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THERMOELASTIC ANALYSIS OF FUNCTIONALLY GRADED HOLLOW CIRCULAR DISC

A thermoelastic boundary value problem of a hollow circular disc made of functionally graded materials with arbitrary gradient is analysed. The steady-state temperature distribution is assumed to be the function of the radial coordinate with prescribed temperature at the inner and outer cylindrical boundary surfaces. The material properties are assumed to be arbitrary smooth functions of the radial coordinate. A coupled system of ordinary differential equations containing the radial displacement and stress function is derived and used to get the distribution of thermal stresses and radial displacements caused by axisymmetric mechanical and thermal loads. General analytical solutions of functionally graded disc with thermal loads are not available. The results obtained by the presented numerical method are verified by an analytical solution. The considered analytical solution is valid if the material properties, except the Poisson ratio, are expressed as power functions of the radial coordinate.

1. Introduction

Functionally graded materials (FGMs) are a class of relatively new and promising materials and they have emerged from the need to optimize material performance (Abondi et al. [1], Hirai [2], Suresh and Martensen [3]). In homogeneous materials the properties are constants, whereas in a FGM the material parameters vary continuously with position usually along one coordinate direction. Functionally graded linearly elastic structures can be considered as non-homogeneous elastic bodies whose material parameters are smooth functions of the position coordinates. Books by Lekhnitskii [4],

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Lomakin [5] and Sarkisyan [6] give solutions to many linearly elastic problem for non-homogeneous bodies.

The exact solutions for displacements and stresses of FGMs in the one-dimensional case of solids spheres and cylinders are presented by Lutz and Zimmerman [7], [8], who considered the non-homogeneous material properties as a linear functions of the radial coordinate. In both papers mentioned above, the thermal loads are caused by uniform heating, that is the temperature difference does not depend on the radial coordinate. Lutz and Zimmerman [7], [8] expressed the governing equations of thermoelasticity in terms of the Lamé parameters instead of Young's modulus and Poisson ratio, so that they were able to find analytical solutions without restricting the Poisson ratio to be constant. Their analytical solutions were represented by infinite power series of the radial coordinate. This type of the solutions was based on the Navier equation and it was derived by the applications of Frobenius theory. An optimal design for functionally graded cylinders according to the distribution of steady-state thermal stresses has been formulated by Tanaka et al. [9]. An analytical solutions for thermal stresses in hollow functionally graded vessels have been derived by Jabbary et al. [10]. A direct analytical method is used to solve the heat conduction and Navier equations under the conditions of plane-strain state by Jabbary et al. [10] assuming material model with simple power law and constant Poisson's ratio. In this case, the Navier equation is reduced to Euler type differential equation, whose solution is readily available. Study by Peng and Li [11] deals with the thermoelastic problem of functionally graded disc with arbitrarily varying material properties. The distribution of thermal stresses and radial displacement can be obtained by solving a Fredholm type integral equation.

In this paper, we consider the thermoelastic problem of a functionally graded disc whose material properties vary arbitrarily along the radial direction. A new numerical approach is presented, which is based on a coupled system of first order ordinary differential equations with variable coefficients. The unknown functions of the system of linear differential equations are the radial displacement and the stress function. General analytical solutions for this type of system of differential equations are not available. By numerically solving the system of differential equations, one can obtain the distribution of the thermal stresses and the radial displacement for arbitrary radial nonhomogeneity. This numerical solution is not restricted to constant Poisson's ratio. An important step to the realization of numerical solution is to transform the solution of derived two-point boundary value problem to an initial value problem. The applied numerical scheme chosen is the Runge-Kutta-Fehlberg method (rfk45). Consequently, we can give a verification of the presented numerical method, assuming power laws of material properties with exception of the Poisson's ratio, which is constant. An analytical solution to the system of differential equations is also presented.

