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ECONOMETRIC BALANCING OF A SOCIAL ACCOUNTING MATRIX UNDER A POWER-LAW HYPOTHESIS

1. INTRODUCTION

Contrary to many other fields, macroeconomics has neglected the link between phenomena and power-law (PL)[1], characterizing non-extensive complex systems within the class of Levy's process laws. In light of recent literature, the amplitude and frequency of macroeconomic fluctuations are not considered to substantially diverge from many other extreme events, natural or human-related, once they are explained in the same time (or space) scale. Following a few recent studies related to applying non-extensive entropy to economics, this study extends the theoretical model (e.g., Bwanakare, 2013a, b; Tsallis, 2004) and proposes a new direction for applications in solving ill-posed inverse problems. In this study, a social accounting matrix (SAM) will be balanced to illustrate this new technique.

In the rest of this introduction, the rationale of applying PL is presented. According to many studies (e.g., Bottazzi, 2007; Champernowne, 1953; Gabaix, 2008), a large array of economic laws take the form of PL, in particular, macroeconomic scaling laws, distribution of income, wealth, the size of cities and firms[2], and the distribution of financial variables such as returns and trading volume. Mantegna, Stanley (1999) have studied the dynamics of a general system composed of interacting units each with a complex internal structure comprising many subunits where they grow in a multiplicative way over a period of 20 years. They found the system following a PL distribution. It is worth noting the similarity of such a system with the internal mechanism of national account tables, like SAMs. Ikeda, Souma (2008) have worked on an international comparison of labour productivity distribution for manufacturing and non-manufacturing firms. A power-law distribution in terms of firms and sector productivity was found in US and Japan data. Testing Gibrat's law of proportionate effect, Fujiwara et al. (2004) have found, among other things, that the upper-tail of the distribution of firm size can be fitted with a power-law (Pareto–Zipf law). In a recent monograph, Bwanakare (2013b) has proposed a theorem linking low-frequency time

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[1] For details about a PL, see, e.g. Gabaix (2008).
[2] See Bottazzi et al. (2007) for a different standpoint on the subject.

series macroeconomic phenomena- and thus input output accounts- with PL distribution. The above citations are not exhaustive.

The central point is that a PL displays, besides its well-known scaling law, a set of interesting characterizations related to its aggregative properties, in that it is *conserved under addition, multiplication, polynomial transformation, minimum and maximum.* Basically, non-extensive (Tsallis) entropy is a thermodynamic concept which, contrary to that of Boltzmann-Gibbs-Shannon, is characterized by complex dependency between elements of non-ergodic systems and independency from initial conditions, fitting power-law a PL distribution (Tsallis, 2009). As opposed to the Gaussian[3] family model, a non-ergodic system suggests that micro-states of the system do not display identical odds of appearing. From the microeconomic prospective[4], this suggests that some economic agents' behaviour does happen more frequently than generally expected- then a heavy queue- and may rely on distant memory and complex correlations. While the Gaussian related Shannon-Kullback-Leibler (SKL) entropy approach is well suited in cases that exhibit limited perturbations, exponential-family phenomena, it remains less appropriate for a class of more complex PL driven shocks, the ubiquity of which, as already mentioned above, now seems evident in nature or social science. Testing PL multifractal properties requires high-frequency series. The higher the series frequency is, the more significant the test outputs about these properties are. The distribution with an exponential tail might correspond to an intermediate stage between a distribution with the PL asymptotics and a very large time lag limit-a Gaussian (Dragulescu, Yakovenko, 2001; Rak et al. 2007). Recently, Nielsen, Nock (2012) have casted exponential family form into PL-related Tsallis non-extensive entropy expression and shown conditions for a closed-form. However, delimiting threshold values for law transition- which is a function of frequency level- is difficult since, to our knowledge, neither a parametric nor non-parametric test yet exists.

Thus, applying Gaussian law systematically could be misleading in the case of some aggregated series and lead in many cases to instable solutions, for example, when a random error diverges enough from the Gaussian model[5] (i.e., with q parameter equaling unity). The methodology presented below fits well with more types of series when applying q-Tsallis entropy. In fact, Gaussian law can be generalized by a class of a few types of Higher-Order Entropy Estimators (Golan, Perloff, 2001; Tsallis, 2009) among which there is Tsallis non-extensive entropy, which presents the valuable additional quality of concavity- then stability- along the existence interval

[3] Then, this law includes all discrete laws converging to normal law. This observation is important for such a study dealing with low frequency time series.

[4] A SAM reflects a general macroeconomic equilibrium based upon microeconomic behaviour of economic agents through an aggregative process.

[5] For instance, data from statistical survey might display systematic errors.

characterizing most real world phenomena. Furthermore, as we will see below, the *q*-Tsallis parameter presents the strong advantage of monitoring complexity of any system. Among rival methods, only it can measure how far a given random phenomenon is from the Gaussian benchmark. Since the generated empirical solution constitutes a converging case of Gaussian law, outputs of the present work should remain qualitatively comparable with those that can be produced by other rival approaches, such as the RAS approach. However, at least two advantages of the proposed technique deserve to be emphasized. The first relates to the possibility of deriving the *q*-Tsallis parameter, thereby allowing for assessment of the complexity level of the analyzed system. The second advantage is from an epistemological standpoint. By proposing the non-extensive entropy approach for balancing a social accounting matrix- which is a one-period time series sample- we extend one of the main laws of modern physics (the generalized second law of thermodynamics) to low frequency economic time series and, thus, propose a new competitive econometric instrument for economic modeling.

