Analysis of the possibility of determining the internal structure of composite material by estimating its thermal diffusivity

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Abstract The aim of this paper is analysis of the possibility of determining the internal structure of the fibrous composite material by estimating its thermal diffusivity. A thermal diffusivity of the composite material was determined by applying inverse heat conduction method and measurement data. The idea of the proposed method depends on measuring the time-dependent temperature distribution at selected points of the sample and identification of the thermal diffusivity by solving a transient inverse heat conduction problem. The investigated system which was used for the identification of thermal parameters consists of two cylindrical samples, in which transient temperature field is forced by the electric heater located between them. The temperature response of the system is measured in the chosen point of sample. One dimensional discrete mathematical model of the transient heat conduction within the investigated sample has been formulated based on the control volume method. The optimal dynamic filtration method as solution of the inverse problem has been applied to identify unknown diffusivity of multi-layered fibrous composite material. Next using this thermal diffusivity of the composite material its internal structure was determined. The chosen results have been presented in the paper.

Keywords: Mathematical model; Direct problem; Inverse heat conduction problem; Composite material; Test stand

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Nomenclature

a	–	thermal diffusivity, m^2/s
$b_{i,j}$	–	coefficients in Eq. (7)
c	–	specific heat, J/kg K
d	–	radial size of the sample, m
k	–	thermal conductivity, W/m K
$\mathbf{F}$	–	function of the state which defines the relation between the vectors of state in two adjacent time steps
$\mathbf{G}$	–	elements of the matrix $\mathbf{G}$ represent derivatives of the state function $\mathbf{F}_i$ with respect to the elements of the state vector y_j
$\mathbf{H}$	–	matrix describes the relationship between measured (observed) and estimated quantities
L	–	number of measurement sensors (thermocouples)
M	–	number of identified parameters
N	–	number of nodal temperatures
$\mathbf{P}$	–	covariance matrix of the estimate errors
R	–	thermal resistance, m^{2}K/ W
$\dot{q}$	–	density of heat flux, W/m^2
$\dot{Q}$	–	heat flux (power of the electric heater), W
t	–	temperature, °C
u	–	volumetric share of fibres in composite material
$\mathbf{V}$	–	covariance matrix of measurements errors
x	–	coordinate in Fig. 1
$\mathbf{y}$	–	vector of state
$\mathbf{z}$	–	vector of observations

Greek symbols

δ	–	axial thickness of the sample, m
Δx_i	–	thickness of ith layer
$\Delta\lambda$	–	time step
ν	–	measurement error
ρ	–	density, kg/m^3
τ	–	time, s
ω	–	maximal disturbance of the measurement result

Subscripts

i	–	refers to the node
k	–	refers to the time step

1   Introduction

As known presently produced composite material are most often multi layered fibrous materials with the reproducible structure. Therefore the thermal diffusivity of these material depends on the thermal properties of the matrix and the fibres as well as the volumetric fraction of the fibres in the

composite material. Hence, knowledge of the true values of thermal diffusivity of materials is very important to determine the volumetric fraction of the fibres in composite material (internal structure of the composite) [1]. Therefore based on the identification of the thermal diffusivity of multilayered fibrous composite material the possibilities of determining its internal structure were presented in the paper.

Determination of the thermal diffusivity of composite material is based on the solution of the inverse heat conduction problem in the investigated sample for given boundary conditions and given geometry [2,3]. The inverse heat conduction problem is generally solved in two stages. In the first stage, based on a suitably formulated mathematical model, the direct heat conduction problem is solved. Auxiliary measurements and their characteristics (for example the temperature field) are also determined. These quantities will be used in the second stage of solving the algorithm.

In the second stage, making use of measurement data and previously determined quantities, the inverse problem is solved and the final quantities are determined.

The specific feature of the considered problem is that the objective function does not depend on the identified parameters in the explicit way. In order to find the relationship between the identified parameters and changes of the objective function we must solve the direct transient heat conduction boundary problem. In this paper to solve the direct boundary problem, the mathematical model has been formulated based on the control volume method. The inverse problem was formulated as an optimisation problem and solved by using the optimal dynamic filtration method. Information about input data required to solve the inverse heat conduction problem was obtained by solving the direct heat conduction problem or from the measurement stand.

The chosen results of research have been presented.