2. Formulation of the governing equations

We consider a functionally graded hollow circular disc as shown in Fig. 1. R_1 and R_2 denote the inner and outer radii of the disc. The temperature difference is $T(r) = t(r) - t_0$, where $t = t(r)$ is the absolute temperature and t_0 is the reference temperature at which the stresses are zero if the disc is undeformed. $T = T(r)$ is obtained from the solution of steady-state heat conduction equation. The strain-displacement relations for axisymmetric plane-stress state are

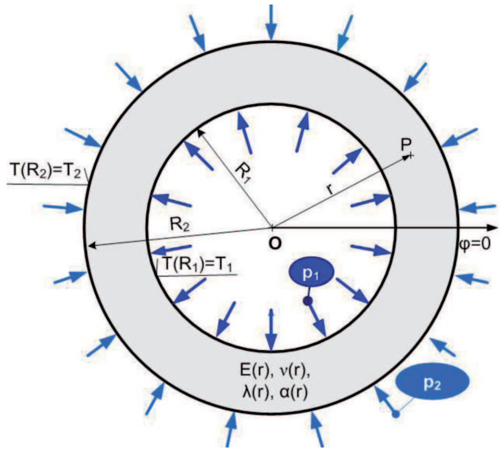

Fig. 1. The hollow functionally graded disc with the mechanical and thermal loads

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\varphi = \frac{u}{r}, \tag{1}$$

where $u = u(r)$ is the radial displacement field and ε_r, ε_φ are the normal strains in radial and circumferential directions, respectively. The stress-strain relationships, assuming that the conditions of plane-stress state are satisfied, have the following forms:

$$\sigma_r = \frac{E}{1-\nu^2}\left[\varepsilon_r + \nu\varepsilon_\varphi - \alpha(1+\nu)T\right], \tag{2}$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}\left[\nu\varepsilon_r + \varepsilon_\varphi - \alpha(1+\nu)T\right], \tag{3}$$

where σ_r and σ_φ are the normal stresses, E is the Young's modulus, ν is the Poisson ratio and α is the coefficient of thermal expansion all of which depend only on the radial coordinate r. The equilibrium equation in radial direction, neglecting the body force and the inertia terms, is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \quad R_1 \leq r \leq R_2. \tag{4}$$

The general solution of Eq. (4) in terms of stress function $V = V(r)$ can be represented as

$$\sigma_r = \frac{V}{r}, \qquad \sigma_\varphi = \frac{dV}{dr}, \quad R_1 \leq r \leq R_2. \tag{5}$$

After some manipulations, from Eqs. (1-3) and (5) we can derive the next system of ordinary differential equations for the displacement field and the stress function

$$\frac{du}{dr} = -\frac{\nu}{r}u + \frac{1-\nu^2}{rE}V + \alpha(1+\nu)T, \tag{6}$$

$$\frac{dV}{dr} = \frac{E}{r}u + \frac{\nu}{r}V - \alpha ET. \tag{7}$$

In Eqs. (6-7) all material properties depend only on the radial coordinate, that is $E = E(r)$, $\nu = \nu(r)$, $\alpha = \alpha(r)$, furthermore $T = T(r)$ is obtained from the solution of heat conduction equation [12].

3. The steady-state heat conduction problem

It is assumed that the temperature differences are given at the inner and outer cylindrical boundary surfaces (T_1 and T_2 respectively), so we have the following first kind thermal boundary conditions:

$$T(R_1) = T_1, \qquad T(R_2) = T_2. \tag{8}$$

The steady-state temperature difference field (without internal heat sources) satisfies the next equation [12]

$$\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{dT}{dr}\right] = 0, R_1 \leq r \leq R_2, \tag{9}$$

where $k = k(r)$ is the thermal conductivity of the functionally graded material. The solution of Eq. (9) under the boundary conditions (8) gives the temperature difference distribution along the radial coordinate

$$T(r) = T_1 + \frac{T_2 - T_1}{\int_{R_1}^{R_2} \frac{1}{\rho k(\rho)} d\rho} \int_{R_1}^{r} \frac{1}{\rho k(\rho)} d\rho, \quad R_1 \leq r \leq R_2. \tag{10}$$

4. Determination of the initial values

The next step is the determination of the initial values for the system of equations (6-7). The stress boundary conditions of the considered thermoelastic problem (Fig. 1) are

$$\sigma_r(R_1) = -p_1, \quad \sigma_r(R_2) = -p_2, \tag{11}$$

which can be expressed in terms of the stress function $V = V(r)$ such as

$$V(R_1) = -p_1 R_1, \quad V(R_2) = -p_2 R_2. \tag{12}$$

In Eqs. (11), (12) p_1 and p_2 are the applied pressures at the inner and the outer cylindrical boundary surfaces.