This paper is organized as follows: Section II is devoted to presenting the link between Kullback-Leibler (K-L) information divergence and non-extensive Tsallis entropy. Section III presents a generalized linear non-extensive entropy econometric model. Then, for empirical applications, a Tsallis cross-entropy econometric model for SAM parameter estimation is presented with details. Section IV proposes parameter area inference for the estimated model. Section V presents the principal theoretical aspects of a SAM structure and its balancing. Section VI presents model outputs, and the last section concludes the paper with some comments and suggestions.

2. Q-GENERALIZATION OF THE K-L RELATIVE ENTROPY AND CONSTRAINING PROBLEM

To derive non-extensive entropy formulation, one first needs to set up the three simplest differential equations and their inverse functions (see Tsallis, 2009) and, next, unify these three cases (without preserving linearity). One then gets:

$$\frac{dy}{dx} = y^q \quad (y(0)=1;\ q \in \Re). \tag{1}$$

We observe that this expression displays power-law distribution form.
Its solution is

$$y = [1 + (1-q)x]^{\frac{1}{1-q}} \equiv e_q^x (e_1^x = e^x)$$

and its inverse function is

$$y = \frac{x^{(1-q)} - 1}{1 - q} \equiv \ln_q x \; (\ln_1 x = \ln x). \tag{2}$$

The above eq. (2) represents the non-extensive (Tsallis) entropy formula[6], which can be explained in logarithmic terms $\ln_q x$ where q stands for the basis. In particular, for q approaching unity, we get the traditional Gibbs-Shannon maximum entropy (Shannon, 1948) upon which the K-L information divergence index (IDI)[7] (Kullback, Leibler, 1951; Maasoumi, 1993) is dually related. The symbol "$y \equiv f(x)$" means y is defined to be the same as $f(x)$ under certain assumptions taken in context. This can be generalized in a straightforward manner as follows (Tsallis, 2009):

$$I_q(p, p^{(0)}) \equiv -\int dx p(x) \ln_q \left[\frac{p^{(0)}(x)}{p(x)} \right] = \int dx p(x) \frac{\left[p(x) / p^{(0)}(x) \right]^{q-1} - 1}{q - 1}$$

or,

$$I_q(p, p^{(0)}) \equiv \sum p_i \frac{\left[p_i / p_i^{(0)} \right]^{q-1} - 1}{q - 1} \tag{3}$$

in discrete cases. Thus, index $I_q(p, p^{(0)})$ stands for the traditional K-L IDI between hypotheses p and $p^{(0)}$, provided that q converges to unity[8]. There exist two main versions of Kullback-Leibler divergence (K-Ld) in Tsallis statistics, namely the usual generalized K-Ld shown above and the generalized Bregman K-Ld (Tsallis et al. 1998). Following Venkatesan, Plastino (2011), problems have been encountered in empirical implementation while trying to reconcile these. In their recent study, the above authors have revealed interesting aspects concerning empirical research when q-generalized cross-entropy is associated with constraining information.

Following recent literature (e.g., Abe, Bagci, 2004; Venkatesan, Plastino, 2011), the generalized Kullback-Leibler defined by eq. 3 could be more consistent with

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[6] Eq. (2) can be optimized under moment restriction and then represents the generalized maximum entropy principle.

[7] See, e.g., Kullback (1968) for a rich definition of this index and its connection with Bayesian formalism.

[8] If we dispose of two systems P and R, the level of q-Tsallis allows for definition of three different entropies. For $q < 1$, the Tsallis entropy becomes a super-extensive entropy where $Sq(P + R) < Sq(P) + Sq(R)$; for $q = 1$, the Tsallis entropy reduces to a standard Gibbs-Shannon extensive entropy where $Sq(P + R) = Sq(P) + Sq(R)$; for $q > 1$, the Tsallis entropy becomes a sub-extensive entropy where $Sq(P + R) > Sq(P) + Sq(P)$.

expectations and the constraints form proposed by Tsallis et al. (1998), known as *q-averages* or *escort distribution*[9]:

$$\langle y_q \rangle = \sum_i \frac{p_i^{\ q}}{\sum_i p_i^{\ q}} y_i \, .$$

3. A GENERALIZED LINEAR NON-EXTENSIVE ENTROPY ECONOMETRIC MODEL

This section applies the results of, e.g., Jaynes (1994) and Golan at al. (1996), to present the model to be later implemented for updating and balancing the social account matrix of the Polish economy. While the argument in criterion function is already known (see eq. 7), we need to reparameterize the generalized linear model which has to play the role of restrictions. Note that this presentation for the present problem is limited to methodological aspects. In fact, elements inside a SAM can be meaningfully presented by columns as the ratio explaining a sector disbursement distribution in favour of the rest of economy sectors. Each coefficient varies between zero and one and the coefficient total by column sums up to unity. Definitely, support space, usually defined a priori for the purpose of reparametrization, coincides with probability space. In this case, the accuracy of estimated parameters is higher as there is non-loss of information from this a priori data (Shen, Perloff, 2001). In any event, to be consistent, let us succinctly present the general procedure of parameter reparametrization in the case of a general inverse linear model:

$$Y = X \cdot \beta + \varepsilon \, , \tag{4}$$

where unknown β parameter values are not necessarily constrained between 0 and 1, which suggests the necessity of reparametrization. The term ε is an unobservable disturbance term, plausibly with finite variance, owing to the nature of economic data, exhibiting observation errors from empirical measurement or from random shocks. These stochastic errors are assumed to be driven by PL, as explained in the introductory section of this document. The variable Y represents a system and X accounts for covariates generating the system through relation parameter matrix β and unobservable disturbance ε to be estimated through observable error components e. Unlike classical econometric models, no constraining hypothesis is needed. In particular, the number of parameters to be estimated may be higher than the observed data points, and the quality of collected information data low. Additionally, as already explained,