2 Formulation of the mathematical model and solution of the direct heat conduction problem

One-dimensional heat conduction in the analysed process (Fig. 1) can be described by a well-known differential equation of the form [4]

$$\rho c \frac{\partial t}{\partial \tau} = k \frac{\partial^2 t}{\partial x^2}, \quad k = \text{idem} . \tag{1}$$
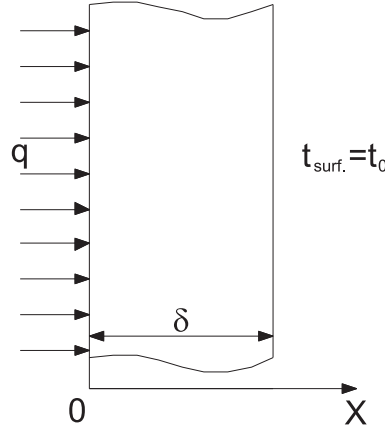
Figure 1. The analyzed sample with given boundary conditions.

The electric heater is located between two identical samples, hence the problem becomes symmetrical and we can analyse only one of the sample. The boundary conditions for that case can be written in the form

$$-k\frac{\partial t}{\partial x}\bigg|_{x=0} = \dot{q} \;, \tag{2}$$

$$t(x,\tau)\bigg|_{x=\pm\delta} = t_{surf.} = t_0 \;, \tag{3}$$

where $x = 0$ denotes the location of the heater, δ is the thickness of the sample and $\dot{q}$ is the density of heat flux.

The initial temperature in the sample is known and kept uniform, so the initial condition for the boundary heat conduction problem has the form:

$$t(x, \tau = 0) = t_0 \;. \tag{4}$$

For the necessity of the Kalman's filter method [5,6], the considered transient boundary value problem (1)–(4) in the discrete form should be written. The discrete mathematical model has been formulated based on the control volume method (Fig. 2).

An example of the energy balance for the node '2' can be written in the form

$$\rho \, c \, \Delta x_2 \, \frac{dt_1}{d\tau} = \rho \, c \, \Delta x_2 \frac{t_2^{k+1} - t_2^k}{\Delta \tau} = \sum_j \dot{Q}_{j2} =$$

$$= \frac{t_1^k - t_2^k}{R_{1-2}} + \frac{t_3^k - t_2^k}{R_{3-2}} = \frac{t_1^k - t_2^k}{\left(\Delta x_1 + \frac{\Delta x_2}{2}\right)/k} + \frac{t_3^k - t_2^k}{\left(\frac{\Delta x_2}{2} + \frac{\Delta x_3}{2}\right)/k} \tag{5}$$
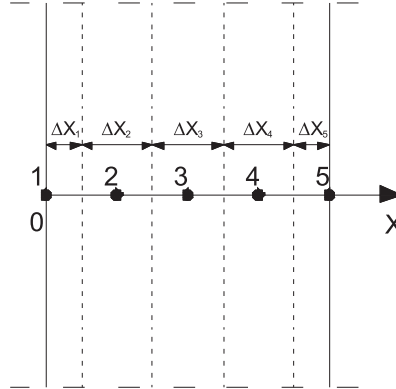
Figure 2. Division of the sample into differential elements.

Introducing the thermal diffusivity a and transforming Eq. (5) the following expression was obtained

$$
\begin{aligned}
t_2^{k+1} = {} & \left(\frac{a\,\Delta\tau}{\Delta x_2 \left(\Delta x_1 + \frac{\Delta x_2}{2} \right)} \right) t_1^k + \\
& + \left[1 - \left(\frac{a\,\Delta\tau}{\Delta x_2 \left(\Delta x_1 + \frac{\Delta x_2}{2} \right)} + \frac{a\,\Delta\tau}{\Delta x_2 \left(\frac{\Delta x_2}{2} + \frac{\Delta x_3}{2} \right)} \right) \right] t_2^k + \\
& + \left(\frac{a\,\Delta\tau}{\Delta x_2 \left(\frac{\Delta x_2}{2} + \frac{\Delta x_3}{2} \right)} \right) t_3^k .
\end{aligned} \tag{6}
$$