At first step we transform the two-point boundary value problem formulated by Eqs. (6), (7) and Eqs. (11), (12) into an initial value problem. This step is required for the realization of the numerical methods. To get the stresses and radial displacement for the considered thermoelastic problem, three numerical solutions will be used with three different initial values. The aim is to look for the suitable value of $u(R_1)$ which provides the validity of the prescribed boundary condition $(12)_2$. At first, we consider two solutions for system of equations (6-7) which are denoted by $[u_1(r), V_1(r)]$ and $[u_2(r), V_2(r)]$. These solutions have the next initial values:

$$u_1(R_1) = u_1 : \textit{arbitrary value}, \tag{13}$$

$$V_1(R_1) = -p_1 R_1, \tag{14}$$

$$u_2(R_1) = u_2 : \textit{arbitrary value, but } u_1 \neq u_2, \tag{15}$$

$$V_2(R_1) = -p_1 R_1. \tag{16}$$

By these solutions we compute u_3 as

$$u_3 = u_1 + \frac{u_2 - u_1}{V_2(R_2) - V_1(R_2)}(-p_2 R_2 - V_1(R_2)). \tag{17}$$

The solution of the thermoelastic boundary value problem formulated by Eqs. (1-4) and Eq. (12) is obtained from the numerical solution of system of equations (6-7) with the initial condition

$$u(R_1) = u_3, \qquad V(R_1) = -p_1 R_1. \tag{18}$$

The validity of this statement follows from the linearity of the considered thermoelastic boundary value problem.

5. An analytical solution

An analytical solution is developed for the case when the distributions of the material properties are assumed to be described with a power law along the radial coordinate as $E(r) = E_0 r^{m_1}, \alpha(r) = \alpha_0 r^{m_2}, k(r) = k_0 r^{m_3}$ and $\nu =$ = constant. For this type of inhomogeneity, there exists exact solution of the two-point boundary value problem defined by Eqs. (6), (7) and Eqs. (11), (12). This analytical solution will be used to check the numerical solution. The general solution of the following homogeneous system of ordinary differential equations

$$\frac{du_h}{dr} + \frac{\nu}{r}u_h - \frac{1-\nu^2}{E_0 r^{m_1+1}}V_h = 0, \tag{19}$$

$$\frac{dV_h}{dr} - E_0 r^{m_1-1}u_h - \frac{\nu}{r}V_h = 0, \tag{20}$$

are as follows

$$u_h = \frac{\lambda_1 + m_1 - \nu}{E_0}C_1 r^{\lambda_1} + \frac{\lambda_2 + m_1 - \nu}{E_0}C_2 r^{\lambda_2}, \tag{21}$$

$$V_h = C_1 r^{\lambda_1 + m_1} + C_2 r^{\lambda_2 + m_1}, \tag{22}$$

where

$$\lambda_1 = \frac{-m_1 + \sqrt{m_1^2 - 4m_1\nu + 4}}{2}, \tag{23}$$

$$\lambda_2 = \frac{-m_1 - \sqrt{m_1^2 - 4m_1\nu + 4}}{2}, \tag{24}$$

and C_1 and C_2 are arbitrary constants which can be obtained from the boundary condition (12). Here we note

$$m_1^2 - 4m_1\nu + 4 = (m_1 - 2\nu)^2 + 4(1-\nu^2) > 0, \tag{25}$$

this means that λ_1 and λ_2 are real numbers because $0 \leq \nu \leq 0.5$. For simplicity it is assumed that $T_2 = 0$. In this case, the temperature change is

$$T(r) = T_1 \frac{r^{-m_3} - R_2^{-m_3}}{R_1^{-m_3} - R_2^{-m_3}}, \quad R_1 \leq r \leq R_2. \tag{26}$$

Next, we seek a particular solution for the system of non-homogeneous differential equations

$$\frac{du_p}{dr} + \frac{\nu}{r}u_p - \frac{1-\nu^2}{E_0 r^{m_1+1}}V_p - \alpha_0 T_1 (1+\nu)\frac{r^{m_2-m_3} - R_2^{-m_3} r^{m_2}}{R_1^{-m_3} - R_2^{-m_3}} = 0, \tag{27}$$

$$\frac{dV_p}{dr} - E_0 r^{m_1-1} u_p - \frac{\nu}{r} V_p + \alpha_0 E_0 T_1 \frac{r^{m_1+m_2-m_3} - R_2^{-m_3} r^{m_1+m_2}}{R_1^{-m_3} - R_2^{-m_3}} = 0. \tag{28}$$