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[9] However, for computational reasons, we have definitely opted in this document for applying the Curado-Tsallis (C-T) constraints [2] of the form:
$\langle y_q \rangle = \sum_i p_i^{\ q} y_i$ where the symbol $\langle \ \rangle$ means that y_q is an average of y_i weighed by pi_q.

to increase precision of such estimated parameters from poor-quality data points, the entropy objective function allows for incorporation of all constraining consistency moments, which act as new Bayesian evidence in the model (Zellner, 1991). Thus, referring to, e.g., Jaynes (1994) and Golan et al. (1996), owing to the maximum entropy principle, each new piece of constraining information will reduce the entropy level of the system depending on the degree of its consistency with the prior.

Taking each β_k (k = 1…K) as a discrete, random variable with compact support (Golan et al. 1996) and $2 < M < \infty$ possible outcomes, one can estimate it by B_k, that is:

$$B_k = \sum_{m=1}^{M} p_{km} v_{km} \ , \ \forall k \in K \ , \tag{5}$$

where p_{km} is the probability of outcome v_{km} and the probabilities must be non-negative and sum up to one. Similarly, by treating each element e_i of e as a finite and discrete random variable with compact support and $2 < M < \infty$ possible outcomes centred around zero, we can express e_i as:

$$e_i = \sum_{j=1..J} r_{nj} z_{nj} \ , \tag{6}$$

where r_n is the probability of outcome z_n on the support space j. We will use the commonly adopted index n, here and in the remaining mathematical formulations, to set the number of statistical observations. It is worth note that the term e can be empirically fixed as a percentage of the explained variable as an a priori Bayesian hypothesis. Posterior probabilities within the support space may display a non-Gaussian distribution class. The element v_{km} constitutes an a priori information provided by the researcher while p_{km} is an unknown probability whose value must be determined by solving a maximum entropy problem. In matrix notation, let us rewrite $\beta = V \cdot P$, with $p_{km} \geq 0$ and $\sum_{k=1}^{K} \sum_{m>2\ldots M} p_{km} = 1$, where again, K is the number of parameters to be estimated and M the number of data points over the support space. Also, let $e = r \cdot z$, with $r_{nj} \geq 0$ and $\sum_{n=1}^{N} \sum_{j>2\ldots J}^{J} r_{nj} = 1$ for N the number of observations and J the number of data points over the support space for the error term. Then, the Tsallis cross-entropy econometric estimator can be stated as:

$$MinH_q(p // p^0, r // r^0, w // w^0) \equiv$$

$$\equiv \alpha \sum p_{km} \frac{\left[p_{km} / p^o{}_{km}\right]^{q-1} - 1}{q-1} + \beta \sum r_{nj} \frac{\left[r_{nj} / r^o{}_{nj}\right]^{q-1} - 1}{q-1} + \delta \sum w_{ts} \frac{\left[w_{ts} / w^o{}_{ts}\right]^{q-1} - 1}{q-1}, \tag{7}$$

s. t.
$$Y = X \cdot \beta + e = X \cdot \sum_{m=1}^{M} v_m \left(\frac{p_{km}{}^{q}}{\sum_{m=1}^{M} p_{km}{}^{q}} \right) + \sum_{j=1}^{J} z_j \left(\frac{r_{nj}{}^{q}}{\sum_{j=1}^{J} r_{nj}{}^{q}} \right), \tag{8}$$

$$\sum\nolimits_{k=1}^{K} \sum\nolimits_{m>2\ldots M} p_{km} = 1, \tag{9}$$

$$\sum\nolimits_{n=1}^{N} \sum\nolimits_{j>2\ldots J}^{J} r_{nj} = 1,$$

$$\sum\nolimits_{t=1}^{T} \sum\nolimits_{s>2\ldots S}^{S} w_{ts} = 1.$$

Additionally, k macro-aggregates can be added to the set of above constraining consistency moments as follows:

$$\sum_i \sum_j \mathrm{H}^{(d)}{}_{ij} T_{ij} = \gamma^{(d)} + \sum_{s=1}^{S} g_s \left(\frac{w_{ts}{}^{q}}{\sum_{t=1}^{T} w_{ts}{}^{q}} \right), \tag{10}$$

where H is a *dxd* aggregator matrix with ones for cells that represent the macro-constraints and zeros otherwise, and γ is the expected value of the aggregate constraint. Once again, g_s stands for a discrete point support space from $s = 2..S$. Probabilities w_{ts} stand for point weights over g_S. The real q, as previously stated, stands for the Tsallis parameter. In the empirical part of this document, the Polish gross domestic product at market and at factor prices will exemplify the above "macro-aggregates".