So the discrete mathematical model of the transient temperature field within the sample can be written in the form of the following system equations

$$
\begin{aligned}
t_1^{k+1} &= (1 + b_{11})t_1^k + b_{12}t_2^k + \gamma_1 \\
t_2^{k+1} &= b_{21}t_1^k + (1 + b_{22})t_2^k + b_{23}t_3^k \\
t_3^{k+1} &= b_{32}t_2^k + (1 + b_{33})t_3^k + b_{34}t_4^k \\
t_4^{k+1} &= b_{43}t_3^k + (1 + b_{44})t_4^k + b_{45}t_5^k \\
t_5^{k+1} &= b_{54}t_4^k + (1 + b_{55})t_5^k
\end{aligned} \tag{7}
$$

where γ includes the boundary conduction data, but coefficients b_{ij} contain

the thermal diffusivity of layers. For node '1' b_{ij} is defined

$$b_{ij} = -\frac{a\Delta\tau}{\Delta x_i(\Delta x_i + \frac{\Delta x_{i+1}}{2})}, \quad \left(\text{for } i = j\right)$$

$$b_{ij} = \frac{a\Delta\tau}{\Delta x_i(\Delta x_i + \frac{\Delta x_{i+1}}{2})}, \quad \left(\text{for } i < j\right)$$

(8)

and for node '2' b_{ij} as

$$b_{ij} = -\left[\frac{a\Delta\tau}{\Delta x_i\left(\Delta x_{i-1} + \frac{\Delta x_i}{2}\right)} + \frac{a\Delta\tau}{\Delta x_i\left(\frac{\Delta x_i}{2} + \frac{\Delta x_{i+1}}{2}\right)}\right], \quad \text{for } i = j,$$

$$b_{ij} = \frac{a\Delta\tau}{\Delta x_i\left(\Delta x_{i-1} + \frac{\Delta x_i}{2}\right)}, \quad \text{for } i > j.$$

(9)

Similary for node '4' b_{ij} is defined as

$$b_{ij} = -\left[\frac{a\Delta\tau}{\Delta x_i\left(\frac{\Delta x_{i-1}}{2} + \frac{\Delta x_i}{2}\right)} + \frac{a\Delta\tau}{\Delta x_i\left(\frac{\Delta x_i}{2} + \Delta x_{i+1}\right)}\right], \quad \text{for } i = j,$$

$$b_{ij} = \frac{a\Delta\tau}{\Delta x_i\left(\frac{\Delta x_i}{2} + \Delta x_{i+1}\right)}, \quad \text{for } i < j,$$

(10)

and for node '5' b_{ij} as

$$b_{ij} = \frac{a\Delta\tau}{\Delta x_i\left(\Delta x_i + \frac{\Delta x_{i-1}}{2}\right)}, \quad \text{for } i > j,$$

$$b_{ij} = -\frac{a\Delta\tau}{\Delta x_i\left(\Delta x_i + \frac{\Delta x_{i-1}}{2}\right)}, \quad \text{for } i = j.$$

(11)

For remaining cases not defined above b_{ij} can be written as

$$b_{ij} = \frac{2a\Delta\tau}{\Delta x_i\left(\Delta x_{i-1} + \Delta x_i\right)}, \quad \text{for } i > j,$$

$$b_{ij} = -\frac{2a\Delta\tau}{\Delta x_i}\left(\frac{1}{\Delta x_{i-1} + \Delta x_i} + \frac{1}{\Delta x_i + \Delta x_{i+1}}\right), \quad \text{for } i = j,$$

$$b_{ij} = \frac{2a\Delta\tau}{\Delta x_i\left(\Delta x_i + \Delta x_{i+1}\right)}, \quad \text{for } i < j,$$

(12)

where:

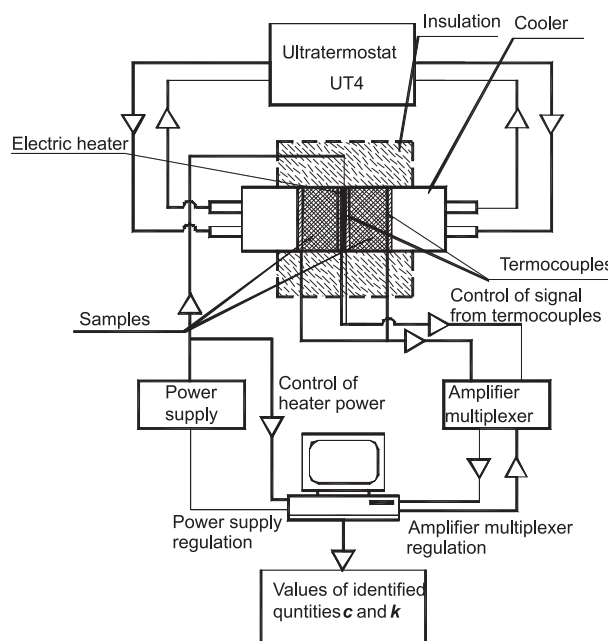|		
t_i^k, t_i^{k+1}	–	temperature in ith node and k and $k+1$ time steps,
a	–	thermal diffusivity determined at each time step ,
$\Delta\tau$	–	length of time step,
Δx_i	–	thickness of the ith layer in the discrete plate.