A simple computation shows that the particular solution for the system of Eqs. (27-28) is as follows

$$u_p = A_1 r^{m_2-m_3+1} + B_1 r^{m_2+1}, \tag{29}$$

$$V_p = A_2 r^{m_1+m_2-m_3+1} + B_2 r^{m_1+m_2+1}, \tag{30}$$

where

$$A_1 = \frac{\alpha_0 T_1 \left[1 - \nu^2 - (1+\nu)(m_1 + m_2 - m_3 - \nu + 1)\right]}{(R_2^{-m_3} - R_1^{-m_3})\left[(m_2 - m_3 + \nu + 1)(m_1 + m_2 - m_3 - \nu + 1) - 1 + \nu^2\right]}, \tag{31}$$

$$A_2 = -\frac{\alpha_0 E_0 T_1 (m_2 - m_3)}{(R_2^{-m_3} - R_1^{-m_3})\left[(m_2 - m_3 + \nu + 1)(m_1 + m_2 - m_3 - \nu + 1) - 1 + \nu^2\right]}, \tag{32}$$

$$B_1 = -\frac{\alpha_0 T_1 R_2^{-m_3} \left[(m_1 + m_2 - \nu + 1)(1+\nu) + 1 - \nu^2\right]}{(R_2^{-m_3} - R_1^{-m_3})\left[(m_2 + \nu + 1)(m_1 + m_2 - \nu + 1) - 1 + \nu^2\right]}, \tag{33}$$

$$B_2 = \frac{\alpha_0 E_0 T_1 R_2^{-m_3} m_2}{(R_1^{-m_3} - R_2^{-m_3})\left[(m_2 + \nu + 1)(m_1 + m_2 - \nu + 1) - 1 + \nu^2\right]}. \tag{34}$$

The complete solution for the system of equations (6-7) in the present case is

$$u(r) = u_h(r) + u_p(r), V(r) = V_h(r) + V_p(r). \tag{35}$$

The constants C_1 and C_2 can be obtained from the stress boundary condition (12) as the solution of the following system of linear equations

$$C_1 R_1^{\lambda_1+m_1} + C_2 R_1^{\lambda_2+m_1} + A_2 R_1^{m_1+m_2-m_3+1} + B_2 R_1^{m_1+m_2+1} = -p_1 R_1, \tag{36}$$

$$C_1 R_2^{\lambda_1+m_1} + C_2 R_2^{\lambda_2+m_1} + A_2 R_2^{m_1+m_2-m_3+1} + B_2 R_2^{m_1+m_2+1} = -p_2 R_2. \tag{37}$$

Solution of system of equations (36-37) gives

$$C_1 = -\frac{(p_1 R_1 + A_2 R_1^{m_1+m_2-m_3+1} + B_2 R_1^{m_1+m_2+1}) R_2^{\lambda_2+m_2}}{R_1^{\lambda_1+m_1} R_2^{\lambda_2+m_2} - R_1^{\lambda_2+m_1} R_2^{\lambda_1+m_1}} + \frac{(p_2 R_2 + A_2 R_2^{m_1+m_2-m_3+1} + B_2 R_2^{m_1+m_2+1}) R_1^{\lambda_2+m_1}}{R_1^{\lambda_1+m_1} R_2^{\lambda_2+m_2} - R_1^{\lambda_2+m_1} R_2^{\lambda_1+m_1}}, \tag{38}$$

$$C_2 = \frac{(p_1 R_1 + A_2 R_1^{m_1+m_2-m_3+1} + B_2 R_1^{m_1+m_2+1}) R_2^{\lambda_1+m_1}}{R_1^{\lambda_1+m_1} R_2^{\lambda_2+m_2} - R_1^{\lambda_2+m_1} R_2^{\lambda_1+m_1}} - \frac{(p_2 R_2 + A_2 R_2^{m_1+m_2-m_3+1} + B_2 R_2^{m_1+m_2+1}) R_1^{\lambda_1+m_1}}{R_1^{\lambda_1+m_1} R_2^{\lambda_2+m_2} - R_1^{\lambda_2+m_1} R_2^{\lambda_1+m_1}}. \tag{39}$$