Above, $H_q(p // p^0,\ r // r^0,\ w // w^0)$ is nonlinear and measures the entropy in the model. Relative entropies of the three independent systems (the three posteriors p, r and w and the corresponding priors p^0, r^0 and w^0, respectively) are then summed up using the weights α, β, δ. These are real positives summing up to unity under the given restrictions. The symbol // is a "distance metric"[10] of divergence information. We need to find the minimum divergence between the priors and the posteriors while the imposed restrictions must be fulfilled. As will be the case in the application below, the first component of the criterion function may concern the parameter structure of the table, the second component errors on column (or row) totals and the last component may concern errors around any additional consistency variable, like the GDP in the

[10] However, note that K-L divergence is not a true metric since it is not symmetric and does not satisfy the triangle inequality.

case below. As has been shown by Tsallis (2009), this form of entropy displays the same basic properties as K-L IDI or relative entropy. The estimates of the parameters and residual are sensitive to the length and position of support intervals of β parameters (eq. 5 and eq. 6) in the context of the Bayesian prior. When parameters of the proposed model are expressed under the form of elasticity or ratios, then the support space should be defined inside the interval between zero and one and will correspond to that of the usual probabilities. In such a case, no reparametrization of parameters is needed. In other cases, support space may be defined between minus and plus infinity, according to intuitive evaluation by the modeller. Additionally, within the same support space, the model estimates and their variances should be affected by the support space scaling effect, i.e., the number of affected point values (Golan et al. 1996). The higher the number of these points, the better the prior information about the system. The weights α, β, δ are introduced into the above dual objective function. The first term of "precision" accounts for deviations of the estimated parameters from the prior (generally defined under a support space). The second and the third terms of "prediction ex post" account for the empirical error term as a difference between predicted and observed data values of the model. As expected, the presented entropy model is an efficient information processing rule which transforms, according to Bayes's rule, prior and sample information into posterior information (Zellner, 1991).

4. PARAMETER CONFIDENCE AREA

In this section we will propose an inference information index $s(a_j)$ as an equivalent to a standard parameter error measure in the case of classical econometrics. An equivalent of determination coefficient R^2 will be proposed, too, under the entropy symbol $S(\text{Pr})$. The departure point is that the maximum level of entropy-uncertainty is reached when non-relevant information-moment constraints are enforced. This leads to a uniform distribution of probabilities over the k states of the system. As we add each piece of informative data in the form of a constraint, a departure from the uniform distribution will result, which means uncertainty shrinkage. Thus, the value of the proposed $S(\text{Pr})$ below should reflect a global departure from the maximum uncertainty for the whole model. Let us follow formulations presented by Golan et al. (1996) and propose a normalized non-extensive entropy measure of $s(a_j)$ and $S(\text{Pr})$. From the Tsallis entropy definition, $S_q > 0$, let us consider now all possible micro-states of the model. This number varies with the number of support space data points i (i = 1..M) and the number of parameters of the model $j(j = 1..J)$. Entropy S_q vanishes (for all q) in the case of $M = 1$; and for $M \succ 1$, q > 0, whenever one of the p_i $(i = 1..M)$ occurrence equals unity, the remaining probabilities, of course, vanish. A global, absolute maximum of S_q (for all q) is obtained, in the case of uniform distribution, i.e., when all $p_i = 1/M$. In such an instance, we have for both systems the maximum entropy equal to:

$$S_q(a_j) = (M^{1-q} - 1)\cdot(1-q)^{-1} \tag{11}$$

and

$$S_q(r) = (n^{1-q} - 1)\cdot(1-q)^{-1}. \tag{12}$$

In eq. 11, n varies with the number of support space data points and the number of observations of the model. We propose below a normalized entropy index in which the numerator stands for the calculated entropy of the system and the denominator displays the highest maximum entropy as shown above (eq. 11 and 12):

$$s(a_j) = p_{ij}\frac{(p_{ij}/p_{ij}{}^{0})^{q-1} - 1}{q-1} / (M^{1-q} - 1)(1-q)^{-1} = p_{ij}\frac{(p_{ij}/p_{ij}{}^{0})^{q-1} - 1}{(1-M^{1-q})} \tag{13}$$

with j varying from 1 to J (number of parameters of the system) and i belonging to M (number of support space points), with $M > 2$; with the total number micro-states, which is obtained by multiplying number of model parameters J by number of support space points M with $M > 2$. Then $s(a_j)$ reports precision on the estimated parameters. Equation 14 reflects the non-additivity Tsallis entropy property for two independent systems. The first term $S(p)$ is related to parameter probability distribution and the second $S(r)$ to error disturbance probability:

$$S(\hat{\mathrm{P}}\mathrm{r}) = [S(\hat{p} + \hat{r})] = \{[S(\hat{p}) + S(\hat{r})] + (1-q)\cdot S(\hat{p})\cdot S(\hat{r})\}, \tag{14}$$

where $S(P) = \sum\sum p_{ij}\frac{(p_{ij}/p^{0}{}_{ij})^{q-1} - 1}{(q-1)(M^{1-q} - 1)}$, and $S(r) = -\sum r_{nf}\frac{(r_{nf}/ro_{nf})^{q-1} - 1}{(q-1)n(1-F^{1-q})}$.

$S(\hat{\mathrm{P}}\mathrm{r})$ is then the sum of normalized entropies related to parameters of the model $S(\hat{p})$, and to disturbance term $S(\hat{r})$. Likewise, the latter value $S(\hat{r})$ is derived for all observations n, with F the number of data points on the support space of estimated probabilities r related to the error term. As it results from the above formulation, the values of these normalized entropy indexes $S(\hat{a}_{ij})$, $S(\hat{\mathrm{P}}\mathrm{r})$ vary between zero and one. Its values, near unity, indicate a poor informative variable- with higher entropy- while lower values are, on the contrary, an indication of a better informative variable about the model. From *information properties* and the above formulation of the q-generalized cross-entropy concept (see eq. 3), the reader can observe that both indexes fulfil basic Fisher-Rao-Cramer information index properties, among them continuity, symmetry, maximum, and additivity.

