

Relationships Between Two Definitions of Fading Memory for Discrete-Time Systems

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Abstract— In this paper, we refer to two definitions of fading memory property, which were published in the literature, for discrete-time circuits and systems. One of these definitions relates to systems working with signals (sequences) defined for both the positive and negative integers, expanding from minus infinity to plus infinity. On the other hand, the second one refers to systems processing sequences defined only for nonnegative integers, that is starting at the discrete-time point equal to zero and expanding to plus infinity. We show here that the second definition follows from the first one. That is they are not independent. Moreover, we also show that if an operator describing a system possesses a fading memory according to the second definition, then its associated operator has this property, too, but in accordance with the first definition.

Keywords— discrete-time systems, time-invariant operators, fading memory property

I. INTRODUCTION

VERY important class of circuit and systems are those that behave as objects with decaying memory property - called also disappearing or vanishing or fading memory. Such a property means that, when the time elapses, an actual behavior of a circuit or system depends less and less upon its behavior in the remote past. Observing objects in a real world, we see that most of them possess such the property. Therefore, it seems to be natural and ubiquitous. Moreover, linear as well as nonlinear systems possess the above property. With regard to linear ones of a continuous time described by a convolution integral, a sufficient condition for possessing a fading memory is an impulse response function of time having decaying properties. Similarly, in the case of nonlinear systems described by a Volterra series, if their so-called nonlinear impulse responses [1] have decaying character their behavior underlies also the property of vanishing memory. In this context, note also that the systems having fading memory can be approximated as those which have simply a finite memory - and the length of this memory can be evaluated.

Notion of fading memory plays a fundamental role in systems theory a long time, but in fact until the recent publications of Boyd and Chua [2], and of Sandberg [3], [4] in the eighties and nineties of the last century, there was no precise mathematical definition of it. And, as shown in the papers mentioned above, there are a few mathematical definitions possible, which differ slightly from each other, but some of them are perfectly equivalent. However, it is not an objective of this paper to discuss all their peculiarities. Regarding this stuff, more can be found in a book [5] of this

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author, having character of a tutorial as well as of a survey of the results published previously (mostly) in the research periodicals. Here, we are concerned on a problem that has been also discussed in [5]. So, to some extent, the material presented here is a repetition of that which can be found in [5]. However, some parts of the material in this paper constitute an refreshment of the above. For example, we use here another argument, expressed by (27), to arrive at the final result of section IV. This is more satisfactory than the argumentation used in [5].

First of all, however, the objective of this conference paper is to make the audience more familiar with the topics related with the ubiquitous fading memory. As said, there exist mathematically advanced papers, and even chapters in books (as for instance in [5]), which seem to be sufficient for the researchers interested in the above concepts. However, the author of this paper is not sure that this is the case and sees the need for more explanation on such for a as conferences.

In particular, in this paper, we derive the relations which exist between the two definitions of fading memory property, presented for discrete-time systems by Boyd and Chua in [2].

Here, we deal with the discrete-time signals that is with the sequences of elements of which values depend upon the discrete-time variable. We treat these sequences (signals) as elements of the space of bounded sequences with the norm

$$\|x\| \stackrel{df}{=} \sup |x(k)|, \quad (1)$$

where the values of k belong to the set of integers $\mathbf{Z} (-\infty \leq k \leq \infty)$ or to the set of nonnegative integers $\mathbf{Z}_+ (k \geq 0)$. The space with the norm given by (1) is called the l^∞ space, or more precisely, $l^\infty(\mathbf{Z})$ or $l^\infty(\mathbf{Z}_+)$ space, when the set of arguments of the sequences $x(k)$ is shown explicitly.

Moreover, in what follows, we will use a delay operator U_τ defined by the following relation

$$(U_\tau x)(k) \stackrel{df}{=} x(k - \tau), \quad (2)$$

where $k, \tau \in \mathbf{Z}$ or $k, \tau \in \mathbf{Z}_+$. Furthermore, we say that an operator N is time-invariant (TI) if

$$(U_\tau Nx)(k) = (NU_\tau x)(k). \quad (3)$$

Additionally, we will choose the value of a delay τ in (2) and (3) such that the resulting sequences will also belong to the space $l^\infty(\mathbf{Z}_+)$ when $x(k) \in l^\infty(\mathbf{Z}_+)$.