Knowing the values of C_1 and C_2 we can get immediately the displacement and stresses as

$$u(r) = \frac{\lambda_1 + m_1 - \nu}{E_0} C_1 r^{\lambda_1} + \frac{\lambda_2 + m_1 - \nu}{E_0} C_2 r^{\lambda_2} + A_1 r^{m_2 - m_3 + 1} + B_1 r^{m_2 + 1}, \quad (40)$$

$$\sigma_r(r) = \frac{V(r)}{r} = C_1 r^{\lambda_1 + m_1 - 1} + C_2 r^{\lambda_2 + m_1 - 1} + A_2 r^{m_1 + m_2 - m_3} + B_2 r^{m_1 + m_2}, \quad (41)$$

$$\begin{aligned} \sigma_\varphi(r) = \frac{dV(r)}{dr} &= C_1(\lambda_1 + m_1) r^{\lambda_1 + m_1 - 1} + C_2(\lambda_2 + m_1) r^{\lambda_2 + m_1 - 1} + \\ &+ A_2(m_1 + m_2 - m_3 + 1) r^{m_1 + m_2 - m_3} + B_2(m_1 + m_2 + 1) r^{m_1 + m_2}. \end{aligned} \quad (42)$$

We note that for plane-strain deformation, when the material properties, expect the Poisson's ratio, are power function of the radial coordinate an exact solution is presented in [10]. The solution given by Jabbari et al [10] is based on the analytical solution of non-homogeneous Navier equation. For plane-strain deformation in terms of radial displacement the Navier equation is as follows

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + (m_1 + 1)\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + \left(\frac{\nu m_1}{1-\nu} - 1\right)\frac{u}{r^2} = \frac{(1+\nu)}{1-\nu}\alpha_0 \left((m_1 + m_2) r^{m_2 - 1} T + r^{m_2} \frac{\mathrm{d}T}{\mathrm{d}r}\right). \quad (43)$$

6. Comparison of the numerical solution with the analytical solution

The following numerical data are used in the presented example:

$$R_1 = 0.5\,\mathrm{m},\ R_2 = 1\,\mathrm{m},\ m_1 = m_2 = m_3 = m = -3,\ \nu = 0.3,\ E_0 = 2 \cdot 10^{11} \frac{\mathrm{N}}{\mathrm{m}^5},$$
$$\alpha_0 = 1.2 \cdot 10^{-6} \frac{1}{{}^\circ\mathrm{Cm}^3},$$
$$T(R_1) = 100^\circ\mathrm{C},\ \ T(R_2) = 0^\circ\mathrm{C},\ \ p_1 = 30\,\mathrm{MPa},\ \ p_2 = 0\ \mathrm{MPa}.$$

The results of the computations are summarized in Table 1, where the definition for the relative error is given by equation (44)

$$e_M(\%) = \left| \frac{M_{analytical} - M_{numerical}}{M_{analytical}} \right| \cdot 100, \quad M(r) = u(r),\ \sigma_r(r),\ \sigma_\varphi(r). \quad (44)$$

The numerical solution is based on the Runge-Kutta-Fehlberg method and it is carried out by the application of Maple 15 *dsolve/numeric/rkf45*. Figures 2 and 3 show the solutions for the displacement fields and the normal stresses in three cases. The solid lines illustrates the previously calculated values (Table 1), the dash lines indicate the solutions for the case when there is

Table 1.
Comparison of the radial stresses and displacement field obtained by numerical and analytical methods

r [m]	solutions	displacement	normal stresses		relative errors		
		u (mm)	σ_r (MPa)	σ_φ (MPa)	e_u (%)	$e_{\sigma r}$ (%)	$e_{\sigma\varphi}$ (%)
0.5	analytic.	0.2544600	–30	–730.7277	$1.559\cdot10^{-5}$	0	$1.737\cdot10^{-5}$
	numeric.	0.2544601	–30	–730.7276			
0.6	analytic.	0.3248146	–66.16169	20.50325	$1.171\cdot10^{-5}$	$0.351\cdot10^{-5}$	$29.02\cdot10^{-5}$
	numeric.	0.3248146	–66.16169	20.50331			
0.7	analytic.	0.3501088	–43.52514	125.4060	$1.061\cdot10^{-5}$	$3.490\cdot10^{-5}$	$2.837\cdot10^{-5}$
	numeric.	0.3501089	–43.52513	125.4060			
0.8	analytic.	0.3546277	–22.64223	115.30513	$1.131\cdot10^{-5}$	$5.284\cdot10^{-5}$	$2.012\cdot10^{-5}$
	numeric.	0.3546277	–22.64222	115.30515			
0.9	analytic.	0.3494365	–8.721734	89.91601	$1.189\cdot10^{-5}$	$17.92\cdot10^{-5}$	$1.934\cdot10^{-5}$
	numeric.	0.3494365	–8.721718	89.91602			
1	analytic.	0.3401298	0	68.02597	$1.410\cdot10^{-5}$	–	$2.082\cdot10^{-5}$
	numeric.	0.3401299	$1.522\cdot10^{-6}$	68.02598			

only thermal loading on the inner curved boundary surface ($p_1 = p_2 = 0$ MPa, $T_1 = 100°$C and $T_2 = 0°$C). Furthermore, the dash-dot lines denote the curves of the displacement field and normal stresses for the mechanical loading in Figs. 2 and 3 ($p_1 = 30$ MPa, $p_2 = 0$ MPa and $T_1 = T_2 = 0°$C).