5. THEORETICAL ASPECTS OF BALANCING A SAM

A SAM is a quadratic table that encompasses information about complex processes of supply and demand of a real, open economy involving, under optimizing behaviors, different economic agents and endowments for a given time period and region. Regarding SAM construction and components (see, e.g., Pyatt, Round, 1985), general equilibrium (e.g., Wing, 2004) implies that respective row and column totals are expected to balance. Conceptually, this model is based on the laws of *product and value conservation* which guarantee conditions of zero profit, market clearance, and income balance (Scrieciu, Blake, 2005). However, different stages of statistical data processing remain concomitant with observation and measurement errors, and the SAM will not balance. This means that an unknown number of economic transaction values within the matrix are inconsistent with the data generating macroeconomic system. For clarity, let us use Table 1 to explain these imbalances, noting, for instance, a difference between the activities row and column totals as follows:

$$(aT + u_1) - (aT + \varepsilon_1) = (u_1 - \varepsilon_1)\,. \tag{15}$$

The term on the left hand side of the above expression stands for the difference between two erroneous and unequal totals of the activity account. Its origin is the plausibly different stochastic errors u_1 and ε_1 on column and row totals, respectively. In Table 1, the first alphabetical letter of symbols inside each cell stands for the first letter of the row (supply) account, and the second letter represents the first letter of the corresponding (demand) column. For instance, in the prototype SAM below, the symbol “ca” stands for the purchases by the *activity sector* of goods and services from the *commodity sector*.

Table 1.

A simplified stochastically non-balanced SAM

	Activities	Commodities	Factors	Institutions	Capital	World	Total
Activities	0	ac	0	Ai	0	aw	aT+ε_1
Commodities	Ca	0	0	Ci	cc	0	cT+ε_2
Factors	Fa	0	0	0	0	0	fT+ε_3
Institutions	Ia	ic	If	Ii	0	iw	iT+ε_4
Capital	0	0	0	Ci	0	cw	cT+ε_5
World	0	wc	0	Wi	0	0	wT+ε_6
Total	aT+u_1	cT+u_2	fT+u_3	iT+u_4	cT+u_5	wT+u_6	

Source: own presentation.

The objective is to find, out of all probability distributions, the one (the posterior) closest to Table 2 (the prior) and ensuring its balance while satisfying other imposed consistency moments and normalization conditions. Referring to Shannon entropy, one may consider post entropy structural coefficients and disturbance errors, respectively, as *signal* and *noise*. The first step consists of computing a priori coefficients by column, from real data from Table 2, by dividing each cell account by the respective column total. Next, we treat these column coefficients as analogous to probabilities, and column totals as expected column sums, weighted by these probabilities (see eq. 7). Coefficient values in initial Table 2 will serve as the starting, best prior estimates of the model. Two other types of priors to initialize the solution concern errors on column totals (eq. 8) and on gross domestic product (GDP) at factor and market prices (eq. 10). GDP variables are added to the model with the purpose of restricting the model to meet consistency macroeconomic relationships for different accounts inside the SAM. The proposed approach combines *non-ergodic Tsallis entropy with Bayes's rule to solve a generalized random inverse problem*. We may optionally consider only some cell values as certain[11] while the rest of the random accounts are unknown. Once again, this is one of the strongest points of the entropy approach over most rival mechanical techniques of balancing national account tables. All row and column totals are not known with certainty. It is apparent that the potential number of degrees-of-freedom of parameters to estimate $n(n-1)$ remains significantly higher than n observed data points (column totals). In the particular case of a SAM, and due to empty cells, that number of unknown parameters may be much lower. Nonetheless, that will not generally prevent us from dealing with an ill-behaved inverse stochastic problem. The next important step is that of initializing the above defined error trough, a reparameterizing process. *A five point support space symmetric around zero* is defined. To scale the error support space to real data, we apply Chebychev's inequality and Three Sigma rule (Golan at al. 1996; Pukelsheim, 1994). Corresponding optimal probability weights are then computed so as to define the prior noise component (Robinson, El-Said, 2000).

6. BALANCING A SOCIAL ACCOUNTING MATRIX OF POLAND AND OUTPUTS

This section presents one of the plausible applications of the non-extensive cross-entropy approach. Readers acquainted with the Shannon entropy approach[12] and its economic applications may know its particular role in recent years for balancing social accounting matrices of many countries (e.g., Miller, Matthews, 2012; Robinson at al., 2000). In the present case, we have used this new technique to balance the

[11] In the present case, only transaction accounts with the rest of the world (import, export, external current balance), plus government commodity consumption accounts are concerned.

[12] We recall here that Shannon-Gibbbs entropy remains a converging case of Tsallis non-extensive entropy.