II. TWO DEFINITIONS OF FADING MEMORY FOR DISCRETE-TIME SYSTEMS

Boyd and Chua formulated in [2, Section III and Appendix A5, respectively] two slightly different definitions of the fading memory property for continuous-time systems. Moreover, they remarked that these definitions can be also used in the case of discrete-time systems, with only re-defining the domain and image spaces of the corresponding operators. In what follows, we recall the above definitions put into a form proper for the type the latter systems. So, in more detail, the first one deals with the systems working on sequences $x(k) \in l^\infty(\mathbf{Z})$ and the second with those working on sequences $x(k)$ belonging to a smaller space $l^\infty(\mathbf{Z}_+)$.

A. Fading Memory Definition No. 1 (FMD1)

Def. 1: A time-invariant operator $N: l^\infty(\mathbf{Z}) \rightarrow l^\infty(\mathbf{Z})$ has fading memory on the subspace B of $l^\infty(\mathbf{Z})$ if there is a decreasing sequence $w: \mathbf{Z}_+ \rightarrow (0,1)$, $\lim_{k \rightarrow \infty} w(k) = 0$, such that for each $x \in B$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for all $v \in B$ the following relation:

$$\sup_{k \leq 0} |x(k) - v(k)| w(-k) < \delta \rightarrow |(Nx)(0) - (Nv)(0)| < \varepsilon \quad (4)$$

holds.

B. Fading Memory Definition No. 2 (FMD2)

Def. 2: A time-invariant operator $N: l^\infty(\mathbf{Z}_+) \rightarrow l^\infty(\mathbf{Z}_+)$ possesses fading memory on the subspace B_+ of $l^\infty(\mathbf{Z}_+)$ if there is a decreasing sequence $w: \mathbf{Z}_+ \rightarrow (0,1)$, $\lim_{k \rightarrow \infty} w(k) = 0$, such that for each $x \in B_+$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for all $v \in B_+$ the following implication:

$$\sup_{0 \leq \tau \leq k} |x(\tau) - v(\tau)| w(k - \tau) < \delta \rightarrow |(Nx)(k) - (Nv)(k)| < \varepsilon \quad (5)$$

holds.

(On this occasion, see a needed correction introduced on the left-side of (5) compared with the corresponding expression in [2].)

One can ask whether really two separate definitions of the fading memory are needed, as used in [2] - one for the $N: l^\infty(\mathbf{Z}) \rightarrow l^\infty(\mathbf{Z})$ operators and the second for the $N: l^\infty(\mathbf{Z}_+) \rightarrow l^\infty(\mathbf{Z}_+)$ ones. In this paper, we will try to answer this question. There are, maybe, some relations between them? However, before starting this, let us observe that there is something analogous in the following question whether the convolution sum

$$y(k) = \sum_{i=0}^{\infty} h(i)x(k-i) \quad (6)$$

for causal sequences (i.e. $x(k) \in l^\infty(\mathbf{Z}_+)$) can be derived from a more general formula

$$y(k) = \sum_{i=-\infty}^{\infty} h(i)x(k-i) \quad (7)$$

for sequences belonging to the space $l^\infty(\mathbf{Z})$, where $h(i)$ stands for an impulse response of a linear time-invariant (LTI) system. Obviously, this can be done by restricting ourselves to a subspace of $l^\infty(\mathbf{Z})$ with the sequences having their elements $x(k) \equiv 0$ for $k < 0$ in (7).

Finally in this section, we point out that besides [2] the FMD2 was also used in [6], [7], and [8].

III. DERIVATION OF FMD2 FROM FMD1

We shall show here, as was also done in [5], that the second definition of fading memory (FMD2), which was thought out for systems considered only for nonnegative times ($k \geq 0$), follows from its more general form (FMD1), as it would so hold. And, to this end, we take into account the definition of FMD2 with the time-invariant operator $N: l^\infty(\mathbf{Z}_+) \rightarrow l^\infty(\mathbf{Z}_+)$ and the sequences x and v belonging to $B_+ \in l^\infty(\mathbf{Z}_+)$.