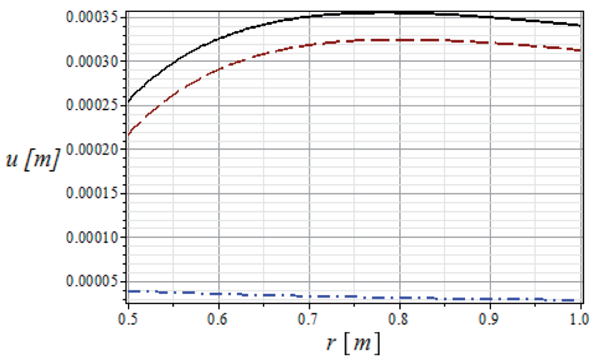

Fig. 2. The radialdisplacement of the thermal (dash line), the mechanical (dash dot line) and the combined thermo-mechanical (solid line) loading

In the next example we consider the case when the power indexes have different values, such as $m_1 = -1.5$, $m_2 = 1.5$ and $m_3 = -3$. Table 2 shows the results of the calculation.

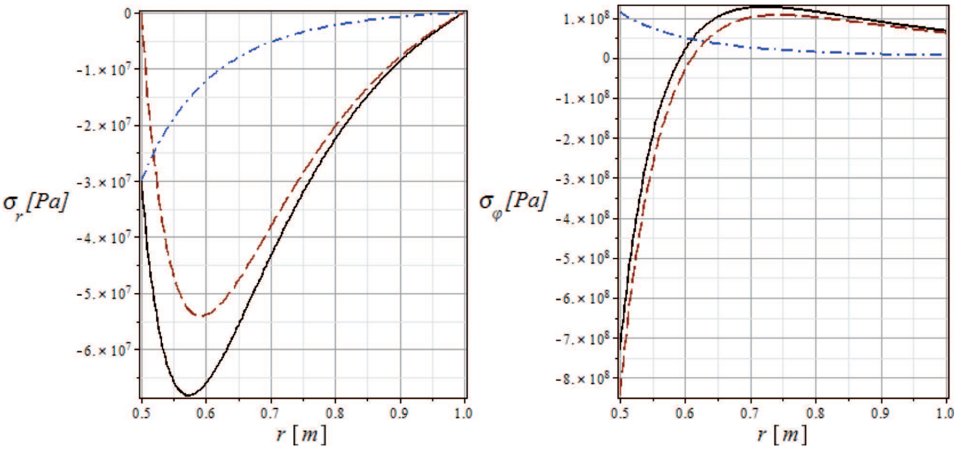

Fig. 3. The normal stresses caused by the thermal (dash line), the mechanical (dash dot line) and the combined thermo-mechanical (solid line) loads

Table 2.
The solutions for the radial stresses and displacement fields when $m_1 = -1.5$, $m_2 = 1.5$ and $m_3 = -3$

r [m]	solutions	displacement	normal stresses		relative errors		
		u (mm)	σ_r (MPa)	σ_φ (MPa)	e_u (%)	$e_{\sigma r}$ (%)	$e_{\sigma\varphi}$ (%)
0.5	analytic.	0.9930158	–30	79.34692	$4.391 \cdot 10^{-5}$	0	$6.216 \cdot 10^{-5}$
	numeric.	0.9930154	–30	79.34687			
0.6	analytic.	0.9606242	–15.22278	42.82697	$4.245 \cdot 10^{-5}$	$2.903 \cdot 10^{-5}$	$7.137 \cdot 10^{-5}$
	numeric.	0.9606238	–15.22279	42.82694			
0.7	analytic.	0.9570188	–8.272115	26.18580	$4.301 \cdot 10^{-5}$	$8.153 \cdot 10^{-5}$	$8.435 \cdot 10^{-5}$
	numeric.	0.9570184	–8.272122	26.18578			
0.8	analytic.	0.9667615	-4.482209	19.04745	$4.309 \cdot 10^{-5}$	$18.88 \cdot 10^{-5}$	$8.981 \cdot 10^{-5}$
	numeric.	0.9667611	–4.482217	19.04743			
0.9	analytic.	0.9744266	–2.002965	17.32734	$4.474 \cdot 10^{-5}$	$47.97 \cdot 10^{-5}$	$8.213 \cdot 10^{-5}$
	numeric.	0.9744262	–2.002974	17.32733			
1	analytic.	0.9622763	0	19.24552	$4.816 \cdot 10^{-5}$	–	$6.397 \cdot 10^{-5}$
	numeric.	0.9622759	$9.136 \cdot 10^{-6}$	19.24551			