3   Description of the test stand

Figure 3. Scheme of the test stand.

The scheme of the test stand is presented in Fig. 3. In the symmetrical system two samples are located on both sides of the electric heater. The heater is made of plastic foil coated with the electricity-resistant layer to minimise its heat capacity. The important technical problem is to measure the temperature of four surfaces of the samples. This is done by means of a thin copper plate with Ni-NiCr thermocouple welded in their centres. Those plates are located between the heater and the samples and on the cooled outside surfaces of the samples. The other effect of applying copper plates is that the temperature of the surfaces is more uniform. The samples are cylinders with a diameter d and thickness δ. The samples, heater and

copper plates are well insulated on the cylindrical surface. The heater is connected to a stabilised electric current-voltage power supply. The power of the heater results from the voltage, U, and current, I, of the heater

$$\dot{Q} = UI \ . \tag{13}$$

The heat flux $\dot{q}$ per surface unit of each sample equals:

$$\dot{q} = \frac{2\dot{Q}}{\pi d^2} \ . \tag{14}$$

The experiment is as follows. After assembling the samples and auxiliary equipment, all functions during the experiment are controled by a special computer program. The computer switches on the power supply, controls and registers the voltage and current, registers the temperatures and stops the procedure if one of the given criteria is reached (i.e., time of the experiment or maximum temperature of the sample). The results of measurements are stored in the memory of the computer.

4   Kalman filter to solve the inverse problem

To involve the discrete Kalman filter procedure into the solution procedure of the inverse heat conduction problem it is necessary to write the discrete transient temperature field within the sample in the form of so called matrix equation of state [5–7]:

$$\mathbf{y}_{k+1} = \mathbf{F}_{k+1,k}(\mathbf{y}_k) \ , \tag{15}$$

where $\mathbf{y}_k$, $\mathbf{y}_{k+1}$ are the extended state vectors, which include the N nodal temperatures and M evaluated coefficients. The product of that causes that $\mathbf{F}_{k+1,k}$ vector valued function of state variables $\mathbf{y}$ is nonlinear. In the case under scruting $M = 1$. So the extended vector of state can be written as:

$$\mathbf{y}^T = [t_1, t_2, ..., t_N, a] = [y_1, ..., y_N, \ddot{y}_{N+M}] \ . \tag{16}$$

The relationship between the results of temperature measurements $\mathbf{z}$ (the so-called vector of observations) in selected L sensors located in the sample and the vector of state $\mathbf{y}$ at a given time step $k + 1$ has the form

$$\mathbf{z}_{k+1} = \mathbf{H}\mathbf{y}_k + \mathbf{v}_{k+1} \ , \tag{17}$$

where the matrix $\mathbf{H}$ of size $L \times M$ consists of the elements equal to unity corresponding to the measured temperatures and with all others elements equal to zero, because only the nodal temperatures are measured. Vector $\mathbf{v}_{k+1}$ represents measurement errors (Gaussian white noise with zero mean) of the covariance matrix $\mathbf{V}_{k+1}$.

Because $\mathbf{F}(\mathbf{y})$ is a nonlinear function of the state variables and discrete linear Kalman filter can not be used, so in order to solve the inverse problem by Kalman filter the function $\mathbf{F}(\mathbf{y})$ must be linearized. After two steps (prediction and filtration – correction) the Kalman filtering process contains the following equations:

- Prediction:

$$\tilde{\mathbf{y}}_{k+1,k} = \mathbf{F}_{k+1,k}(\tilde{\mathbf{y}}_k) \tag{18}$$

and the covariance matrix of the prediction estimate errors can be written as

$$\mathbf{P}_{k+1,k} = \mathbf{G}_{k+1,k}\mathbf{P}_{k,k}\mathbf{G}^T_{k+1,k} , \tag{19}$$

where $\mathbf{G}$ is the square matrix of the size $((N + M)(N + M))$. The elements of matrix $\mathbf{G}$ represent derivatives of the state function F_i with respect to the unknown quantity y_j

$$G_{i,j} = \frac{\partial F_i(y_1, ...y_N, y_{N+1}, ..., y_{N+M})}{\partial y_j} . \tag{20}$$