Now, to be able to use the definition FMD1, we must first redefine the operator N and the sequences on which it works. For this, we introduce sequences x_f and v_f , with the discrete-time arguments assuming the values from the whole set \mathbf{Z} , as

$$x_f(k) = x(k) \text{ for } k \geq 0 \text{ and } 0 \text{ for } k < 0 \quad (8)$$

and

$$v_f(k) = v(k) \text{ for } k \geq 0 \text{ and } 0 \text{ for } k < 0, \quad (9)$$

respectively. Note that the sequences x_f and v_f such defined can be viewed as the extensions of the sequences taken from the space $l^\infty(\mathbf{Z}_+)$ to the corresponding ones belonging to the space $l^\infty(\mathbf{Z})$.

Further, with the use of the extended sequences x_f and v_f , we can define in the following way a time-invariant operator N_f

$$(N_f x_f)(k) \stackrel{df}{=} (Nx)(k) \text{ for } k \geq 0 \text{ and } 0 \text{ for } k < 0 \quad (10)$$

that works on such sequences. The sequence x_f in (10) stands for all the sequences given by (8), which form a subspace B_f of the space $l^\infty(\mathbf{Z})$.

Assume now that the operator N_f given by (10) exhibits the fading memory property, therefore, we can apply to it the FMD1. However, before, let us consider the time-shifted sequences $U_{-k} x_f(l)$ and $U_{-k} v_f(l)$ of which form is illustrated in Fig. 1 (for the first of them).

Applying the definition given by (4) to the operator N_f working on the sequences $U_{-k} x_f(l)$ and $U_{-k} v_f(l)$, we get

$$\sup_{l \leq 0} |U_{-k} x_f(l) - U_{-k} v_f(l)| w(-l) < \delta \rightarrow \rightarrow |(N_f U_{-k} x_f)(0) - (N_f U_{-k} v_f)(0)| < \varepsilon \quad (11)$$

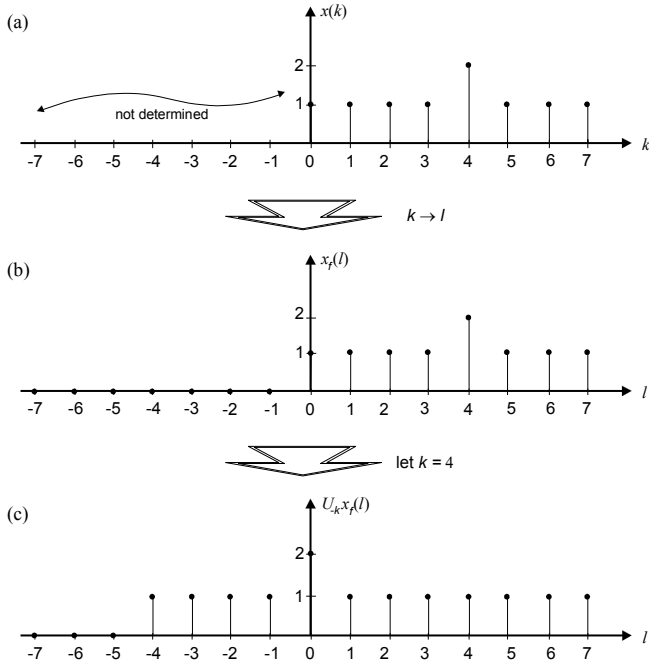


Fig. 1. (a) Original sequence belonging to the space $l^\infty(\mathbf{Z}_+)$, (b) the sequence $x_f(l) \in l^\infty(\mathbf{Z})$ created from $x(k)$, (c) the time - shifted sequence $U_{-k} x_f(l)$; this figure was taken from [5].