Table 1 and 2 show a good agreement between the results computed by the presented numerical and the analytical methods. The stress field in the case of the numerical solution can be obtained from the next equations

$$\sigma_r = \frac{V}{r}, \qquad \sigma_\varphi = \frac{dV}{dr} = E_0 r^{m-1} u + \frac{\nu}{r} V - \alpha_0 E_0 r^{2m} T, \tag{45}$$

according to Eqs. (5), (7) and the dependences of E and α from the radial coordinate. Some different values of power index $m = m_1 = m_2 = m_3$ ($m =$ $= -2, -1, 0, 1, 2$), the graphs of the normalized temperature difference function (T/T_1), normalized radial displacement (u/R_2) and normalized stresses ($\sigma_r/p_1, \sigma_\varphi/p_1$) are shown in Figures (4-7).

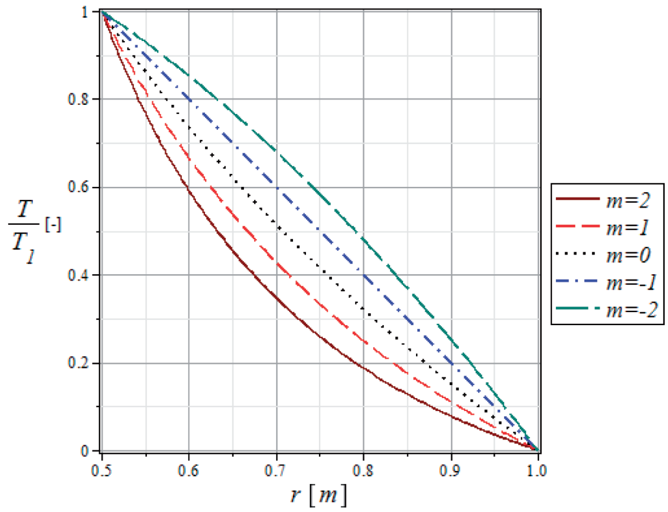

Fig. 4. The radial distribution of the normalized temperature change

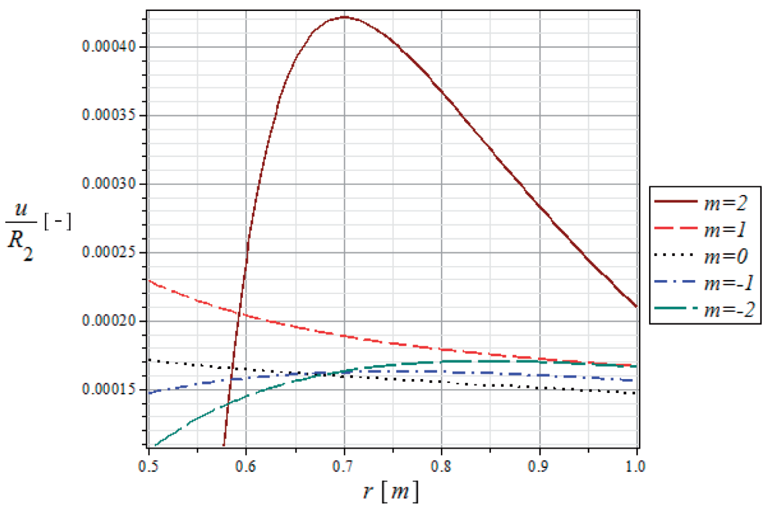

Fig. 5. The radial distribution of the normalized radial displacement

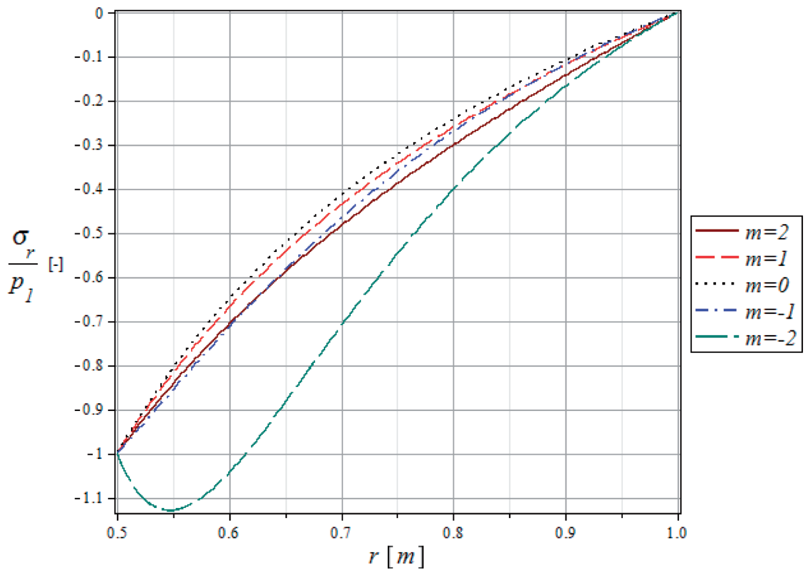

Fig. 6. The radial distribution of the normalized radial stress

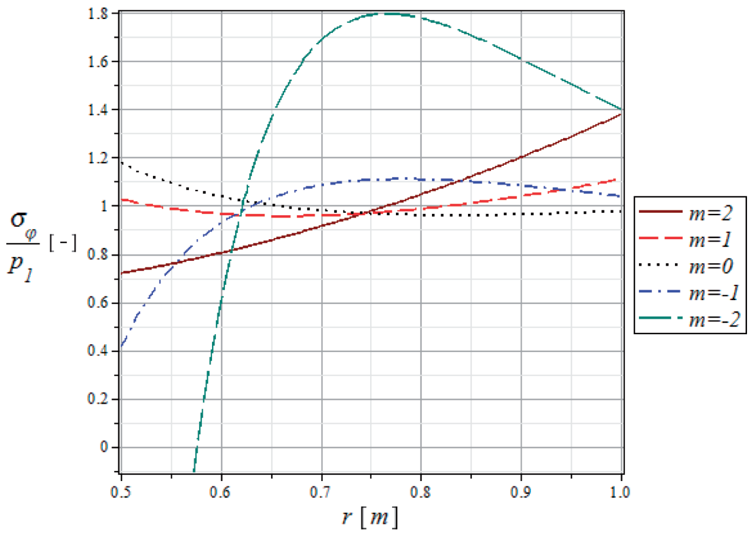

Fig. 7. The radial distribution of the normalized circumferential stress

7. Conclusions

A new numerical method has been presented to treat the thermoelastic problem of a functionally graded hollow cylindrical disc with arbitrarily varying material properties along the radial direction. A coupled system of ordinary differential equations for the radial displacement and stress function are derived, which constitutes a two-point boundary value problem with the prescribed stress