Polish SAM of 2005. Technically, the problem of cross-entropy is to find a new set of SAM coefficients (posteriors) that minimize the so-called Kullback-Leibler (1951) divergence measure of the Tsallis "cross-entropy" (CE) between the prior (the initial, unbalanced SAM) and the posteriori SAM, under given restrictions. These are related to data moments, normalization condition, or any other a priori information presenting consistency with posterior probabilities in the criterion function (see eq. 7–10). For the model computations, we have used the GAMS code and the solver Minos5. Table 2 and Table 3 present the non-balanced and the post entropy-balanced SAM, respectively. The statistical data used come from the Polish Main Public Office of Statistics (http://www.stat.gov.pl/gus/), and from EUROSTAT (www.eurostat.eu). In Table 2, the number values in the total column marked in bold are related to the non-balanced sectors. As suggested in the preceding section, such imbalances and inconsistencies mainly result from the complexity of economic information gathering at country scale, where various institutions constitute different and contradictory sources of information. Furthermore, other human error during statistical table compilation remains plausible. In their recent work, trying to balance the Polish SAM for 2010, Tomaszewicz, Trębska (2013) have noticed the lack of direct data values of current and capital transfers in the case of Polish statistical data. As explained in the celebrated work of Golan et al. (1996), and based on various simulations, entropy formalism acts as a Bayesian efficient processing rule. Then, independent of the prior information level, when new data (new evidence) is consistent with the data generating process, the entropy formalism allows the estimator to quickly converge toward the minimum variance. However, in the real world, the data generating system is unknown and the assessment of a new methodology may rely on mere opinion. In fact, an official balanced SAM may still contain many conflicting errors, for instance, those related to the selected closure rule. There are other SAM balancing techniques. The RAS approach remains the most popular among them. In a recent, thorough study on the comparative performance of cross-entropy and RAS techniques, Chisari at al. (2012) concluded that cross-entropy had a more general character for the reasons listed below:

a. It does not need all the new totals of rows or columns (although prediction will be less accurate).
b. It does not need a balanced initial matrix (the sum of rows could be more/less than the sum of columns).
c. New rims could contain an error term.
d. New rims can be non-fixed parameters.
e. Many values on the final matrix could be fixed (not necessarily a parameter).
f. It allows non-linear constraints.

Referring to their simulation outputs, the authors propose *a rule of thumb consisting of preferring the RAS method if and only if no constraint or one constraint is enforced*. This seems to explain why the RAS approach continues to be successfully applied in different prediction studies. In a recent study conducted by Bwanakare

Table 2.

Initial unbalanced Polish SAM (2005)

	aAct	pCom	Labor	Capital	Pollfees	Hou	Ent	GRE	CapAc	RoW	Total
aAct	0.0	160.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	36.5	196.6
pCom	108.6	0.0	0.0	0.0	0.0	71.6	0.0	7.8	18.9	0.0	207.0
Labor	35.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	35.2
Capital	50.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	50.5
Pollfees	2.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.3
Hou	0.0	0.0	31.7	27.9	0.0	0.0	7.2	27.1	0.0	1.8	95.7
Ent	0.0	0.0	0.0	24.8	0.0	0.0	0.0	0.0	0.0	0.0	24.8
GRE	0.0	10.3	0.0	0.0	2.3	22.4	7.0	0.0	0.0	0.0	42.0
CapAc	0.0	0.0	0.0	0.0	0.0	6.5	11.0	0.5	0.0	0.9	18.9
RoW	0.0	37.2	0.0	0.0	0.0	0.0	1.9	0.0	0.0	0.0	39.1
Total	196.6	207.7	31.7	52.7	2.3	100.5	27.2	35.5	18.9	39.1	

Source: own compilation.

Table 3.

Balanced, post non extensive entropy Polish SAM (2005); weight equals (0.05; 0.94; 0.01)

	Aact	Pcom	Labor	Capital	Pollfees	Hou	Ent	Gre	Capac	Row	Total
Aact		160.2								36.5	196.7
Pcom	109.4					71.13		7.85	18.94		207.3
Labor	33.46										33.46
Capital	51.61										51.61
Pollfees	2.272										2.272
Hou			33.46	25.62			6.9	30.3		1.8	98.1
Ent				25.99							25.99
Gre		9.848			2.272	20.13	6.52				38.76
Capac						6.843	10.6	0.61		0.86	18.94
Row		37.2					1.94				39.13
Total	196.7	207.3	33.46	51.61	2.272	98.1	26	38.8	18.94	39.1	

Source: own compilation.

(2013b) consisting of balancing the EU input output matrix, the author- after having applied only a single constraint- found the outputs from the RAS approach slightly better compared with those from the cross-entropy technique. Thus, the conclusion from that study seems to support the one presented above by Chisari at al. (2012). However, this suggestion does not seem to be consistent with the investigations done by Robinson, El-Said (2000) on the Mozambique economy. These authors have found that the RAS and Shannon entropy approaches produce the same performance when no additional restriction is imposed. More investigations are needed to contradict or confirm the findings of the authors mentioned in this paragraph. Nevertheless, taking its stochastic characteristics into account, cross-entropy potentially has a higher performance than the RAS approach, particularly when statistical data are known with uncertainty.