Using next the definition (2) of the delay operator in the expression on the left-hand side of (11), we can rewrite it as

$$\sup_{l \leq 0} |x_f(l+k) - v_f(l+k)| w(-l) < \delta \quad (12)$$

Using a new variable $l+k = \tau$ in (12), we arrive at

$$\sup_{\tau-k \leq 0} |x_f(\tau) - v_f(\tau)| w(k-\tau) < \delta \quad (13)$$

And, because $x_f(\tau) = v_f(\tau) = 0$ for $\tau < 0$, see Fig. 1(b), (13) assumes the form

$$\sup_{0 \leq \tau \leq k} |x_f(\tau) - v_f(\tau)| w(k-\tau) < \delta \quad (14)$$

Finally, taking into account the definitions (8) and (9) in (14), we can rewrite it as

$$\sup_{0 \leq \tau \leq k} |x(k) - v(k)| w(k-\tau) < \delta \quad (15)$$

Now, let us return to the right-hand side of the implication in (11). Note first that the expression on the right-hand side of (11) can be rewritten, by virtue of (2), as

$$|(N_f x_f)(k+0) - (N_f v_f)(k+0)| < \varepsilon \quad (16)$$

or equivalently as

$$|(N_f x_f)(k) - (N_f v_f)(k)| < \varepsilon \quad (17)$$

Observe now that in our above derivations we assumed that the time-shifting parameter (delay) k to be greater or equal to zero. Therefore, having this fact in mind and the definition of the operator N_f given by (10), we can rewrite (17) as

$$|(Nx)(k) - (Nv)(k)| < \varepsilon, \quad k \geq 0 \quad (18)$$

We are now in a position to summarize the results of our derivation. So, taking into account the achieved results (15) and (18) for the left- and right-hand sides of implication (11), and comparing them with the definition (5), we conclude that in fact the definition FMD2 can be derived from the first one, FMD1. In other words, the definition FMD2 is only a specific variant of the definition FMD1, derived for sequences defined only for nonnegative times ($k \geq 0$).

IV. THE REVERSE PROBLEM

Let us also ask whether a derivation of a seeming to be more general FMD1 from a more specific FMD2 is possible. In this section, we shall try to answer such the question.

However, before starting formulation of the problem, let us illustrate it on a simple example. And to this end, consider as before two descriptions for LTI discrete-time systems: first "a more specific", given by (6), and second, "a more general", given by (7). Note that to get (7) from (6), we must first extend both the domain and image of the operator described by (6), for example, by letting the lower summation limit to be equal to $-\infty$ and the discrete-time variable k starting from $-\infty$, too. By doing this, we arrive in an extended operator working on the sequences x_f . In the second step, we postulate equality between the values "produced" by that extended operator based on (6) with the values received from (7) for the same input sequences x_f and time-shifted $x_{f\tau}$. Obviously, we fulfil the above requirements in our example.

Similarly, we shall first extend the operator N occurring in FMD2 for solving the problem formulated in this section. Note that the operator N_f cannot play such a role because it assumes identically zero values for the negative time instants.

A proper extended operator of $N : l^\infty(\mathbf{Z}_+) \rightarrow l^\infty(\mathbf{Z}_+)$ will be an operator having the same form as N , but allowing to work with the sequences from the space $l^\infty(\mathbf{Z})$. Let us call it as N_g ; it should have the following property

$$(N_g x_f)(k) \stackrel{df}{=} (Nx)(k) \text{ for } k \geq 0 \text{ and } (N_g x_f)(k) \text{ for } k < 0. \quad (19)$$

Moreover, $(N_g x_{f\tau})(k)$ should be well defined. Let us recall the FDM2 for the operator N for $k = 0$. We have then

$$\sup_{0 \leq \tau \leq 0} |x(\tau) - v(\tau)| w(0-\tau) < \delta \rightarrow |(Nx)(0) - (Nv)(0)| < \varepsilon, \quad (20)$$

which can be further rewritten as

$$|x(0) - v(0)| w(0) < \delta \rightarrow |(Nx)(0) - (Nv)(0)| < \varepsilon \quad (21)$$

Obviously, the values $x(0)$ and $v(0)$ are equal to $x_f(0)$ and $v_f(0)$, respectively. Hence, (21) can be also expressed as

$$\left| x_f(0) - v_f(0) \right| w(0) < \delta \rightarrow \left| (N_f x_f)(0) - (N_f v_f)(0) \right| < \varepsilon. \quad (22)$$