- Correction:

$$\tilde{\mathbf{y}}_{k+1} = \tilde{\mathbf{y}}_{k+1,k} + \mathbf{K}_{k+1}\big[\mathbf{z}_{k+1} - \mathbf{H}\tilde{\mathbf{y}}_{k+1,k}\big] , \tag{21}$$

where $\mathbf{K}_{k+1}$ is the so-called Kalman gain matrix and can be expressed as

$$\mathbf{K}_{k+1} = \mathbf{P}_{k+1,k}\mathbf{H}^T\big[\mathbf{H}\mathbf{W}_{k+1,k}\mathbf{H}^T + \mathbf{V}_{k+1}\big]^{-1} . \tag{22}$$

The covariance matrix of estimate errors has the form:

$$\mathbf{P}_{k+1} = \mathbf{P}_{k+1/k} - \mathbf{P}_{k+1/k}\mathbf{H}^T[\mathbf{H}\mathbf{P}_{k+1/k}\mathbf{H}^T + \mathbf{V}_{k+1}]^{-1}\mathbf{H}\mathbf{P}_{k+1/k} . \tag{23}$$

The calculation procedure starts at the time $k = 0$ for the given *a priori* initial vector $\mathbf{y}_0$ and covariance matrix $\mathbf{P}_{0,0}$. At each time step the algorithm uses the measurement results $\mathbf{z}_{k+1}$ from the number of L sensors (thermocouples) located within the sample.

5 Selected results of the research

At first the accuracy of the proposed method was verified by a numerical experiment [8]. To solve the inverse problem for numerical experiment results of 'measurements' t_i^{meas} were obtained by adding a noise term $(\omega\nu)$ to the results of solution of the direct boundary heat conduction problem (for given values of a) t_i according to the relationship

$$t_i^{meas} = t_i + \omega\nu \; , \tag{24}$$

where ν is the standard deviation of measurement errors and ω is maximal disturbance of the measurement result. For normally distributed errors (Gaussian distributed noise) with 99% confidence for the measured data, ω lies in the range $-2.576 \leq \omega \leq 2.576$, and the value of ω is calculated by a random generator.

In the numerical experiment the shape of the samples was assumed as the cylinders with a diameter $d = 72$ mm, thickness $\delta = 9.1$ mm and density of material $\rho = 1700$ kg/m^3. The power of the heater varied during each experiment. The example value of power, P, is 2.49 W, with a heat flux $\dot{q} = 306.12$ W/m^2. The measurement sensors were located on the internal $(x = 0)$ and external $(x = \delta)$ surfaces of the samples. The simulated measurement data were obtained using the simulated exact results obtained from the solution of the direct heat conduction problem and disturbed by the different error ν, Eq. (21).

The temperature within the sample was measured with time step $\Delta\tau = 1$ s. The influence of the measurement errors $v = 0.01 \div 0.1$ K on the results of identification was examined, but results presented below were obtained for the error $\nu = 0.05$ K.

The thermal property assumed for the numerical experiment is diffusivity, $a = 1.1439 \times 10^{-7}$ m^2/s. The following entry value of the searched parameter was assumed: $a = 1.1029 * 10^{-7}$ m^2/s. After calculations by means of the Kalman filter method the following result was obtained: $a = 1.144 \times 10^{-7}$ m^2/s.