The main purpose of the figures displayed below is to put emphasis on some model output characteristics through selected parameters or indices. In particular, the impact of q-Tsallis variation and weights in criterion function on computed outputs is underscored. Increasing this parameter is equivalent to a kind of "complexifying" of interrelations between economic actors or sectors inside the economy (Foley, Smith, 2008), such as reinforcing competitive conditions. Three distinct weight components (eq. 7) {(0.94;0.05;0.01)_p; (0.333; 0.334; 0.333)_nw; (0.05; 0.94; 0.01)_w1} have been assigned in the entropy criterion function and each weight inside each set corresponds, respectively, to distribution of SAM coefficients, column totals, and GDP disturbance errors. GDP accounts deserve relatively lower importance as they are only connected with a limited number of SAM accounts (production factors and tax income). Then, symbols _p, _nw, and _w1 on the right hand side of each of the above weight set underscore the dominant probability in each set. In particular, the _nw corresponds to the case equivalent weights. Figure 1 compares model goodness according to weights assigned to different components in the criterion function, for different *q* lying inside Gaussian attractor interval [1-5/3]. Increasing weights on the parameter probability component should enhance post-entropy SAM coefficient precision while worsening error estimation, thus at the cost of model ex-post-prediction (Golan at al., 1996). As has already been said, the model entropy encompasses statistical losses in the parameter space (precision) and in the sample space (prediction). Analytically, it can be directly shown that Lagrange multipliers stand for implicit *nonlinear* function of weights imposed in the generalised cross-entropy criterion function. Changes in weights thus alter the corresponding optimal solution value. In general, as in most constrained optimisation problems, *smaller* Lagrange multipliers for a *q* cross-entropy formulation should imply smaller impact of constraints on the objective, at least for *q* around unity, i.e., the Gaussian case. The above defined three weight types correspond, respectively, to three goodness indices "S(Pr)": $good_{-p}, good_{-nw}, good_{-w1}$, where S(Pr) is the total normalized entropy of the system (eq. 14). This index then tells us, given the unbalanced prior SAM, to what extent new evidence reflected in constraining moment conditions and the estimated model has discriminated in favour of the balanced post entropy SAM for different levels of the q-Tsallis parameter. In the present model, its highest value is around 0.99 once higher weight has been imposed on column total errors (_w1) for a *q*- parameter evolving around unity. We recall that this inference index varies between zero and one. Figure 2 analyses the precision-prediction loss trade-off between the two random sources of model sensitivity by the above selected weights and different *q*-Tsallis parameters. We compare two extreme weighting cases A= {(0.94; 0.05; 0.01)_p and B=(0.05; 0.94; 0.01)_w1}. The symbol "P*PI shrink*" is a precision index for each q-Tsallis parameter. To get the measure, we first calculate the relative differences (in absolute value) between the SAM post-entropy probability from cases A and B. Next, we calculate the arithmetical divergence mean by summing up, in absolute values, those differences divided by the number of structural probabilities being parameters within the table.

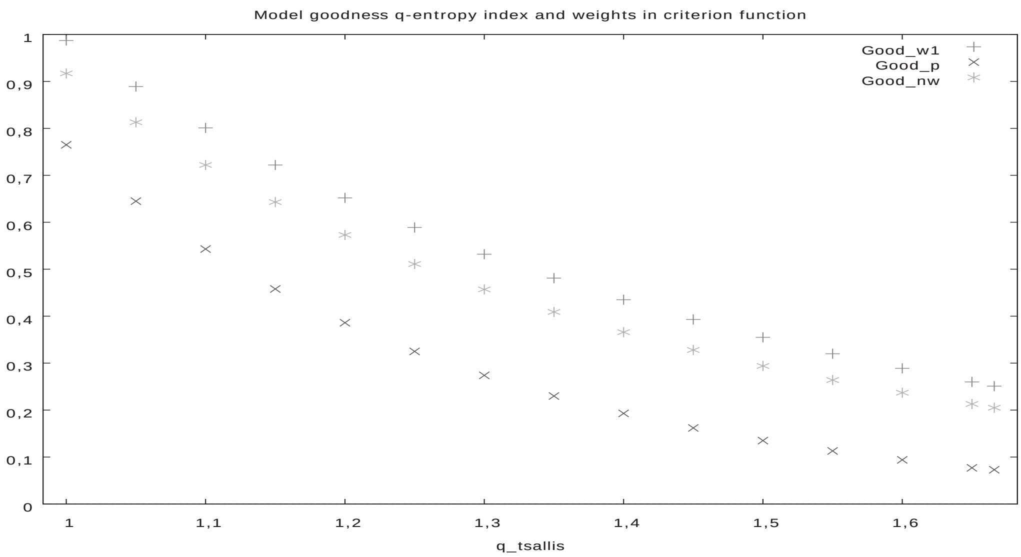

Figure 1. Model of goodness q-entropy index and weights in criterion function

The next prediction index values, "Sigma shrink," are obtained in the same way as "*PI shrink*" described above with the difference that, in this last case, attention is drawn to standard disturbance error affecting column totals. As we can observe, reducing weights on the SAM probability component in favor of the column total errors component relatively increases information divergence related to SAM coefficients between the prior and the posterior. Impact of such a weight change is to reduce standard disturbance error on column totals. This is described by Figure 2, where the best outputs are reflected by values at the beginning of the curve in the south-eastern corner. We notice, in the present case, a higher sensitivity of error component to weight change than the one from SAM coefficients. The index varies between approximately 0 and 0.9 while in the last case it varies between -0.12 and zero.