boundary conditions. This two-point boundary value problem is transformed into an initial value problem, whose numerical solution gives the radial displacements and radial and hoop stresses. The results of the numerical computations are verified by an analytical solution of the derived two-point boundary value problem.

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Analiza termosprężysta wydrążonej kolistej tarczy gradientowej

S t r e s z c z e n i e

W artykule analizowano problem termosprężystej wartości brzegowej dla wydrążonej, kolistej tarczy gradientowej o dowolnym gradiencie materiału. Przyjęto, że rozkład temperatury w stanie ustalonym jest funkcją współrzędnej promieniowej, z założoną z góry temperaturą na wewnętrznej i zewnętrznej powierzchni. Założono, że właściwości materiału są dowolnymi funkcjami gładkimi współrzędnej promieniowej. Właściwości sprężyste opisano systemem zwyczajnych równań różniczkowych zawierających przemieszczenia promieniowe i funkcje naprężenia. Równania zostały wykorzystane do wyznaczenia rozkładów naprężeń termicznych i przemieszczeń promieniowych powodowanych przez osiowosymetryczne obciążenia mechaniczne i termiczne. Ogólne rozwiązania analityczne dla tarcz gradientowych poddanych obciążeniom termicznym nie są znane. Wyniki uzyskane w zaprezentowanej metodzie numerycznej zweryfikowano przez porównanie z rozwiązaniem analitycznym. Rozwiązania analityczne, rozważane przez autorów, są słuszne gdy właściwości materiałowe, z wyjątkiem współczynnika Poissona, można opisać funkcjami potęgowymi współrzędnej promieniowej.