Note that because of the properties of the sequences x_f and v_f given by (8) and (9), respectively, we can write the following

$$\sup_{k < 0} \left| x_f(k) - v_f(k) \right| w(-k) \equiv 0 < \delta. \quad (23)$$

So, taking both (22) and (23) together, we finally arrive at

$$\sup_{k \leq 0} \left| x_f(k) - v_f(k) \right| w(-k) < \delta \rightarrow \left| (N_f x_f)(0) - (N_f v_f)(0) \right| < \varepsilon. \quad (24)$$

Comparison of (24) with (4) shows the latter implication is nothing else than the FMD1, however, restricted to the sequences x_f . Hence, this is not fully satisfactory result. In what follows, we will try to achieve more. To this end, let us consider the operator N_g defined in a descriptive way above. We shall check whether this operator has a fading memory in the sense of the FMD1, when the original operator $N : l^\infty(\mathbf{Z}_+) \rightarrow l^\infty(\mathbf{Z}_+)$ possesses in the sense of the FMD2. And, let us start considering quite formally the FMD2 given by (5). We rewrite this definition introducing a new variable $\tau' = \tau - k$, what leads to

$$\begin{aligned} \sup_{-k \leq \tau' \leq 0} \left| x(k + \tau') - v(k + \tau') \right| w(-\tau') < \delta \rightarrow \\ \rightarrow \left| (Nx)(k) - (Nv)(k) \right| < \varepsilon \end{aligned} \quad (25)$$

Further, observe that (25) can be rewritten in the following way

$$\begin{aligned} \sup_{-k \leq \tau' \leq 0} \left| x(k + \tau') - v(k + \tau') \right| w(-\tau') < \delta \rightarrow \\ \rightarrow \left| (Nx)(k+0) - (Nv)(k+0) \right| < \varepsilon \end{aligned} \quad (26)$$

Using in (26) the notation of the extended time-shifted sequences introduced before together with the definition of the extended operator N_g , we arrive at

$$\begin{aligned} \sup_{-k \leq \tau' \leq 0} \left| x_{f(-k)}(\tau') - v_{f(-k)}(\tau') \right| w(-\tau') < \delta \rightarrow \\ \rightarrow \left| (N_g x_{f(-k)})(0) - (N_g v_{f(-k)})(0) \right| < \varepsilon \end{aligned} \quad (27)$$

Take now into account any sequence x belonging to some ball B of the space $l^\infty(\mathbf{Z})$ such that this sequence is identical with a certain sequence $x_{f(-k)}(\tau)$ for all $\tau \geq -k$. Further, choose such a k that the following

$$\begin{aligned} \sup_{-\infty \leq \tau' \leq k} \left| x(\tau') - v(\tau') \right| w(-\tau') < \\ < \sup_{-k \leq \tau' \leq 0} \left| x_{f(-k)}(\tau') = x(\tau') - v_{f(-k)}(\tau') = v(\tau') \right| w(-\tau') \end{aligned} \quad (28)$$

will hold. Thereby, we arrive in such a situation that, from the point of view of the left-hand side inequality in (27), the sequences x and $x_{f(-k)}$ belonging to some ball B of the space $l^\infty(\mathbf{Z})$ will be indistinguishable. So, finally, it follows from (27) that

$$\sup_{\tau \leq 0} \left| x(\tau) - v(\tau) \right| w(-\tau) < \delta \rightarrow \left| (N_g x)(0) - (N_g v)(0) \right| < \varepsilon \quad (29)$$

holds for all the sequences of the ball B of the space $l^\infty(\mathbf{Z})$. Further, this allows us to conclude that when the operator $N : l^\infty(\mathbf{Z}_+) \rightarrow l^\infty(\mathbf{Z}_+)$ possesses the fading memory in the sense of the FMD2, then its extended operator N_g possesses the fading memory in the sense of the definition FMD1.

V. REMARK

Closely related topics to that of fading memory are the relations between the original expansions for linear and nonlinear operators and their associated ones. However, this is a material for another paper.

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