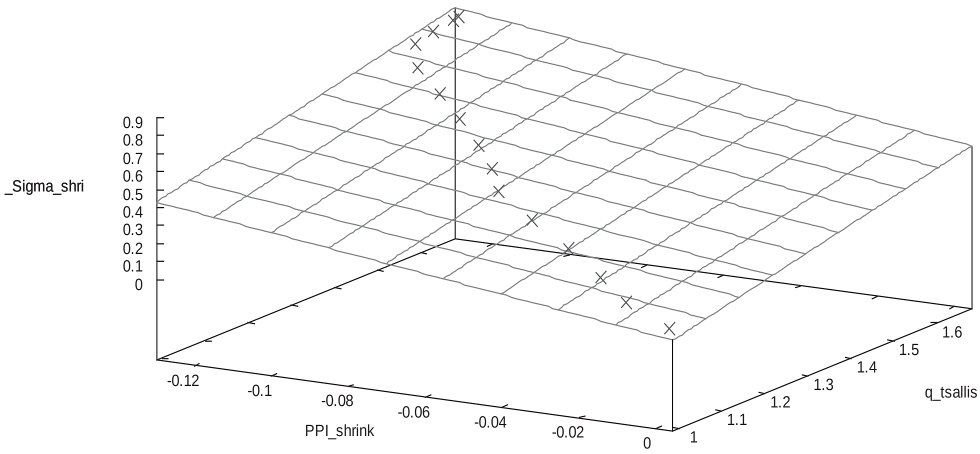

Figure 2. Precision and prediction loss tradeoff due to weight change in c.f. for different q_ parameters

7. CONCLUDING REMARKS

This paper aims at extending applications of a non-extensive entropy approach to modeling generalized inverse problems in the case of stochastically balanced systems. A Polish SAM, as a case study, has been optimally balanced. However, because the existing SAM represents only an approximation of the unknown true values of the macroeconomic transactions, it is difficult to accurately assess outputs of the estimated model. We found optimal outputs for q-Tsallis close to unity, suggesting the Gaussian structure of the SAM. Statistical inference indices proposed in this paper have been used to analyze the tradeoff between parameter precision and sample prediction for different weights in the objective function and different q-Tsallis complexity parameters. Superiority of the proposed approach should rely essentially on its generalizing attributes owing to its non-extensivity, conceptually ensuring solutions less prone to initial conditions. We suggest more investigations in other economies and other fields, particularly those in countries with different economic structures.

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EKONOMETRYCZNE ZBILANSOWANIE MACIERZY RACHUNKOWOŚCI SPOŁECZNEJ POD HIPOTEZĄ PRAWA POTĘGOWEGO

S t r e s z c z e n i e

Względna Entropia Shannon-Kullback-Leibler (SKLCE) jest szczególnie przydatna przy rozwiązaniu problemu odwrotnego systemu ergodycznego. Choć empiryczne zastosowanie podejścia Shanon-Gibbsa spotkało się ostatnim czasem ze znacznym sukcesem, cierpi jednak cały czas ze względu na charakter hipotezy ergodycznej, ograniczając wszystkie mikroelementy systemu pojawianiem się identycznego prawdopodobieństwa. Niniejszy artykuł ma na celu rozszerzenie zastosowania nieekstensywnego modelu względnej entropii (NECE) dla zbilansowania losowych macierzy wyjścia-wejścia. Model ten postuluje, że działalność ekonomiczna cechuje się długookresową pamięcią kompleksowych interakcji między podmiotami gospodarczymi lub między sektorami. Stosując własności skalowania prawa potęgowego budujemy model, który z powodzeniem zbilansuje polską macierz rachunkowości społecznej cechującą się równowagą ogólną Warlasa. Zaproponowano wnioskowanie statystyczne dla przedziału ufności indeksów informacji. Zaobserwowano, że zwiększenie wag komponentów składnika losowego dualnego kryterium funkcji prowadzi do większych wartości parametru q-Tsallisa, zaś zmniejszenie tych wag przybliża wartość parametru q-Tsallis'a do jedności. Przewagą podejścia entropii Tsallis'a nad innymi konkurującymi metodami jest możliwość uogólnienia modelu Gaussowskiego, ze względu na to, że bierze ono pod uwagę istnienie rozkładu grubego ogona. Dzięki cechom parametru q-Tsallis'a możliwą staje się również ocena kompleksowości systemu statystycznego.

Słowa kluczowe: q-uogólniana dywergencja informacji Kullback-Leibleir'a, macierz rachunkowości społecznej

ECONOMETRIC BALANCING OF A SOCIAL ACCOUNTING MATRIX UNDER A POWER-LAW HYPOTHESIS

Abstract

Shannon-Kullback-Leibler cross-entropy (SKLCE) is particularly useful when ergodic system inverse problems require a solution. Though empirical application using the Shanon-Gibbs approach has recently met with notable success, it suffers from its ergodicity, constraining all micro-states of the system to appear with identical odds. The present document aims at extending applications of a *non-extensive cross-entropy model* (NECE) for balancing an input output stochastic system. The model then postulates that economic activity is characterized by long run complex behavioural interactions between economic agents and/or economic sectors. Applying scaling property of a Power-law we present a model which successfully balances a Polish national social accounting matrix (SAM) expected to exhibit Warlasian general equilibrium features. The *Rao-Cramer-Kullback* inferential information indexes are proposed. We note that increasing relative weight on the disturbance component of the dual criterion function leads to higher values of the *q-Tsallis complexity index* while smaller disturbance weights produce q values closer to unity, the case of Gaussian distribution.

The great advantage of the approach presented over rival techniques is its allowing for the generalisation of Gaussian law enabled by its capability of including heavy tall distributions. The approach also constitutes a powerful instrument for the assessment of complexity in the analysed statistical system thanks to the q-Tsallis parameter.

Keywords: q-Generalization of K-L information divergence, social accounting matrix