinear inductances $\mathbf{L}_n(\mathbf{i}, \varphi)$, defined by formulas (9) and (10) is symmetric in a similar manner as the matrix on inductances for a linear magnetic circuit. As such, when the magnetic state of the magnetic circuit will be maintained for the given pair of currents (i_1, i_2) , the winding inductances can be found from classical formulas for the linear case

$$L_{n,k} = \frac{\psi_n}{i_k}.$$

By collecting the values for a selected part of space of currents (i_1, i_2) at a given value of angle φ , the nonlinear inductances can be found. However, these should be approximated as the three dependent functions for the case of two windings

$$L_{1,1}^n = A_{2,0} + (A_{4,0} \cdot i_1^2 + 2A_{3,1} \cdot i_1 \cdot i_2 + A_{2,2} \cdot i_2^2), \quad (21a)$$

$$L_{1,2}^n = A_{1,1} + (A_{3,1} \cdot i_1^2 + 2A_{2,2} \cdot i_1 \cdot i_2 + A_{1,3} \cdot i_2^2), \quad (21b)$$

$$L_{2,2}^n = A_{2,2} + (A_{2,2} \cdot i_1^2 + 2A_{1,3} \cdot i_1 \cdot i_2 + A_{0,4} \cdot i_2^2). \quad (21c)$$

In these three functions all coefficients which are necessary for writing up the formulas for the function describing the co-energy, the linkage fluxes and the dynamic inductances are present.

Comparing the three described algorithms, it can be stated that the number of functions, which have to be determined is different for each algorithm. For the simplest case of two windings, and starting with the co-energy only one two-variable function should be estimated. The linkage flux approach requires the two mutually related functions to be determined. The direct computing of nonlinear inductances increases the number of functions to be determined to three. Besides of that, the determined functions contain different ranks of powers of

currents in each algorithm respectively. Considering the same level of approximation (i.e. the two first terms of the Taylor series only) of the co-energy function, currents have ranks of powers up to four. In formulas for linkage fluxes, the maximal ranks of powers of currents is three and in formulas for the nonlinear inductances it equals two. It means that the co-energy is much more sensitive with respect to currents than linkage fluxes and linkage fluxes are more sensitive than nonlinear inductances. Consequently, the coefficients obtained, based on the nonlinear inductances could be determined more exactly than those based on the linkage fluxes and can be determined more accurately based on the co-energy. It is worth noticing that there is a balance between accuracy and the number of functions, which should be set up very carefully.

4. Problem of arbitrary number of windings

Determining inductances for the arbitrary, say N -windings, is even more complicated since all mathematical operations have to be carried out in the N -dimensional space of currents (i_1, i_2, \dots, i_N) . The number of functions, which should be determined, depends on the kind of initial data in computations. Only the algorithm starting with the co-energy always requires one function of N variables. The algorithm based on the values of the linkage fluxes requires N functions that are mutually dependent of N variables. The algorithm based on the values of the nonlinear inductances needs $(N + 1)N/2$ mutually dependent functions of N variables. Additionally, in order to obtain realistic approximations more than two first terms of the Taylor series should be taken into account. In order to take control over those difficulties the higher order forms can be used as in [5, 10]. Formulas below show relationships between the co-energy function, the linkage fluxes functions and the matrices of dynamic and nonlinear inductances

$$E_{\text{mo}}(\varphi, \mathbf{i}) = \frac{1}{2} \mathbf{i}^T \cdot \mathbf{A}_2(\varphi) \cdot \mathbf{i} + \frac{1}{4} \mathbf{i}^T \cdot \mathbf{A}_4(\varphi, \mathbf{i}) \cdot \mathbf{i} + \frac{1}{6} \mathbf{i}^T \cdot \mathbf{A}_6(\varphi, \mathbf{i}) \cdot \mathbf{i} + \dots,$$

$$\Psi(\mathbf{i}, \varphi) = (\mathbf{A}_2(\varphi) + \mathbf{A}_4(\varphi, \mathbf{i}) + \mathbf{A}_6(\varphi, \mathbf{i}) + \dots) \cdot \mathbf{i},$$

$$\mathbf{L}_d(\mathbf{i}, \varphi) = \mathbf{A}_2(\varphi) + 3 \cdot \mathbf{A}_4(\varphi, \mathbf{i}) + 5 \cdot \mathbf{A}_6(\varphi, \mathbf{i}) + \dots,$$

$$\mathbf{L}_n(\mathbf{i}, \varphi) = \mathbf{A}_2(\varphi) + \mathbf{A}_4(\varphi, \mathbf{i}) + \mathbf{A}_6(\varphi, \mathbf{i}) + \dots,$$

where: $\mathbf{i} = [i_1, i_2, \dots, i_N]$ is the vector of currents and $\Psi(\mathbf{i}, \varphi) = [\psi_1(\mathbf{i}, \varphi), \psi_2(\mathbf{i}, \varphi), \dots, \psi_N(\mathbf{i}, \varphi)]$ is the vector of linkage fluxes.

The more detailed forms of matrices $\mathbf{A}_2(\varphi)$, $\mathbf{A}_4(\varphi)$, $\mathbf{A}_6(\varphi)$, ... can be found in [2, 5, 10]. These formulas allow writing up the co-energy or the linkage fluxes or nonlinear inductances as functions of currents (i_1, i_2, \dots, i_N) using the same coefficients that should be found from data obtained from the magnetic field computations.

5. Conclusions

The new definition of nonlinear inductances of electrical machine windings presented in this paper leads to symmetrical inductance matrix, similarly as in case of linear magnetic circuit. Detailed considerations that lead to such definition are shown by the example of two windings. However, the nonlinear inductances can only be determined from the data obtained from magnetic field computations. The three different algorithms have been described. Respective algorithms can start with values of co-energy or from values of linked fluxes but also nonlinear inductances can be directly computed. The paper describes such algorithms by using the elementary example of two windings in nonlinear magnetic circuit. Finally, the extension of the above procedure for an arbitrary number of windings has been proposed.

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