Next, using the real measurement data for different content of the fibres in the composite material and the worked out computer program, the thermal diffusivities were determined (the program is worked out by autor in Delphi environment). The average temperatures of heated and cooled surfaces of the samples for exemplary real measurement data are shown in Fig. 4. The results obtained by means of the proposed algorithm have been presented in the Tab. 1 and in Fig. 5.
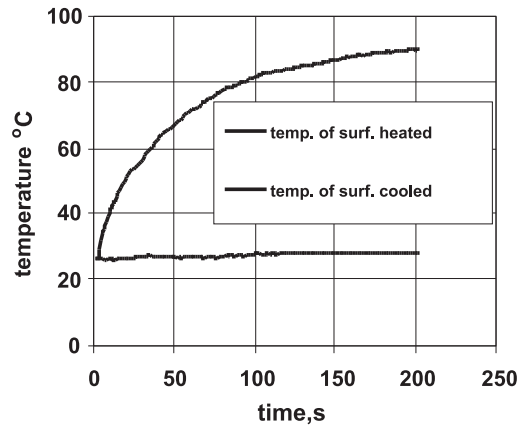
Figure 4. Temperature distribution on the surfaces of the sample as a function of time.
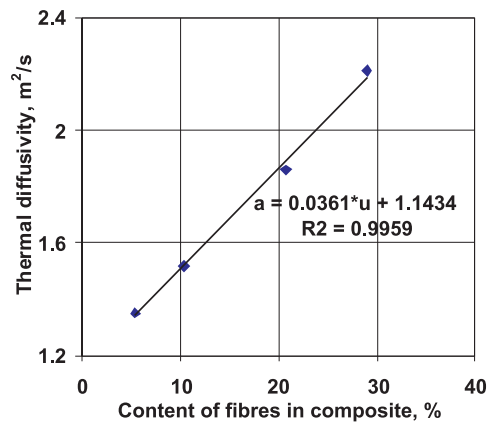

Figure 5. Thermal diffusivity a as a function of a volumetric share of a fibres u in the composite material, R2 is the correlation coefficient.

Table 1. Results of estimation of the thermal diffusivity.

No. of sample	Content of fibres in composite material, [%]	Thermal diffusivity $a \times 10^7$, [m²/s]
1	5.41	1.35
2	10.40	1.52
3	20.78	1.86
4	28.91	2.21

The appointed values of thermal diffusivity of samples with a thickness 9.1 mm and different contents of fibres in composite material provided the basis for determination of the mathematical relation between thermal diffusivity and share of fibres in the composite. In the presented diagram it is visible that this dependence is linear and has the form: $a = 0.0361U + 1.1434$. The values of the coefficients of this line were determined by applying the method of linear regression. The appointed function describes well the relation between diffusivity and content of fibres, which confirms the value of the correlation coefficient.

6   Final conclusions

The final conclusions may be listed as follows:

- The inverse heat conduction solution problem based on the combination control volume method, measurement data and the Kalman filter method constitutes a very effective tool for identification of the thermal diffusivity of the fibrous composite materials.

- The method enables us to estimate the accuracy of identification parameter, what is a very important feature of the algorithm.

- The proposed approach makes it possible to short the time of experiments which is another advantage of this method.

- On the basis of the obtained results we can affirm, that exists a possibility to determine the share of fibres in the composite material based on the determined thermal diffusivity.

References

[1] HOSSELMAN D.P.H., DONALDSON K.Y.: *Effects of detector nonlinearity and specimen size on the apparent thermal diffusivity of NIST 8425 graphite.* Int. J. Thermophys. **11**(1990), 3, 573–585.

[2] BECK J.V., ARNOLD K.: *Parameter Estimation in Engineering and Science*, Wiley, New York 1977.

[3] KUCYPERA S.: *Determination of thermophysical parameters of orthotropic materials based on the solution the inverse heat conduction problems.* XIII Symposium on Heat and Mass Transfer, Koszalin-Darłówko 2007, Vol. 2, 633-Õ640 (in Polish).

[4] ÖZISIK M.N., ORLANDE H.R.: *Inverse Heat Transfer.* Taylor & Francis, New York 2000.

[5] SCARPA F., MILANO G.: *Kalman smoothing technique applied to the inverse heat conduction problems.* Numer. Heat Transfer B-Fund. **28**(1995), 79–96.

[6] SKOREK J., KUCYPERA S.: *Dynamic filtration based on the inverse approach for determination heat conductivity and specific heat of solids.* In: Proc. 34th ASME National Heat Transfer Conf., Symposium on Inverse Thermal Problem, CD, Pittsburgh 2000.

[7] MOULTANOVSKY A.V, REKADA M.: *Inverse heat conduction problem approach to identify the thermal characteristics of super-hard synthetic materials.* Inverse Prob. Eng. **10**(2002), 1, 19–39.

[8] MZALI F., ALABOUCHI F., NASRALLAH S. PETIT D.: *Optimal experiment design and thermo-physical characterisation of a plastically deformed solid. Inverse Problem.* Design and Optimization Symposium, Miami, Floryda, April 2007, 16–28.